SOURCE AND CORRECTION OF ERRORS AFFECTING MULTIPLEX MAPPING

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IN ANY process or method of mapping by stereo-photogrammetrical means there are many sources contributing to the final error. A map prepared by a method in which these errors are ignored does not compare in accuracy with a map prepared by the same method if these errors are known in source, location, and magnitude, and steps are taken to eliminate or compensate for them. Therefore, to name a process or particular plotting instrument is not to name an accuracy.

It was with the above in mind that the work on which this paper is based was performed by personnel in the Chattanooga office of the U. S. Geological Survey-Tennessee Valley Authority joint mapping program. The need became apparent for a detailed study of errors and methods of handling them as soon as work with the wide-angle lens was undertaken, as the distortion of this lens is such that the resultant errors would otherwise be greatly in excess of those which could be tolerated under the specifications as set up for accurate topographic mapping.

Increased accuracy may be attained by reducing the flight height, but as this would increase the cost materially, it would only be considered as a last expedient.

No attempt is made to enumerate or discuss all of the many sources of errors affecting a map plotted by means of the Multiplex, nor is consideration given here to the personal error, as this will vary with the individual and, to some extent, will always be present.

The problem was divided into two parts: first, that of finding the total or final error, and second, that portion which each step contributes toward the total error.

This discussion of mapping, by means of the Multiplex, divides naturally into four major sections as follows:

1. The Aerial Camera and Lens
2. The Aerial Film
3. The Diapositive Printer
4. Multiplex Projectors

Of the four named above, one item in particular is discussed in detail, that of camera lens distortion and the resultant errors as they affect the final map, since this was found to be the largest contributing source of error.

The problems considered in this paper are approached along lines that the problems naturally take when the Multiplex is primarily considered. However, the same method of approach is applicable to any plotting method where conjugate images are projected or observed through lenses which do not completely compensate for the camera lens distortion, thereby leaving a residual, or differential distortion to be considered.

The total error was determined by selecting several stereoscopic models in which an abundance of control existed, or was established, so distributed as to give the desired information and establish a pattern for the errors. Models were also selected which contained reservoirs having many miles of meandering shore line which could be used as profiles of equal elevation, against which errors in the model could be established. The results found proved quite similar to those established by mathematical analyses which are described later in this paper.

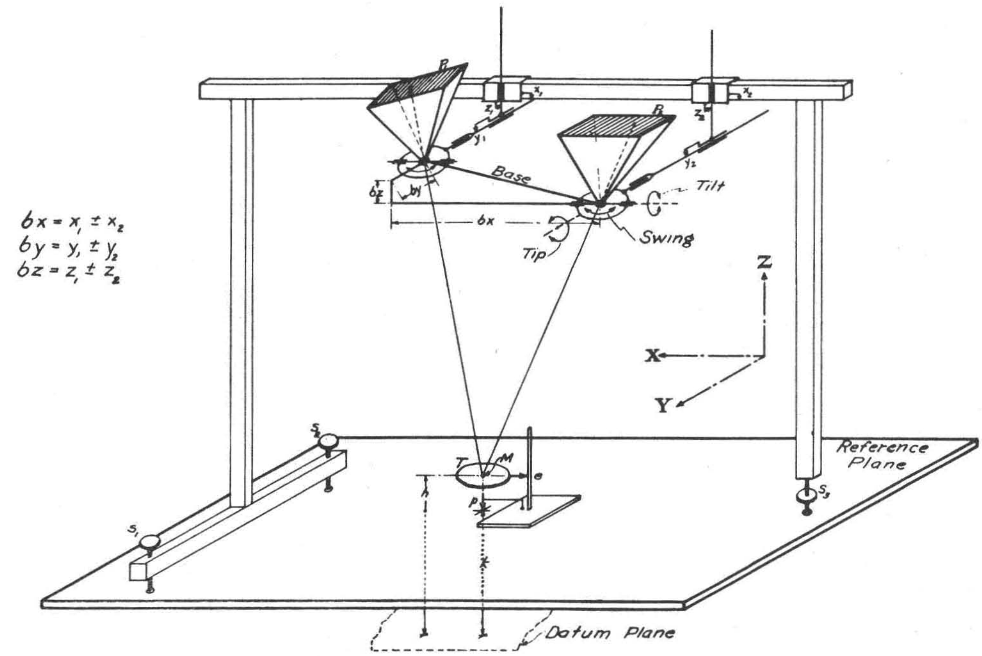


FIG. 1. SCHEMATIC DIAGRAM OF THEORY OF MULTIPLEX.

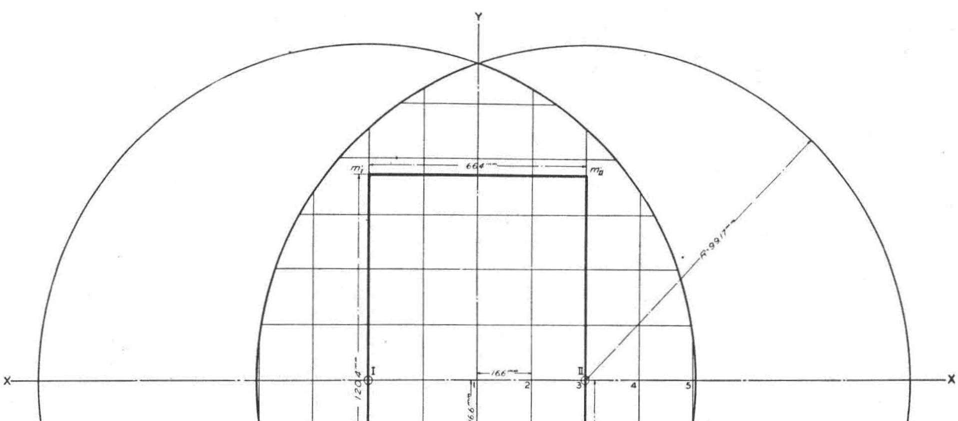


FIG. 2. OVERLAPPING PHOTOGRAPHS, I AND II, SHOWING TOTAL STEREOSCOPIC AREA, NEAT MODEL $m_I m_{II} m_{II}' m_I'$, AND GRID PATTERN OF POINTS USED IN CALCULATIONS, WITH DIMENSIONS.

1. The Aerial Camera and Lens

When the aerial camera is carefully adjusted and calibrated, mechanically and optically, to precision standards, the only remaining source of large errors is that of lens distortion. This lens distortion may be compensated for, to a large degree, by using a suitable lens having opposite distortion to that of the camera lens, in one of several possible locations. The amount and effect of the distortion which finally exists in the stereoscopic model (differential lens distortion) are determined in a manner described in detail below.

In order to analyze the performance of the camera and lens as a unit, a complete calibration report such as may be procured from the National Bureau of Standards is necessary. Such a report was obtained, and values as measured by the National Bureau of Standards were used as representing the distortion of this particular camera lens, No. 1876038. It is assumed, in this part of the discussion, that there are no sources of error other than those resulting from the differential distortion of the camera lens.

The X and Y-directions have been taken as horizontal, or parallel to sea level; X being the direction of the line of flight, and Y being perpendicular to X. The Z-direction is perpendicular to X and Y, or vertical. Refer to Fig. 1.

Throughout this paper the model is considered to be that of a horizontal plane in nature with photographs truly vertical and flight height equal. Tilts and differences in flight heights usually encountered in stereoscopic plotting are considered small enough to have slight effect on the final result. The effect of relief is considered briefly in the last part of this discussion.

True relative orientation of a stereoscopic pair is attained when the two projected angle-true cones of rays occupy the exact relative scale positions of their respective positions in nature. If there were no distortion present in the camera lens, all pairs of conjugate image rays would then intersect and each intersection would then locate some point in the model having the same relative position, or X, Y, and Z co-ordinates, as the corresponding points in nature.

With the above orientation maintained and distortion in the camera lens now considered, many of the conjugate image rays will fail to intersect, even though the projectors occupy the exact relative scale positions of the camera stations in space. For any given pair of conjugate image rays, whether they intersect or not, there is only one elevation at which they have no X-separation or X-parallax. In cases where the rays do not intersect, a point, p''', Fig. 7, midway between the two rays at this elevation will be considered as their virtual intersection. The amount of this failure of intersection is indicated in Fig. 7 by y_p which is the Y-parallax at this point in the model. It is shown in Table II that there is no y_p of more than .26 mm. at any point in the model.

Besides causing failure of some pairs of conjugate image rays to intersect, distortion in the camera lens also causes the intersections, actual or virtual, to be removed from their true relative locations in the model. The amount by which any point in the model is removed from its true relative position is the error at that point. This error may be resolved into X, Y, and Z components.

As has been stated, in the case of true relative orientation of the projectors, the distortion in the camera lens causes y_p. There is no y_p at the points directly beneath or in a line joining the two projected principal points of the projectors but there is an equal and symmetrical amount, Δy, at each of the four corners of the model. This parallax at the four corners is automatically removed during the usual process of attaining apparent relative orientation by tipping the projectors equal amounts but in opposite directions, Fig. 6. This applies to most stereoscopic plotting instruments which have differential lens distortion and

TABLE I. CALCULATION OF VERTICAL ERROR DUE TO LENS DISTORTION, INCLUDING EFFECT OF TIP—MODEL SCALE = 1:15,840

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Point	Distance from I mm.	D_1 mm.	D_{1x} mm.	Distance from II mm.	D_2 mm.	D_{2x} mm.	$\frac{qx_1^2}{Z^2}$	$\frac{qx_2^2}{Z^2}$	Tip ΔX_1 $\frac{qx_1^2}{Z^2}+q$ mm.	Tip ΔX_2 $\frac{qx_2^2}{Z^2}+q$ mm.	$\Sigma x =$ $(\Delta x_1+\Delta x_2+D_{1x}$ $+D_{2x}-2q)$ mm.	Absolute Vertical Error ΔZ $\left(\frac{Z}{B}\cdot\frac{Z}{f}\cdot\Sigma x\right)$ mm.	Vertical Error ΔZ_a Adjusted for Datum (ΔZ—.038) mm.	Error in Feet
1	33.2	−.051	−.051	33.2	−.051	−.051	.007	.007	−.071	−.071	−.116	−.630	−.668	−34.7
2	49.8	−.029	−.029	16.6	−.038	−.038	.016	.002	−.080	−.066	−.085	−.461	−.499	−26.0
3	66.4	+.026	+.026	00.0	.000	.000	.028	.000	−.092	−.064	−.002	−.011	−.049	− 2.6
4	83.0	+.051	+.051	16.6	−.038	+.038	.044	.002	−.108	−.066	+.043	+.234	+.196	+10.2
5	99.5	−.050	−.050	33.2	−.051	+.051	.064	.007	−.128	−.071	−.070	−.380	−.418	−21.7
6	37.2	−.050	−.045	37.2	−.050	−.045	.007	.007	−.071	−.071	−.104	−.565	−.603	−31.4
7	52.5	−.021	−.020	23.4	−.048	−.034	.016	.002	−.080	−.066	−.072	−.390	−.428	−22.2
8	68.5	+.031	+.030	16.6	−.038	.000	.028	.000	−.092	−.064	+.002	+.011	−.027	− 1.4
9	84.5	+.051	+.050	23.4	−.048	+.034	.044	.002	−.108	−.066	+.038	+.206	+.168	+ 8.7
10	46.8	−.037	−.026	46.8	−.037	−.026	.007	.007	−.071	−.071	−.066	−.358	−.396	−20.6
11	59.8	+.005	+.004	37.2	−.050	−.022	.016	.002	−.080	−.066	−.036	−.195	−.233	−12.1
12	74.3	+.044	+.039	33.2	−.051	.000	.028	.000	−.092	−.064	+.011	+.060	+.022	+ 1.0
13	89.4	+.048	+.045	37.2	−.050	+.022	.044	.002	−.108	−.066	+.021	+.114	+.076	+ 4.0
14	59.8	+.005	+.003	59.8	+.005	+.003	.007	.007	−.071	−.071	−.008	−.043	−.081	− 4.2
15	70.2	+.036	+.026	52.5	−.021	−.007	.016	.002	−.080	−.066	+.011	+.005	−.033	− 1.7
16	83.0	+.051	+.041	49.8	−.029	.000	.028	.000	−.092	−.064	+.014	+.076	+.038	+ 2.0
17	96.8	+.015	+.013	52.5	−.021	+.007	.044	.022	−.108	−.066	−.026	−.141	−.179	− 9.3
18	74.3	+.044	+.020	74.3	+.044	+.020	.007	.007	−.071	−.071	+.026	+.141	+.103	+ 5.3
19	83.0	+.051	+.031	68.5	+.031	+.008	.016	.002	−.080	−.066	+.021	+.114	+.076	+ 4.0
20	93.6	+.036	+.026	66.4	+.026	.000	.028	.000	−.092	−.064	−.002	−.011	−.049	− 2.6
21	89.4	+.048	+.018	89.4	+.048	+.018	.007	.007	−.071	−.071	+.022	+.119	+.081	+ 4.2
22	96.8	+.015	+.008	84.5	+.051	+.010	.016	.002	−.080	−.066	.000	.000	−.038	− 2.0
m	89.6	+.047	+.035	60.2	+.006	.000	.028	.000	−.092	−.064	+.007	+.038	.000	0.0

At m, $D_{1y} = .032$ and $D_{2y} = .006$

$$y = -.026$$

$$q = \frac{y \cdot Z^2}{xy} = \frac{-.026 \cdot 99.2^2}{66.4 \cdot 60.2} = -.064$$

Columns 1–12 are at photo scale; columns 13 and 14 are at model scale.
In columns 10 and 11, minus indicates divergent tip.

the assumption of true orientation cannot be made at the start from the point of view of what actually happens in practice. It can be shown that tip (See Fig. 1) is the only adjustment which will remove this parallax. Apparent relative orientation, as achieved in practice by observation and adjustment in the stereoscopic model, is generally assumed to be attained when the y_p under each projector and at the four corners of the model is eliminated. This deviation of the projectors from the true relative positions of the two camera stations in space causes an additional deformation of the model in the nature of a parabolic cylinder with its axis in the Y-direction, Fig. 13. These elevation errors vary from 0 to 12 feet, in this particular case, at a scale of 1:15,840. These errors, although due to the tip adjustment, are, in reality, a direct result of the distortion in the camera lens, and therefore must be included in the calculations. Although the error caused by this tip may be small in some cases it varies considerably in models of different shapes, or base-height ratios and with different lenses.

This effect of tip is given prominence in this paper because it is a natural consequence of the process of relative orientation. It also has a definite bearing on "extension" or "bridging" with the Multiplex as it makes a primary contribution to the cumulative upward or downward errors in elevations of points, as extension progresses from the starting model. As this tip causes a constant convergent or divergent relationship between each pair of projectors, it can be shown that the error caused by the small tip angles usually encountered varies nearly as the square of the distance out from a starting model.

A basis of numerical values (all of which are given in millimeters) was obtained from the most efficient spacing, with regard to base length, accuracy, coverage of exposures, and shapes of models for a 90° camera using circular pictures. Fig. 2 shows the total and also the net size of the stereoscopic model, at the negative scale, and the numerical designation of a number of points in the lower right quadrant. Since, by symmetry, the effect of distortion is similar in the other quadrants, errors computed for the lower right quadrant actually give the errors at symmetrical points in the other three quadrants. Table I was computed, using the dimensions given in Fig. 2 and the distortion as measured by the National Bureau of Standards for Topogon Lens, No. 1876038 of Zeiss manufacture, Fig. 3.

In Fig. 7, the rays shown by solid lines represent the correct position of a pair of conjugate image rays intersecting at p. The dash-dot lines represent the positions of the rays as a result of distortion (D_1 and D_2) and the long-dash short-dash lines show the positions after tip (β). The short-dash lines show the final position after applying the scale correction q to each projector. The quantities to be found are the vertical error, Δz, the y_p, and the horizontal errors H_y and H_x. Quantities Δz_a which have been tabulated in the computations differ from Δz only by a datum change sufficient to make Δz_a equal zero at the corners of the model. The curved dashed lines in the plan, Figs. 6 and 7, show the effect of tip in a horizontal plane. For purposes of computing this effect of tip, it is desirable to derive an equation which relates the X and Y-movements of any point projected on a horizontal plane as the projector is tipped a small amount. This equation is:

$$\Delta x = \frac{\Delta y(x^2 + z^2)}{xy} \tag{1}$$

which is derived in connection with Fig. 4.

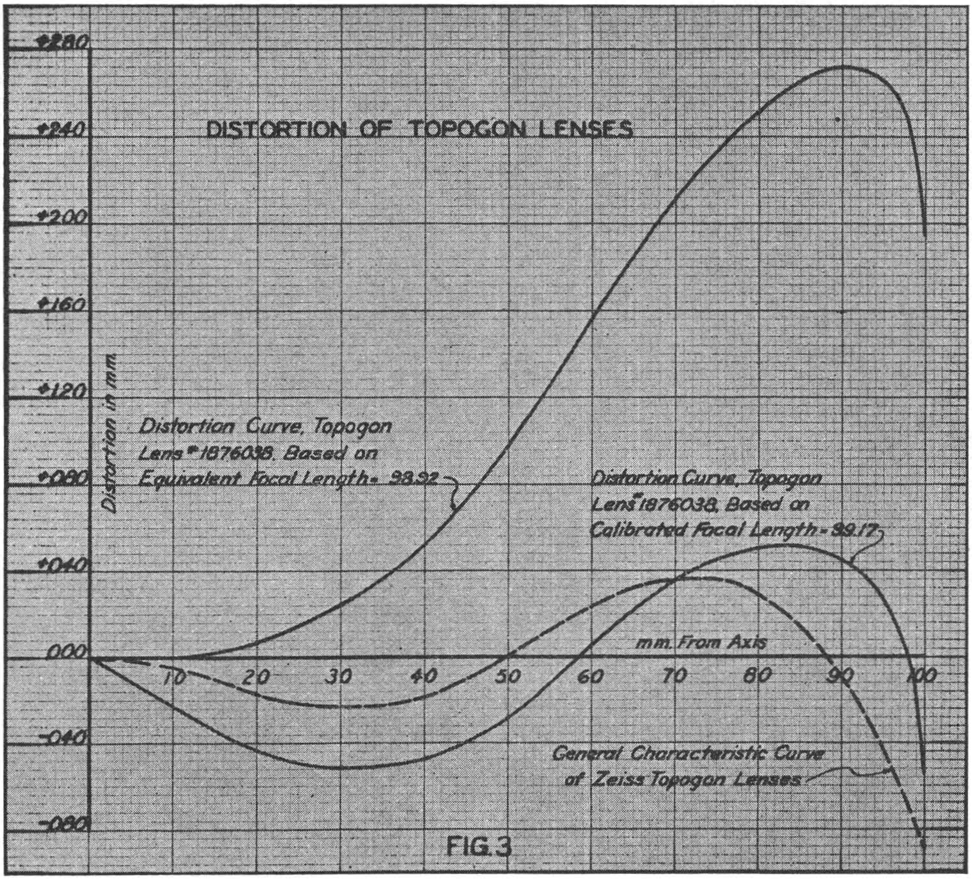


FIG. 3. LEGEND ON CUT.

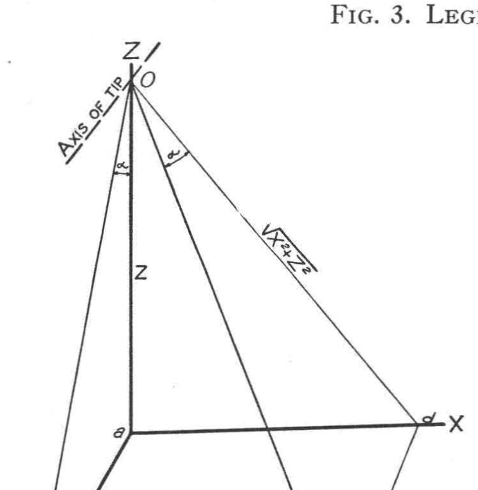


FIG. 4. RELATION BETWEEN THE X AND Y MOVEMENTS OF ANY POINT, CAUSED BY A SMALL AMOUNT OF TIP.

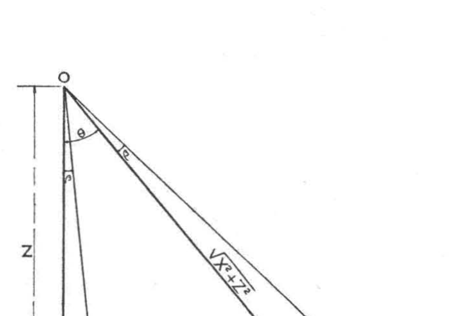


FIG. 5. MOVEMENT ΔX OF ANY POINT IN A MODEL DUE TO TIPPING THE PROJECTOR THROUGH ANGLE β.

Fig. 6 is a diagram showing the effect of removing y_p at the two lower corners, m_I and m_{II}, of the model. Parallax will be simultaneously removed at the other two corners, m_I' and m_{II}', of the neat model because the distortions are symmetrical. In Fig. 6, m_I and m_{II} are points where conjugate image rays would intersect if there were no distortion of any kind, and the model horizontal and flat. Points *I* and *II* represent the projectors with the several rays con-

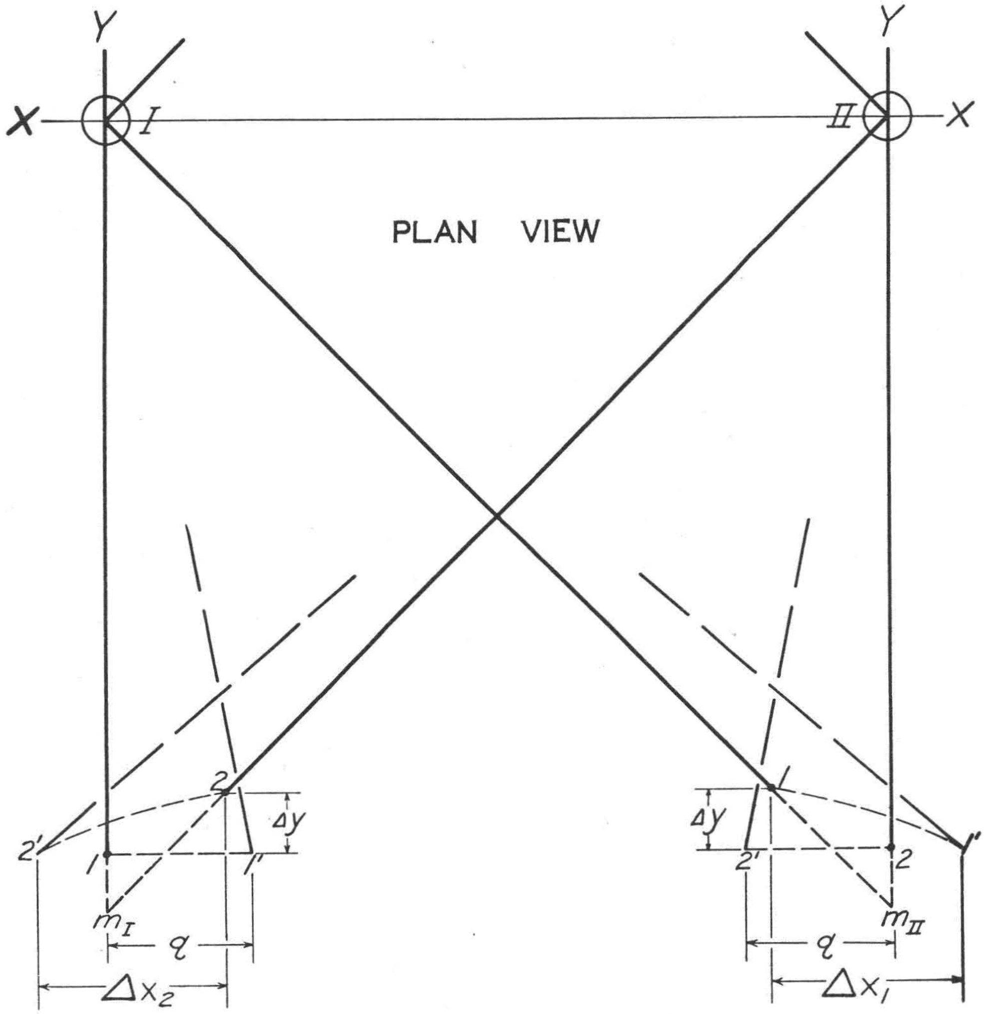


FIG. 6. EFFECT AT MODEL CORNERS OF TIPPING OUT *Y*-PARALLAX CAUSED BY LENS DISTORTION.

cerned projected to an *XY*, or horizontal plane, through m_I and m_{II}. The distortion of ray $\overline{Im_{II}}$, is $\overline{1m_{II}}$, in this plane, 1 being the point where the ray now intersects the horizontal plane. Points m_{II}, 1, 1′, 2, and 2′ are all in a horizontal plane through m_{II}. The distortion of ray $\overline{IIm_{II}}$ is $\overline{2m_{II}}$, 2 being the point where this ray now intersects the plane. The difference in *Y*-components of $\overline{1m_{II}}$ and $\overline{2m_{II}}$, Δy, is equal to the y_p at m_{II}, for all practical purposes. This fact is seen from Fig. 7. In this figure, the failure of the short-dashed rays to intersect causes y_p, which is the amount $p'p''$. Point p''' is midway between p' and p''. Because the angle between the two short-dashed rays, in the view along the X-axis, is very small, the y_p can be measured at the elevation of point p (which is at the elevation of m_{II}), with negligible error. Actually, in practice, the Multiplex operator will observe and eliminate the y_p at an elevation slightly above or below the XY plane containing points m_I and m_{II} because it will be

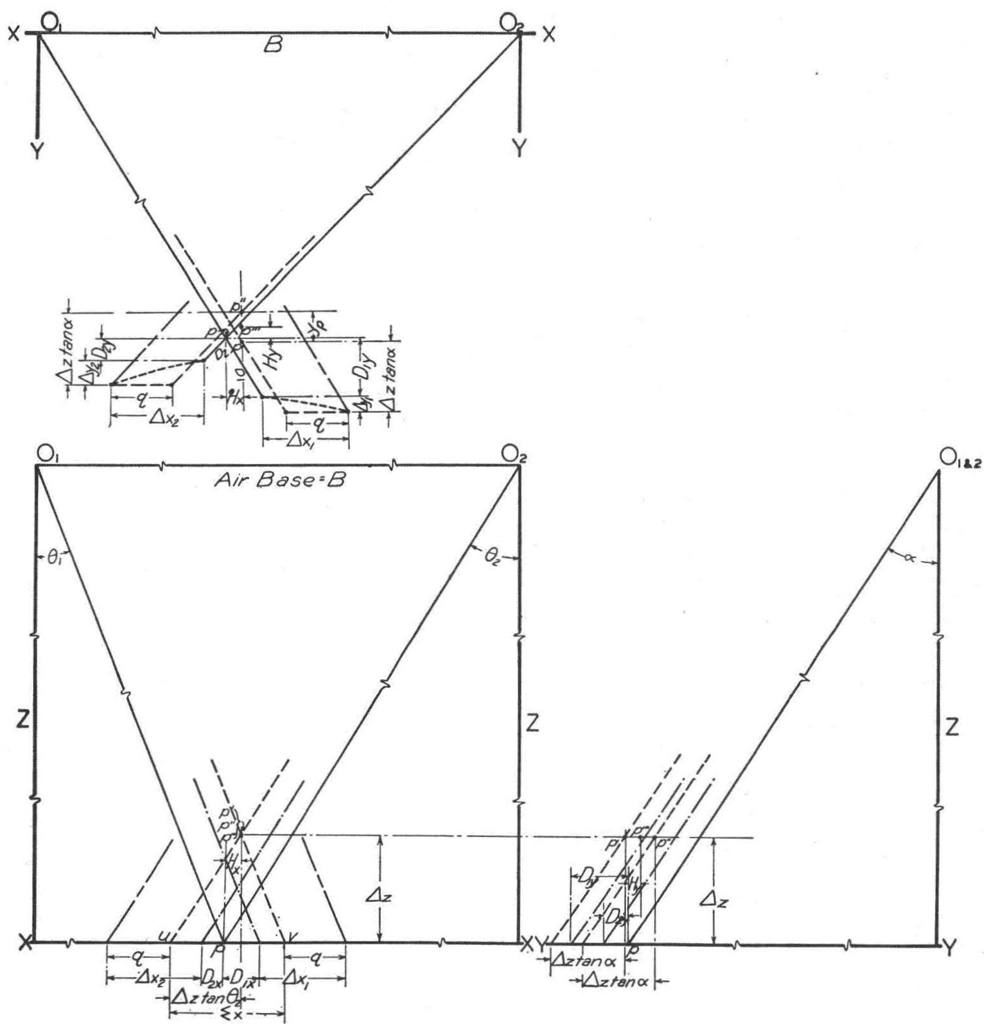


FIG. 7. EFFECTS OF LENS DISTORTION.

at this elevation that the two conjugate image rays intersect a horizontal plane at two points which have no x_p. The difference caused by this fact may be safely neglected. Therefore, in Fig. 6, Δy can be assumed equal to the y_p.

This amount, Δy, will be eliminated by tipping projector I until the intersection of the ray with the horizontal plane moves to point 1′, which has no Y-separation from point 2. The point moves along a curve pp', which is a hyperbola (conic section), as shown in Fig. 4. Since the same situation exists at m_I, projector II will also be tipped an equal amount, causing point 2 to move to point 2′. At m_{II}, points 2 and 2′ have a negligible Y-separation because the curve in Fig. 4 has a slope of zero, $(dy/dx=0)$ at points along the axes, and point 2 is on, or stays near, the Y-axis for projector II.

From the values of m in Table II, Δy is computed to be .026 mm. q is found to be −.064 from Eq. 3. Δx values at points having other X-co-ordinates are computed from Eq. 2. These values are entered in columns 10 and 11, Table I. Column 12 is the algebraic sum of values in columns 4, 7, 10, and 11, minus the constant $2q$, which represents the change in base length, necessitated by tip, such that the scale of the model will remain the same.

Equation 4 shows that the vertical error, Δz, is a constant proportion (Z/B) of $\sum x$. Δz_a in column 14 is the vertical error relative to point m, assuming point

TABLE II. CALCULATION OF Y-PARALLAX NO TIP—MODEL SCALE 1:15,840

Point	D_{1y}	D_{2y}	y_p Photo Scale	y_p Model Scale
1	.000	.000	.000	.000
2	.000	.000	.000	.000
3	.000	.000	.000	.000
4	.000	.000	.000	.000
5	.000	.000	.000	.000
6	−.022	−.022	.000	.000
7	−.007	−.034	.027	.098
8	+.008	−.038	.046	.167
9	+.010	−.034	.044	.160
10	−.026	−.026	.000	.000
11	+.003	−.045	.048	.174
12	+.020	−.051	.071	.258
13	+.018	−.045	.063	.229
14	+.004	+.004	.000	.000
15	+.026	−.020	.046	.167
16	+.031	−.029	.060	.218
17	+.008	−.020	.028	.102
18	+.039	+.039	.000	.000
19	+.041	+.030	.011	.040
20	+.026	+.026	.000	.000
21	+.045	+.045	.000	.000
22	+.013	+.050	.037	.134
m	+.032	+.006	.026	.094

m and the other three corners of the neat model to be set on vertical control with a resultant vertical error of zero at the four corners of the model.

The combinations of quantities which give the horizontal errors are indicated in Fig. 7, and the values are tabulated in Tables III and IV. The vertical errors are shown graphically in Fig. 8, which is a representation of the vertical errors (due to distortion in the camera lens and tip resulting from it) at different points throughout the model, when the four corners are correctly set on control points. These errors are plotted as ordinates on latitudinal and longitudinal sections through the model. Points of equal vertical error are connected by curves, giving the distortion contours shown in the figure. Sections are drawn with positive values up and to the right.

If tip were not to be considered, the vertical error may be calculated by the formula:

$$\Delta z = \frac{Z}{f}\cdot\frac{Z}{B}\,(D_1x_1/L_1 + D_2x_2/L_2) \quad \text{(See Fig. 7)}$$

where D_1 and D_2 are the distortions as taken from the distortion curve, L_1 and

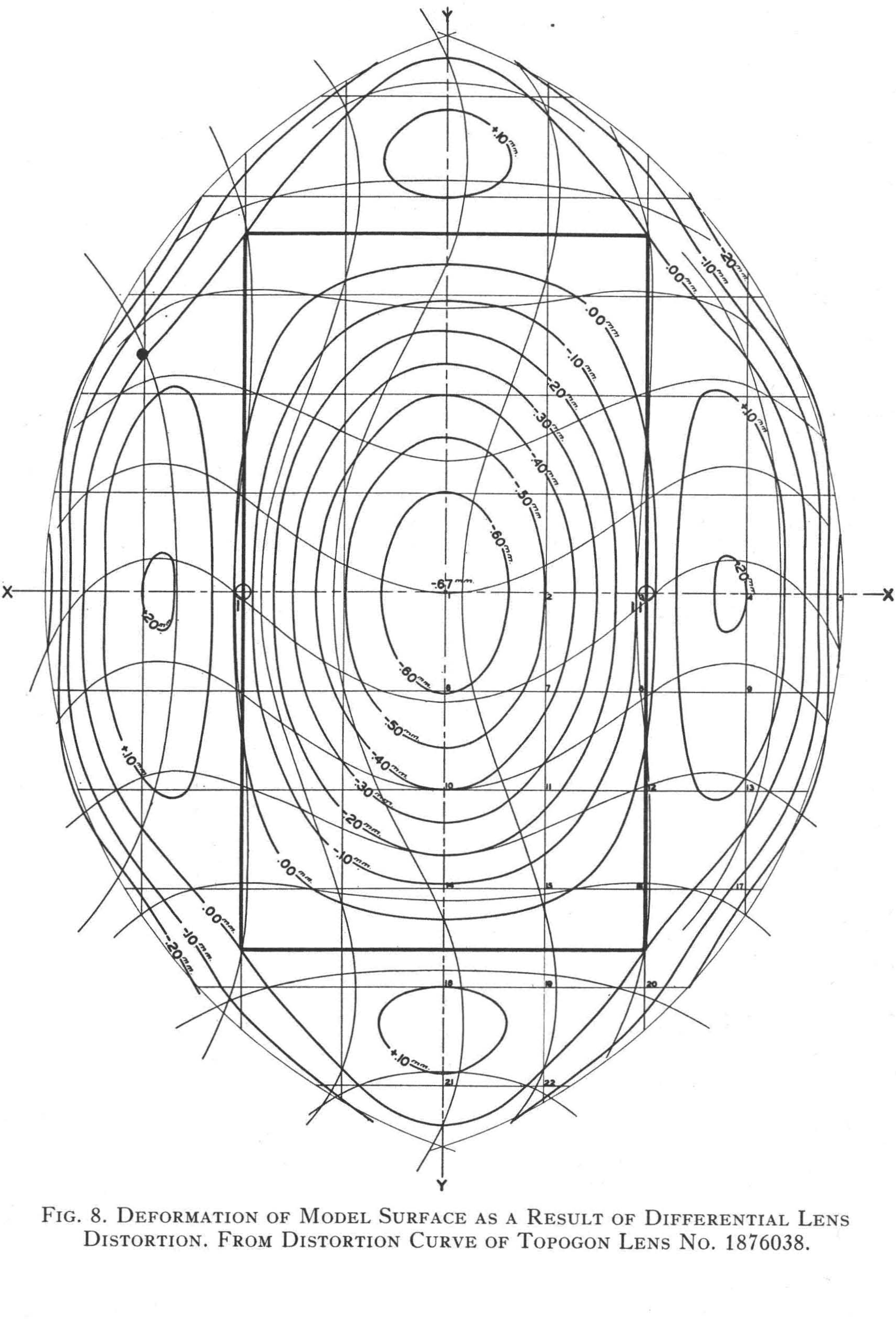


FIG. 8. DEFORMATION OF MODEL SURFACE AS A RESULT OF DIFFERENTIAL LENS DISTORTION. FROM DISTORTION CURVE OF TOPOGON LENS NO. 1876038.

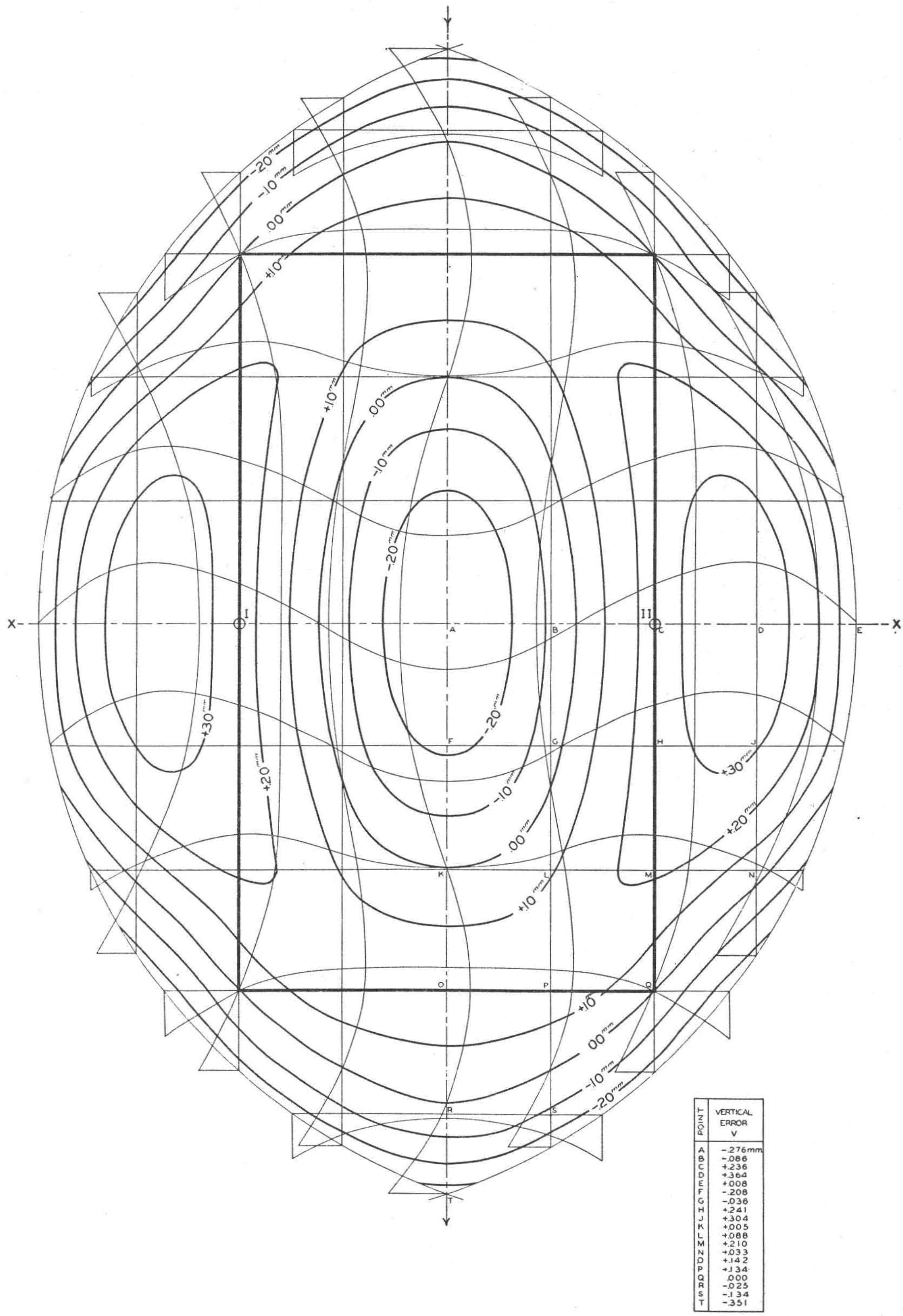
TABLE III. CALCULATION OF H_y

Point	D_{1y} Component mm.	D_{2y} Component mm.	Δy_1 mm.	Δy_2 mm.	$-2\Delta z \tan \alpha$	$2H_y$ (Eq. 6)	H_y Photo mm.	H_y Model mm.	H_y Feet
1	.000	.000	.000	.000	.000	.000	.000	.000	0.0
2	.000	.000	.000	.000	.000	.000	.000	.000	0.0
3	.000	.000	.000	.000	.000	.000	.000	.000	0.0
4	.000	.000	.000	.000	.000	.000	.000	.000	0.0
5	.000	.000	.000	.000	.000	.000	.000	.000	0.0
6	−.022	−.022	−.004	−.004	+.052	.000	.000	.000	0.0
7	−.007	−.034	−.006	−.002	+.036	−.013	−.006	−.024	−1.5
8	+.008	−.038	−.007	.000	−.001	−.038	−.019	−.069	−3.6
9	+.010	−.034	−.009	+.002	−.019	−.050	−.025	−.092	−4.8
10	−.026	−.026	−.007	−.007	+.066	.000	.000	.000	0.0
11	+.003	−.045	−.012	−.004	+.036	−.022	−.011	−.040	−2.1
12	+.020	−.051	−.015	.000	−.011	−.057	−.028	−.103	−5.4
13	+.018	−.045	−.018	+.004	−.020	−.061	−.030	−.111	−5.8
14	+.004	+.004	−.010	−.010	+.012	.000	.000	.000	0.0
15	+.026	−.020	−.018	−.005	−.001	−.018	−.009	−.033	−1.7
16	+.031	−.029	−.022	.000	−.021	−.041	−.020	−.074	−3.9
17	+.008	−.020	−.027	+.005	+.039	+.005	+.002	+.009	+0.5
18	+.040	+.040	−.014	−.014	−.052	.000	.000	.000	0.0
19	+.041	+.030	−.024	−.007	−.042	−.002	−.001	−.004	−0.2
20	+.026	+.026	−.029	.000	+.004	+.027	+.013	+.049	+2.6
21	+.045	+.045	−.018	−.018	−.054	.000	.000	.000	0.0
22	+.013	+.050	−.030	−.009	.000	+.024	+.012	+.044	+2.3
m	+.032	+.006	−.026	.000	−.012	.000	.000	.000	0.0

L_2 are the lengths of the horizontal projections of the rays, and x_1 and x_2 are the x-components of L_1 and L_2 respectively; f is the calibrated focal length of the camera, and Z and B are the flying height and air base respectively.

TABLE IV. CALCULATION OF H_x

1	2	3	4	5	6	7	8
Point	D_{2x}	Δx_2	$-q$	$-\Delta Z \tan \theta_2$	H_x (Eq. 7) Photo mm.	H_x Model mm.	H_x Feet
1	−.051	−.071	+.064	+.058	.000	.000	0.0
2	−.038	−.066	+.064	+.021	−.019	−.069	−3.6
3	.000	−.064	+.064	.000	.000	.000	0.0
4	+.038	−.066	+.064	+.011	+.047	+.171	+8.9
5	+.051	−.071	+.064	−.035	+.009	+.033	+1.7
6	−.045	−.071	+.064	+.052	.000	.000	0.0
7	−.034	−.066	+.064	+.018	−.018	−.065	−3.4
8	.000	−.064	+.064	.000	.000	.000	0.0
9	+.034	−.066	+.064	+.010	+.042	+.153	+8.0
10	−.026	−.071	+.064	+.033	.000	.000	0.0
11	−.022	−.066	+.064	+.009	−.015	−.054	−2.8
12	.000	−.064	+.064	.000	.000	.000	0.0
13	+.022	−.066	+.064	+.005	+.025	+.091	+4.7
14	+.003	−.071	+.064	+.004	.000	.000	0.0
15	−.007	−.066	+.064	.000	−.009	−.033	−1.7
16	.000	−.064	+.064	.000	.000	.000	0.0
17	+.007	−.066	+.064	−.006	−.001	−.004	−0.2
18	+.020	−.071	+.064	−.013	.000	.000	0.0
19	+.008	−.066	+.064	−.005	+.001	+.004	+0.2
20	.000	−.064	+.064	.000	.000	.000	0.0
21	+.018	−.071	+.064	−.011	.000	.000	0.0
22	+.010	−.066	+.064	.000	+.008	+.029	+1.5
m	.000	−.064	+.064	.000	.000	.000	0.0



POINT	VERTICAL ERROR V
A	-.276mm
B	-.086
C	+.236
D	+.364
E	+.008
F	-.208
G	-.036
H	+.241
J	+.304
K	+.005
L	+.088
M	+.210
N	+.033
O	+.142
P	+.134
Q	.000
R	-.025
S	-.134
T	-.351

FIG. 9. DEFORMATION OF MODEL SURFACE AS A RESULT OF DIFFERENTIAL LENS DISTORTION. FROM GENERAL CHARACTERISTIC CURVE, TOPOGON LENSES.

2. The Aerial Film

As the aerial film is the sole medium by which the information recorded by the aerial camera is imparted to all future steps in the process, it is extremely important that every means be taken to insure the proper selection, processing and preservation.

The film used is selected as having the best balance of film base, grain and emulsion. The film base is a nitrate topographic base, as this still seems to have a slight advantage, for this work, over an acetate base. Although the emulsion chosen does not have the finest grain procurable, it was chosen as having the finest grain commensurate with the speed necessary to provide a slightly over-exposed-underdeveloped negative which gives a more even density over the entire negative and cuts down contrast. This is very important as it makes the problem of illumination much easier and helps to provide a diapositive free from either extremely light or dark areas and results in more detail.

The process of developing and drying the film is carried out with great care and the film is then stored in a repository, where temperature and humidity are carefully controlled, until glass diapositives are made from it. No other use of the film is allowed until its record is transferred to these glass diapositives.

Even with the best treatment, the film base does not expand or shrink at the same rate in both the X and Y-directions. This is handled in the following manner. A large number of negatives are measured, under conditions of temperature and humidity identical to those which exist when diapositives are printed, and values are found for X and Y-distances, as against those measured on a glass plate exposed in the camera. Tilt and "crab," as usually encountered in stereoscopic plotting, have a negligible effect on errors caused by differential shrinkage. The change in each direction is considered uniform. The percentage change, in the X-direction, thus found is applied to the "f" value (calibrated focal length chosen) and this adjusted value is then used in the calibration of the diapositive printer. It can be shown that this eliminates all horizontal errors, due to this source, in the X-direction (H_x), as well as errors in elevations, or the Z-direction, which would otherwise be present in the final model. The horizontal errors of points in the Y-direction (H_y) are now considered. From the large number of measurements on the film as described above, the differential shrinkage or expansion is established. Under the above conditions this was found to be $+.11\%$ in the Y-direction as against the X-direction. If the above conditions are fulfilled, the horizontal error in the model in the Y-direction is then equal to .0011 times the Y-distance out from the $X-X$ line, through points I and II, in Fig. 2. In the case of a model scale of 1:15,840, the largest error within the neat model in the Y-direction, due to shrinkage, would be .238 mm., or 12 feet, and would apply to all points along $\overline{m_I m_{II}}$ and $\overline{m_I' m_{II}'}$. As this is purely a known scale change in the Y-direction, it may be handled as such, and the map sheet shifted section by section during plotting.

Relief does not further affect this error due to shrinkage, and points having a given X and Y-value will plot in the same horizontal position regardless of elevation or Z-value.

The errors, due to differential film shrinkage as set forth above, are present in either this form or another in most stereo-photogrammetric plotting methods using film, though the size of the error may vary.

3. The Diapositive Printer

The diapositive printer receives the record which the film contains and re-

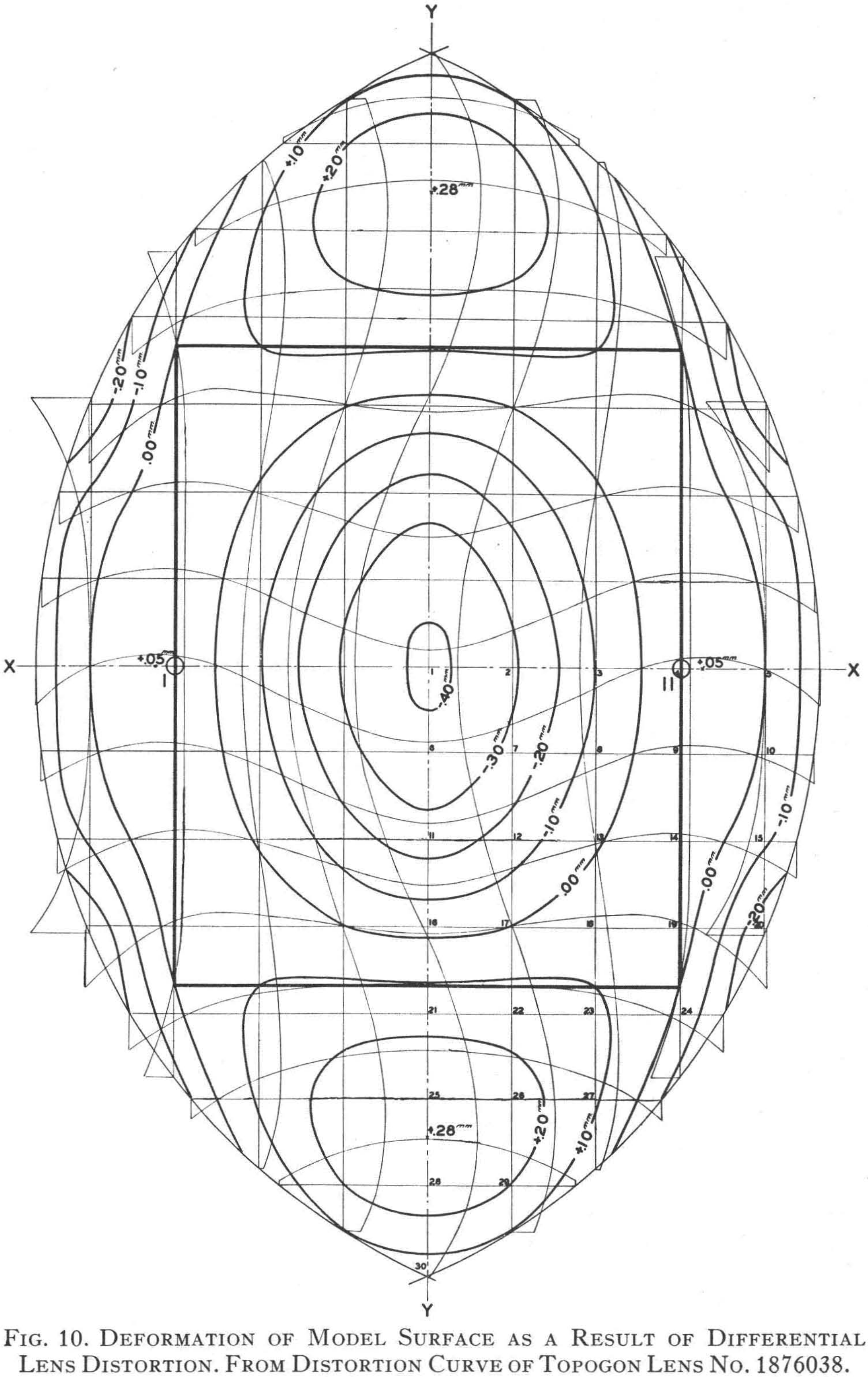


FIG. 10. DEFORMATION OF MODEL SURFACE AS A RESULT OF DIFFERENTIAL LENS DISTORTION. FROM DISTORTION CURVE OF TOPOGON LENS NO. 1876038.

cords this information on a glass diapositive for use in the projectors. This diapositive must be a precise scale reduction of the aerial negative and must preserve as much of the detail in the negative as possible. The printer must therefore be set so as to produce a diapositive of the correct scale and, at this scale, have the greatest possible definition. The reason for the exact scale requirement is given under Section 4. By very careful adjustment the scale is held to $\pm .025\%$, which satisfies the metric accuracy required, and the definition is kept to limits as imposed by the resolution of the emulsion.

As all information on the negative passes through the printer lens, the amount of error from this source must be determined. This was accomplished by procuring a very accurate grid, of 5 millimeter spacing, and using this in place of the aerial negative in the printer. The resultant diapositive was then measured and found to be a true scale reduction within the limits of a comparator having a least reading of .001 mm. Therefore, this printer lens was considered to contribute no measurable distortion throughout its angular field.

If the surface of the emulsion or the glass plate is not flat, the resultant errors may be greater than could be tolerated, and therefore, to reduce this source of error, steps were taken to procure diapositive plates having flat surfaces on which an emulsion of even thickness is coated. This error, due to a possible unevenness of emulsion or glass surface, could be entirely eliminated by use of a printer lens having a focal length such that rays would strike the diapositive, during exposure, at the same angle at which they will be projected from the diapositive in the projector.

4. The Multiplex Projectors

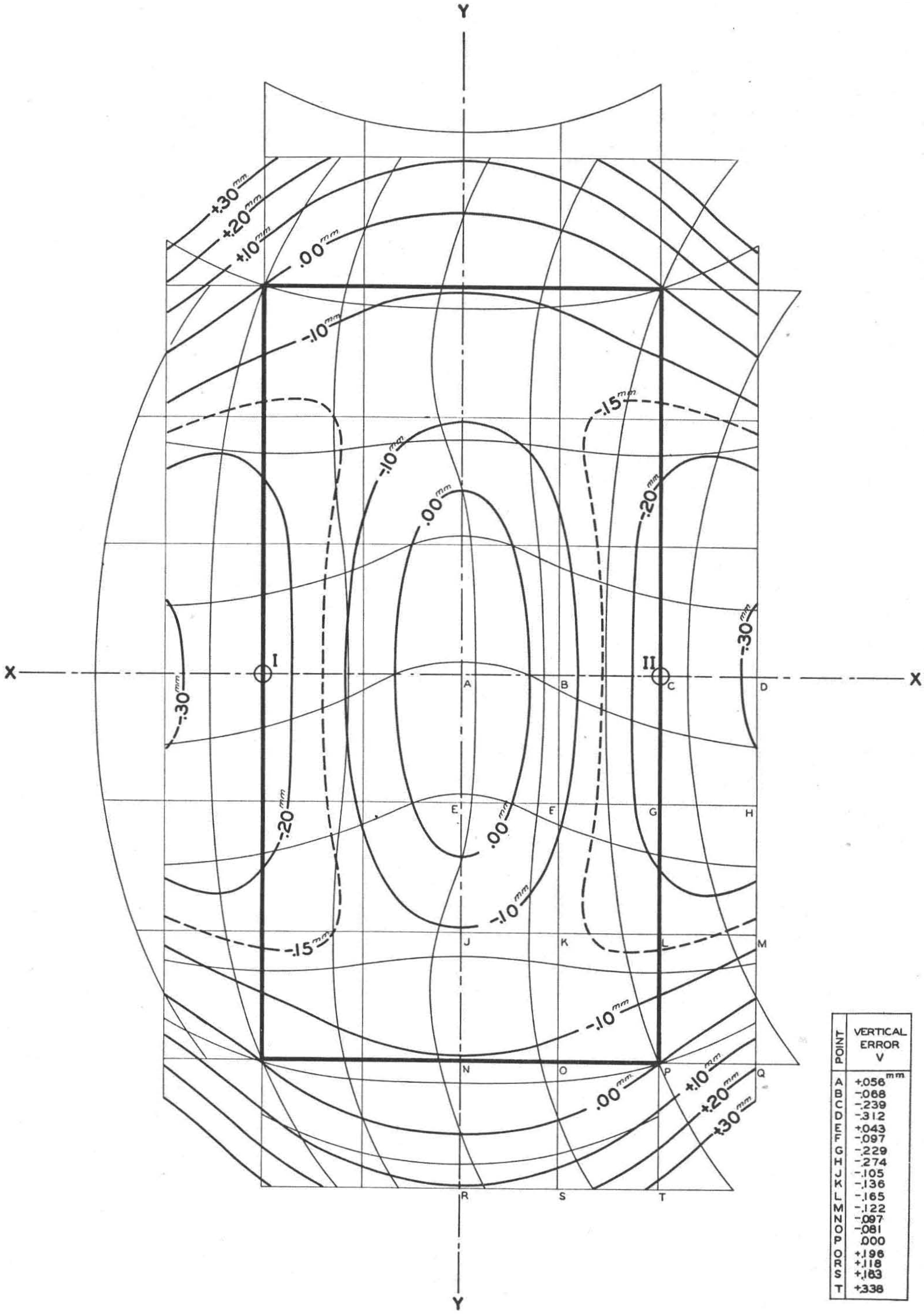
These projectors represent the aerial camera in miniature and, therefore, must be carefully adjusted and calibrated, especially so because they are in miniature. The ratio of diapositive scale to model scale averages 1:16.5, and any error is magnified in the same ratio.

It must be possible to use a diapositive in any one of many projectors; therefore, the projectors must be calibrated to the same values and standards, so as to make them interchangeable. Only a few of the more important steps are considered here in so far as they affect the final error.

The aerial film is actually exposed in a plane in the camera which gives the best average definition over the negative as a whole. The distance between this plane and the internal perspective center is the principal distance. When the principal distance is chosen for the projectors to represent the principal distance of the aerial camera, the reduction ratio of the film to the diapositive is determined. If this ratio is varied a small amount in the diapositive printer, it changes the scale of the diapositive. If this diapositive is then used in one of the projectors, the projected cone of rays is geometrically changed and the distribution of distortion is affected. This fact may be made use of, and a variation in the ratio made to produce a diapositive which corresponds to an accepted calibrated focal length of the aerial camera.

As previously stated, under Section 2, the proper calibrated focal length must also take into account the film shrinkage.

The difference in results obtained using calibrated focal length as against equivalent focal length is shown in Table V. The effect in the model of using a calibrated focal length is a change in the vertical scale of the model, providing the horizontal scale is kept constant. The distortion pattern of a horizontal plane model is negligibly affected by the use of calibrated focal length, but when relief is present in a model, the amount and distribution of errors are changed,



POINT	VERTICAL ERROR V
A	+.056mm
B	-.068
C	-.239
D	-.312
E	+.043
F	-.097
G	-.229
H	-.274
J	-.105
K	-.136
L	-.165
M	-.122
N	-.097
O	-.081
P	.000
O	+.196
R	+.118
S	+.183
T	+.338

FIG. 11. DEFORMATION OF MODEL SURFACE AS A RESULT OF DIFFERENTIAL LENS DISTORTION. FROM DISTORTION CURVE OF BAUSCH & LOMB LENS HAVING 75° COVERAGE.

and therefore in some cases a calibrated focal length may be used to give a better mean or average vertical scale.

In order to find the amount of error for which the projector is responsible, it was decided to procure and use two near perfect grids of one millimeter spacing as diapositives. When using these grids as diapositives, in several different pairs of projectors which have been properly calibrated and adjusted, the wide-angle or 90° Multiplex projectors are accurate enough mechanically and optically to produce a stereoscopic model which will be correct or flat within the limits of the operator's perception (± 0.05 mm.) throughout the neat area of the model.

The projectors, therefore, were considered to have no measurable effect on lens distortion within the neat model.

Table V. Effect of Relief

A tabulation of vertical errors (Δz_a) at points designated in Fig. 2, assuming three values of relief for each point and showing a comparison of results obtained by using two different values of "*f*."

Note: Values in this table are in feet, as they affect elevations in nature when the model scale is 1:15,840.

Relief	C.F.L. (99.17 mm.)			E.F.L. (98.92 mm.)		
Point	−2000 ft.	0 ft.	+2000 ft.	−2000 ft.	0 ft.	+2000 ft.
1*	−46.4	−34.7	−25.0	−41.4	−34.7	−30.0
2*	−38.2	−26.0	−14.6	−33.2	−26.0	−19.6
3*	−13.6	− 2.6	+ 3.9	− 8.6	− 2.6	− 1.1
4	+ 6.7	+10.2	+ 7.3	+11.7	+10.2	+ 2.3
5	+ 4.6	−21.7	—	+ 9.6	−21.7	—
6*	−43.0	−31.4	−20.9	−38.0	−31.4	−25.9
7*	−35.1	−22.2	−11.0	−30.1	−22.2	−16.0
8*	−11.7	− 1.4	+ 4.1	− 6.7	− 1.4	− 0.9
9	+ 6.0	+ 8.7	+ 6.9	+11.0	+ 8.7	+ 1.9
10*	−33.3	−20.6	− 9.2	−28.3	−20.6	−14.2
11*	−24.4	−12.1	− 3.1	−19.4	−12.1	− 8.1
12*	− 6.8	+ 1.0	+ 4.3	− 1.8	+ 1.0	− 0.7
13	+ 2.5	+ 4.0	−17.3	+ 7.5	+ 4.0	−22.3
14*	−16.8	− 4.2	+ 4.3	−11.8	− 4.2	− 0.7
15*	−12.0	− 1.7	+ 4.5	− 7.0	− 1.7	− 0.5
16*	− 3.0	+ 2.0	+ 1.0	+ 2.0	+ 2.0	− 4.0
17	− 2.7	− 9.3	—	+ 2.3	− 9.3	—
18*	− 2.4	+ 5.3	+ 7.9	+ 2.6	+ 5.3	+ 2.9
19*	− 1.6	+ 4.0	+ 4.3	+ 3.4	+ 4.0	− 0.7
20*	− 3.3	− 2.6	—	+ 1.7	− 2.6	—
21	+ 1.8	+ 4.2	−11.5	+ 6.8	+ 4.2	−16.5
22	− 0.3	− 2.0	—	+ 4.7	− 2.0	—
*m**	− 2.7	0.0	−16.7	+ 2.3	0.0	−21.7

The means and averages of values for points marked with an (*) are:

Mean	−24.5	−14.7	− 8.5	−19.0	−14.7	−13.5
Average	−18.4	− 9.2	− 4.4	−13.4	− 9.2	− 9.4

It is to be seen that in this particular case, and with the relief assumed, that the E.F.L. (or even some value of "*f*" slightly less) gives the most consistent average errors.

Conclusion

A. The diapositive printer and projectors which are used in this office are free from distortion within the limits of accuracy required.

B. The unequal shrinkage or expansion of aerial film in the *X* as against the *Y*-direction causes a final error which can be confined to the *Y*-direction, the magnitude of which is easily determined.

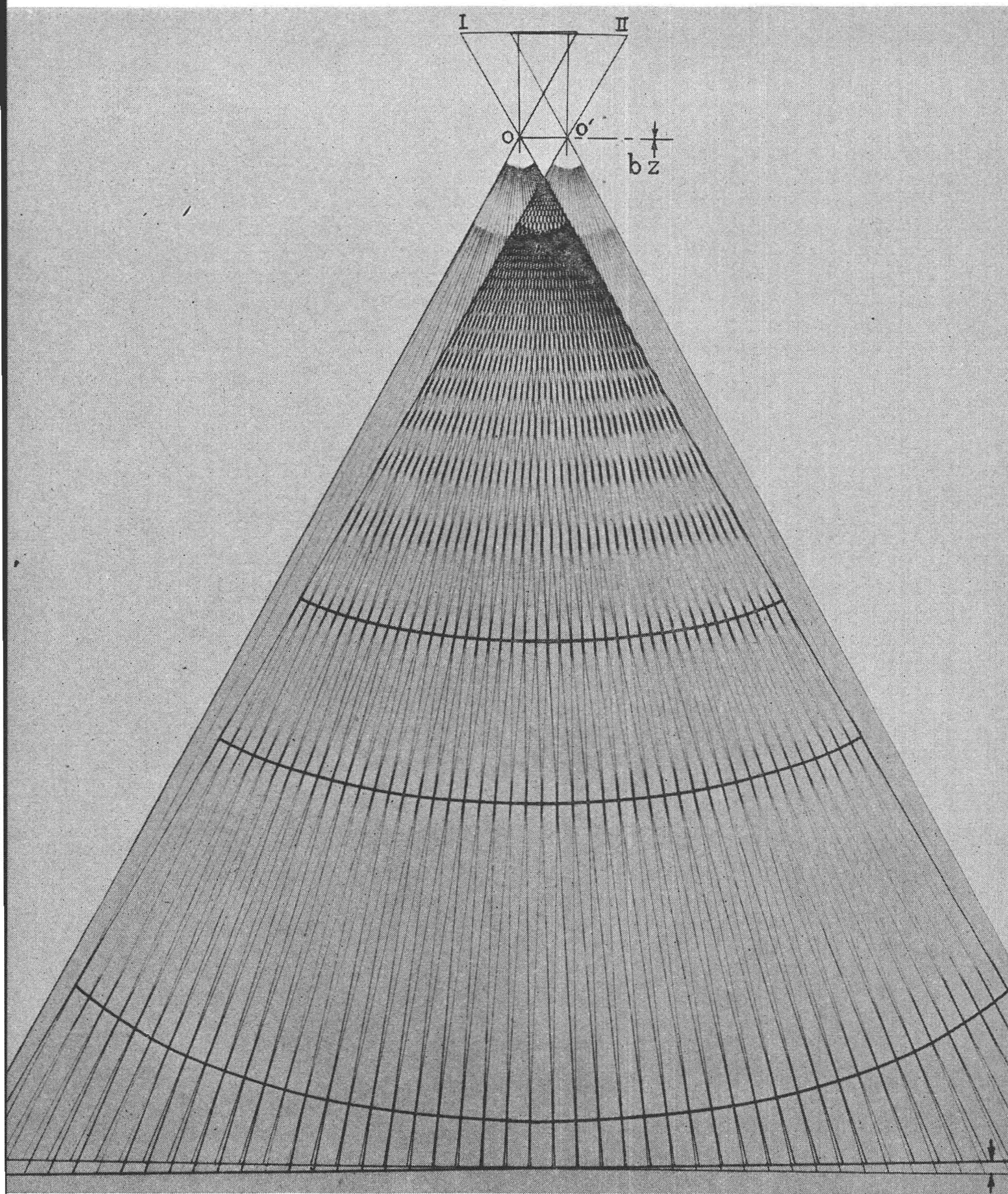


FIG. 12

C. The aerial camera can be adjusted and calibrated to standards of metrical precision which will give an accuracy suitable to present mapping needs.

The aerial camera lens was found to be the major source of error. This will also be true of most wide-angle lenses of present manufacture. The errors, as shown by Tables I, II, III, IV, and V, are present in the X, Y, and Z-directions in varying amounts. These errors are of such a nature and amount that it would not be feasible to use this lens for accurate topographic mapping, unless some means is used to eliminate or reduce these errors.

So far as can be ascertained, no other satisfactory method has been devised for handling this differential lens distortion than that described here. Figs. 8, 9, 10, and 11 show the vertical errors expressed in the form most convenient to use. These curves representing equal vertical errors are drawn directly on the map sheet and as the plotting of the contours progresses, the index, or height of the floating mark, is varied an amount corresponding to the error indicated by these curves. This adds difficulty to the process of mapping. Whether the amount of lens distortion is large or small the presence of relief so complicates the use of these curves as to make compensation very difficult due to the fact that the correction varies with each change in relief.

D. In the particular case considered in this paper the differential lens distortion is the total camera lens distortion since the projectors and diapositive printer lenses have been found to be distortion-free for their purposes. It is particularly desirable to maintain the projector lenses free from distortion since in this case they can be used in conjunction with any camera lens.

E. Since each diapositive reduction printer must be calibrated metrically to the camera with which it is used, the printer seems the logical place to compensate for the distortion of that particular camera lens.

The U. S. Geological Survey has on order at present projectors and a reduction printer of new design. The reduction printer will have incorporated a lens designed to compensate to a large degree for the distortion of the camera lens.

It appears from information received concerning the printer lens now being made that the effect of distortion may be expected to be reduced by approximately one-half. The distortion not compensated for (residual distortion) will be handled in the same manner as in the past, until such time as a printer lens or other device, having still better compensation, or a camera lens having less distortion or a different type of distortion more readily compensated, can be obtained.

F. The manufacturers of lenses and photographic materials still have a great field of endeavor in which any improvement is a direct contribution to the accuracy and economy of stereo-photogrammetric mapping.

Definitions of Terms

x_1—Horizontal distance from Projector I measured parallel to the line of flight.

x_2—Horizontal distance from Projector II measured parallel to the line of flight.

y—Horizontal distance from the Air Base. y is perpendicular to x.

m—The correct position of a point at the corner of a neat model without relief, when there is no distortion in the lens.

m-plane—A horizontal plane through m.

z—Vertical distance from the m-plane. z is perpendicular to x and y.

Z—Flight height.

B—Air Base.

y_p—Y-parallax—amount of separation of the projections of conjugate image points on the horizontal plane in which they have no X-separation.

Δx_1—x-movement of the trace, on the plane of the photograph, of any ray, caused by tipping Projector I.

Δx_2—x-movement of the trace, on the plane of the photograph, of any ray, caused by tipping Projector II.

Δy_1—y-movement of the trace, on the plane of the photograph, of any ray, caused by tipping Projector I.

Δy_2—y-movement of the trace, on the plane of the photograph, of any ray, caused by tipping Projector II.

Δz—Vertical error at a point, resulting from lens distortion.

Δz_a—Adjusted vertical error at a point, obtained when the datum is changed so that the vertical errors at the corners become zero.

q—x-movement of the trace, on the plane of the photograph, of any ray in the YZ plane, when a tip motion is applied to the projector from which the ray emanates.

H_y—Net horizontal error in the y-direction, resulting from lens distortion.

H_x—Net horizontal error in the x-direction, resulting from lens distortion.

Σx—Resultant x-separation of conjugate image rays at any point in the m-plane, after tip has been applied to remove the y-parallax at the corners, and the scale has been adjusted.

D_1—Distortion of a ray emanating from Projector I.

D_2—Distortion of a ray emanating from Projector II.

D_{1x}—x-component of D_1.

D_{2c}—x-component of D_2.

D_{1y}—y-component of D_1.

D_{2y}—y-component of D_2.

θ_1—Angle which the projection of any ray from Projector I on the XZ plane makes with the vertical.

θ_2—Angle which the projection of any ray from Projector II on the XZ plane makes with the vertical.

α—Angle which the projection of any ray on the YZ plane makes with the vertical.

β—Angle of tip.

Equation for the Lateral Movement, Δx, of any Point in a Model Due to a Small Amount of Tip, β

In Fig. 5, O is the lens point. $\overline{Oc}$ is the projection on an XZ plane of the ray to any point, c. $\overline{ce}$ is drawn so that triangle Oce is similar to triangle Oab. Then:

$$\frac{p}{q} = \frac{\sqrt{x^2 + Z^2}}{Z} = \frac{1}{\cos \theta}.$$

In triangle cgd,

$$\Delta x = \frac{p'}{\cos (\theta + \beta)}.$$

Since β is a very small angle, $\cos (\theta+\beta)$ may, for all practical purposes, be considered equal to $\cos \theta$. Also, we can assume p' equals p. Then:

$$\Delta x = \frac{p}{\cos\theta} = \frac{p\sqrt{x^2 + Z^2}}{Z} = \frac{q(x^2 + Z^2)}{Z^2} = \frac{qx^2}{Z^2} + q. \tag{2}$$

Combining equations (1) and (2),

$$\Delta x = \frac{\Delta y(x^2 + Z^2)}{xy} = \frac{q(x^2 + Z^2)}{Z^2}.$$

From which,

$$q = \frac{\Delta y \cdot Z^2}{xy}. \tag{3}$$

RELATION BETWEEN THE X AND Y MOVEMENTS OF ANY POINT CAUSED BY A SMALL AMOUNT OF TIP, β

In triangle Oap', Fig. 4, $\tan \alpha = S/Z$

In triangle Odp,

$$\tan \alpha = \frac{y}{\sqrt{x^2 + Z^2}}.$$

Therefore,

$$\frac{S}{Z} = \frac{y}{\sqrt{x^2 + Z^2}}.$$

Point p is the intersection of ray Op with a horizontal plane. As the projector is tipped, point p, whose co-ordinates are x and y, moves on the curve $p'p$. The equation of this curve is given by the above relation between x and y. Angle α between any given ray, Op, and the XZ plane, remains constant as the projector is tipped.

Solving the above equation for y, we obtain:

$$y = \frac{S}{Z}\sqrt{x^2 + Z^2}.$$

Differentiating y with respect to x, we have:

$$\frac{dy}{dx} = \frac{S}{Z} \cdot \frac{1}{2}\left(\frac{2x}{\sqrt{x^2 + Z^2}}\right) \text{ (}S\text{ and }Z\text{ are constants).}$$

But, since

$$\frac{S}{Z} = \frac{y}{\sqrt{x^2 + Z^2}}$$

$$\frac{dy}{dx} = \frac{xy}{x^2 + Z^2}.$$

For small amounts of tip, we can say:

$$\frac{\Delta y}{\Delta x} = \frac{xy}{x^2 + z^2},$$

and,

$$\Delta x = \frac{\Delta y(x^2 + z^2)}{xy} \cdot \tag{1}$$

Equations for Computation of Errors

Referring to Fig. 7, the lines $p'''u$ and $p'''v$ are, for all practical purposes parallel to the lines O_{2p} and O_{1p}, respectively. Therefore the triangles $p'''uv$ and pO_2O_1 may be considered as similar triangles. From the similarity of these triangles,

$$\frac{\Delta z}{\Sigma x} = \frac{Z}{B} \cdot$$

So that,

$$\Delta z = \frac{Z}{B} \Sigma x. \tag{4}$$

But, it can be seen from Fig. 7 that Σx is equal to $(D_{1x}+D_{2x}+\Delta x_1+\Delta x_2-2q)$. Substituting this value of Σx in Equation (4), and applying the factor Z/f to convert from photograph scale to model scale, we get:

$$\Delta z = \frac{Z}{f} \cdot \frac{Z}{B} (D_{1x} + D_{2x} + \Delta x_1 + \Delta x_2 - 2q). \tag{5}$$

Referring again to Fig. 7, the value of H_y is determined as follows:

$$H_y = 1/2(\Delta z \tan \alpha - \Delta y_2 - D_{2y} + \Delta y_1 + D_{1y} - \Delta z \tan \alpha) - (\Delta y_1 + D_{1y} - \Delta z \tan \alpha)$$
$$= 1/2(- D_{1y} - D_{2y} - \Delta y_1 - \Delta y_2 + 2\Delta z \tan \alpha), \text{ where } \tan \alpha = y/Z.$$

Since it is obvious from the figure that H_y is negative in this case, we apply the minus sign and finally get:

$$H_y = 1/2(D_{1y} + D_{2y} + \Delta y_1 + \Delta y_2 - 2\Delta z \tan \alpha). \tag{6}$$

To convert to model scale, the factor Z/f is applied to the right-hand member of the equation.

Fig. 7 shows further that the value of H_x may be expressed as follows:

$$H_x = D_{2x} + \Delta x_2 - \Delta z \tan \theta_2 - q, \quad \text{where} \quad \tan \theta_2 = x_2/Z. \tag{7}$$