REVIEW OF MR. R. O. ANDERSON'S *APPLIED PHOTOGRAMMETRY* (THIRD EDITION) AND *RIGOROUS ANALYSIS OF THE SCALE POINT AND TILT FORMULAS**

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EDITOR'S NOTE: "This review not only outlines the material presented by Mr. Anderson in his books on photogrammetry but also is a detailed discussion of some of the highlights which, if thoroughly understood, would alleviate many worries that come to an engineer attempting to comprehend the aspects of photogrammetry. Special attention is given the 'scale point,' which if employed in photographic tilt and scale determination, results in an extremely easy and simplified method. The review contains many features that not only should be of interest to readers of the books but also should be of much assistance in understanding the subject matter. These works are both 'raked over the coals' and 'elevated on a pedestal' and as the conclusion points out, Mr. Anderson deserves much acclaim for his contribution to the advancement of photogrammetry.

"Mr. Kingsley has worked hard on this review and deserves the plaudits of every one interested in this phase of photogrammetry. It is hoped that liberal discussion will follow as there is no better way to advance a branch of science."

MUCH more discussed than read and much more read than understood, Mr. Anderson's works actually comprise a major contribution to the photogrammetric field. Although the words "equivalent elevation" and "scale point" are everyday terms to many of us, their derivation and full use are not widely comprehended. In addition to a brief outline of the subject matter, the review will attempt to bring out the relation of the principles to practice, and discuss the presentation of the material.

First of all in *Applied Photogrammetry* a section is devoted to a procedure for scale determination which contains many of the features that are normally used. Following this, with the aid of a number of excellent graphical representations, the elementary relationships between flying height, focal length, scale, and relief displacements are developed. The equivalent elevation, which is used in correcting a scale-check line for relief is defined and two proofs are given together with the formula that may be used. The formula for the convergence correction for the equivalent elevation is also demonstrated. The ratio equation is then built up as a relationship between actual and theoretical flying heights or actual and theoretical scales. This ratio equation is very commonly used. A number of examples are given, displaying what can be obtained by the ratio equation. The tilt of the photograph is not considered in the material concerning equivalent elevation but relief and tilt are considered separately, which is the most desirable means of showing the effects of either.

In the chapter dealing with tilt, the tilt displacement formulas, together with the relationships of lines in tilted and non-tilted planes are developed in very good form. A proof is offered for the tilt formula, by which the tilt may be determined as a function of the focal length and ratio change per inch on the photograph. The scale point is described and several "analogical" proofs are given. Following this are a number of problems with complete calculations which demonstrate how ratios, scales, flying heights and tilts may be obtained. In order to obtain tilt accurate to the "n*th* degree" by these methods successive solutions

* Copies can be obtained from the author whose address is 401 Pound Bldg., Chattanooga, Tennessee.
Applied Photogrammetry (Third Edition) .... $3.00 each
Rigorous Analysis of the Scale Point and Tilt Formulas .... $1.50 each

are involved. The examples serve to point out the accuracy obtainable by first, second, and third solutions under the conditions of extreme tilt and relief.

A section is devoted to a compact and explicit description of the graphical ray method of control, commonly referred to as the "hand templet" system; also, formulas and calculations are shown for an analytical method of extending control. The latter method requires measuring the angles between pass points.

In the booklet *Rigorous Analysis of the Scale Point and Tilt Formulas*, Mr. Anderson found in the works of Brooks Taylor, a formula which is employed in proving the tilt formula and scale point. Taylor's theorem of progression is used to much advantage in furnishing the reader a clearer understanding of the derivation of the formula for locating the scale point. The entire procedure necessary to obtain correct tilts from the scale point method is demonstrated. A number of examples with calculations serve as a help in learning the method. The last part of the book points out how tilts may be obtained by stereo-correspondence deficiency with Abram's Contour Finder.

Mr. Anderson's books in several instances have been criticized, perhaps unjustly, because some of the statements do not seem readily comprehensible, and some of the proofs seem to lack what it takes to make them wholly convincing. Acknowledging that every writer has a right to his own method of expression to which he becomes attached, a small amount of this criticism is justifiable and is not denied by the author. Often when one works extensively with a particularly specialized subject he may make use of a group of words which will mean very much more to him than they would to an unfamiliarized reader. In part, this may help to explain this situation. In some instances, seemingly unorthodox methods are used to propound a proof since the answer and the hypothesis are deployed through an algebraic routine until an identity is reached. Nevertheless and admittedly, a proof is constituted, and this writer does not mean to imply that this might not be the easiest way. Moreover, technical publications cannot be consummated page by page like novels since every variable is a character, every equation a plot, and every proof a short story.

The material brought out with respect to equivalent elevation is quite complete. There are a number of methods whereby the effect of relief can be eliminated from the comparison between photographic and ground distance. First, the terminals of the line may be brought to their undisplaced positions, or, to the position where they would have been if they were at the selected datum elevation, and a new line length computed for comparison with the ground length. Secondly, the projection of the line on the ground may be found at the selected datum and this new ground length compared with the photographic distance. Thirdly, an elevation may be obtained at which the relationship between photographic distance and ground distance will be the correct scale at that elevation (excluding tilt). This is the equivalent elevation method. It is a simple matter to convert scale prevailing at a particular elevation to datum scale. Of course, the effect of tilt will appear in the scale for all three methods. No one method is more correct than the other since when tilt is present the terminals undergo a certain displacement and the position of the nadir point is unknown. When very accurate work is desired successive solutions will remedy the situation, since the first tilt determination may be used to very closely locate the nadir point and the non-tilted positions of the terminals of the line. Under normal conditions of tilt and relief, very rarely would it be necessary because of this factor to make a second determination of equivalent elevation to obtain an accuracy of 0.1 per cent providing, of course, the terminal elevations are correct. Of the three methods mentioned the equivalent elevation method is at least ten times faster. The normal time required to compute an equivalent elevation having the terminals established with their elevations, is about thirty seconds. As brought out in the book, the equivalent elevation formula is not quite correct, but the correction formula is given. This is called the convergence correction and really makes up for the discrepancy existing when the quantity *one* is expressed as equal to *one* times the cosine of a small angle. It might appear as a technical malfeasance to make this approximation in the middle of a proof but the algebraic reduction was made much more simple, with no loss in accuracy, since the convergence correction formula was devised. Mr. Anderson states that seldom is the convergence correction necessary. Experience has substantiated this, since considering all terrain for purposes of an average, in not more than 3 per cent of the instances is it necessary to make the convergence correction for equivalent elevation to obtain an accuracy within 0.1 per cent having correct terminal elevations. Of course, supporting this statement is the fact that seldom are field parties asked to travel over terrain of rapidly changing elevation and if co-ordinated control is available, long scale check lines are usually obtainable, which practically eliminates a convergence correction.

Much of the material in Mr. Anderson's books deal with the relationship brought about by tilt of the negative plane. As mentioned previously the work covering the relationship between line lengths on the tilted and non-tilted photograph, and the development of the tilt displacement formulas is in excellent form. Two separate calculations of the displacement on the raised side do not show precisely the same value but this is due to insufficient accuracy in the substituted values rather than a discrepancy in the formulas since the formulas are identities. In the development of the tilt formula, where the sine of the tilt angle is shown equal to the focal length times the ratio change per inch, the final formula is correct but a slight discrepancy exists in the proof, i.e., the cosine of the tilt angle has been confused with the relationship between the tilted and non-tilted planes. It is understood that Mr. Anderson plans to present several different proofs of the tilt formula in the near future. Also, in this section the reader should not be misguided by the inference that ratio change should be prorated over the non-tilted interval because it is parallel to the measured ground distance. Perhaps this was not meant as a reason but in any event this comes into being because of the manner in which the ratio equation is set up. This is brought out later in the review and appears elsewhere in the books.

Treatment of the proof of the scale point in *Applied Photogrammetry* is mostly analogical in that the effect of elevation changes has been substituted for the effects of tilt. The proofs that are shown resemble those for equivalent elevation. It would be inappropriate to declare such proofs right or wrong but since elevations actually have no bearing on the scale point, these proofs are in a way inapplicable and misleading or at least not the most desirable. Mr. Anderson, however, has not left the problem unsolved, for in *Rigorous Analysis* a definite, comprehensive proof has been evolved which verifies the practical usage of the dropped perpendicular method of locating the scale point and gives all the "trimmings" that, when applied, will permit obtaining "tilts to the minute," limited, of course, in practice by the accuracy of the elevations and measurements.

In *Applied Photogrammetry* it is brought out that greater accuracy for tilts may be obtained if scale points are displaced to their non-tilted position, after first being located by the dropped perpendicular to the scale-check line from the iso-center. Since the position of the iso-center and the amount of tilt are unknown at the outset, successive solutions are necessary. However, with this adjusting the precise value of tilt is not obtained, but when using the equation developed in *Rigorous Analysis* the exact position of the scale point in the non-tilted plane is located, which will serve as a basis for determining the correct tilt. In this equation the first part is identical to that developed in *Applied Photogrammetry* but the last part does not yield the same effect (except in special cases), as displacing scale points to a non-tilted position. The necessity of having scale points in the non-tilted position is apt to appear mystifying. Fundamentally, it is because of the manner in which the ratio equation has been set up. The ground length divided by the photographic length (L/p) appears in the numerator of the ratio equation and this relationship after elevation correction is made then exists as a function of a non-tilted length divided by the tilted length which relationship may be represented as E_1/E. Now, since the tilted length is equal to the non-tilted length plus or minus the tilt displacement ($E_1 = E \pm e$), the amount of ratio over to represent the effect of tilt develops as $\pm e/E$. Since the fundamental equation, sine $t = e/EE_1$ which is developed in the book must be satisfied, and the ratio equation as set up produces the ratio pertaining to tilt as $\pm e/E$, it then becomes obvious that scale points must lie in a non-tilted plane, since by substitution in the tilt equation, the non-tilted length exists in the denominator. If the ratio equation were inverted, scale points would exist in the tilted plane and the use of a different formula for the scale point would be necessary.

The expression e/E actually represents the per cent of the tilted length required to convert this length to what would have existed had not the photograph been tilted (non-tilted length). Mr. Anderson designates e/E as "ratio change induced by tilt." This appears to be somewhat of a misnomer since this is more specifically the ratio change necessary to correct from the already tilted plane. Thus it seems that it might be more correctly called "ratio change effected by restitution."

The scale point is not easy to define nor is it easy to comprehend. A common concept is that the scale of a line on a photograph prevails at the scale point, which definition does not satisfy the mathematically curious. Mr. Anderson defines the scale point as follows, "The scale point of a given line is the point on the photograph at which the photographic scale equals the tilt axis scale plus, or minus, the scale change induced by tilt over the normal interval from the tilt axis to the position of the scale point of that line." The term, photographic scale should be interpreted as the horizontal ground distance in feet divided by the measured photographic length in inches between the image points conforming with the ground terminals of the check lines; also, the statement is meant to exclude the effect of any elevation changes. Whether this definition is ample for rapid comprehension is conjectural, but it would appear that a little more explanation of the nature and characteristics of the scale point would have helped the books. It would seem that a reader, whose occupation is not specifically linked with technical photogrammetrical work would appreciate a more complete picture of the applicability of the scale point in the process of tilt and scale determination, and what it actually represents. The books, however, do go to considerable length in pointing out how to locate the scale point under varying conditions in line with any desired accuracy.

Without a doubt, the scale point has a tremendous significance in any work where distances on a photograph are compared with distances on the ground. Perhaps a few statements in this review would help to unveil some of the mystery that shrouds the scale point. Consider two lines on the ground, having equal lengths and equal elevations at their terminals and lying at right angles to each other. Assume also a tilted photograph taken of these lines with the extended image of one of the lines passing through the iso-center, and the tilt axis perpendicular to the radial line. If the photograph were not tilted the lines would appear equal in length, but because of tilt the line lengths change and the radial line changes approximately twice as fast as the line parallel to the tilt axis. This can be proved by computing the displacements at the terminals of the lines and comparing the computed distances. The reason that the radial line does not change exactly twice as fast is because tilt displacements do not vary exactly as the square of the distance from the tilt axis to the displaced, image. The center of these lines is in the same position on the photograph but each line has a *different* scale! It then appears obvious that the scale of a line also becomes a function of the direction of the line. Therefore, since it is impossible to add apples and oranges, it becomes desirable to represent scales on a comparable basis. The solution to this problem lies in the theorem that scale parallel to the axis of tilt is a straight line function of the vertical distance from a horizontal plane, passing through the external node of the lens, to the plane of the negative. A change in scale parallel to the axis of tilt caused by varying the distance from the tilt axis effects a change in this vertical distance which is exactly comparable to a change in effective focal length. Therefore if for any line the difference between its length on the tilted photograph and its hypothetical non-tilted length is represented as effective focal length change, scale parallel to the axis of tilt comes into being at some position in a plane where change of effective focal length may be represented. The relationship between the tilted and non-tilted lengths establishes the amount the focal length must change, and this change then becomes the altitude of a triangle in which the angle opposite this altitude is the angle of tilt and the hypotenuse is the distance in the medium plane from the iso-center to the position at which scale parallel to the axis of tilt may be represented, and this position is the *scale point.* This is a method of explanation. You ask, "what does this mean?" It means that if the scale of a line is considered to apply at the scale point, scale parallel to the axis of tilt is expressed, and this is scale on a comparable basis. By this method tilts may be determined, and scales pro-rated anywhere in the medium plane, to determine datum scales parallel to the axis of tilt. "How do you locate the scale point?" Several proofs are possible to divulge a very easy working formula or rule. Mr. Anderson in his book "*Rigorous Analysis*, etc." has presented the development of a working formula in very good form. The only difference in the treatment presented for a radial line and any other line, is that in the case of lines other than radial, a tilt convergence correction may be made for extremely accurate work. Mr. Anderson develops the applicable formula. In practice, this correction is practically never made.

The "grandeur" of the scale point lies in the fact that no matter what the tilt nor where the axis of tilt, the approximate location of the scale point is a ten-second operation. To be more specific, the approximate position of the scale point can be located by the following rule which appears in *Applied Photogrammetry*. The midpoint of the scale check line bisects the distance between the foot of a perpendicular (to check line or its extension) dropped from the principal point, and the position of the scale point. Regarding the correct position of the scale point it should be said that if the above rule is applied using the iso-center as the point from which to drop the perpendicular, the position obtained is the initial receiving position for all tilts, and that the correct position is a shift from this position, depending upon the amount of tilt in the photograph. This signifies that the location of the initial receiving position is not a function of tilt itself. This is brought out in *Rigorous Analysis* and it is adeptly proved that this position is a function of the first term of Taylor's theorem of progression which was set up for tilt displacements.

Naturally, one is interested in the question "of what consequence is the shift from where the approximate position of the scale point is located by a dropped perpendicular from the principal point, to the correct position, obtainable after the "trimmings" are applied?" Considering the amount of tilt usually existing in "vertical" photography, and if one is only interested in flying heights or scales parallel to tilt axis at iso-center within one or two tenths of 1% and tilts within five or ten minutes, this shift is rather insignificant. In fact, in practice very rarely is anything used but the dropped perpendicular from the principal point. Quite often the effect of shifting scale points to their correct position is compensating, especially if scale check lines are more or less balanced about the center of the photograph. The average shift of scale points is not more than 0.5 inch for a 5° tilt. Actually in practice it is desirable to be familiar with the extent of shift for certain degrees of tilt. Generally, the shifting of scale points to their correct positions has a proportionally greater effect on the swing than on the amount of resultant tilt.

If for a photograph several scale check lines were available a slight inconsistency not traceable to measuring errors would be noticed in the scale at certain positions when different "scale point triangles" are used in prorating, unless the scale points exist in their correct positions. This is true because it is theoretically incorrect to prorate scale or ratios between scale points unless they are located in the non-tilted plane. The non-tilted plane is applicable in this instance due to the manner in which the ratio equation has been established in *Applied Photogrammetry*. Mr. Anderson presents an extended discussion of this particular subject in "*Rigorous Analysis*," etc.

The application of the "trimmings" is a matter of successive solutions, since the corrections are based on the tilt itself, which must of necessity be computed to make the corrections. This is accomplished by first determining tilt and position of iso-center without any corrections and then applying trimmings to obtain a more correct tilt to obtain more correct trimmings and so on until the value of tilt converges to within the accuracy required.

The use of the scale point affords an easy and practical means of determining tilts, and datum scales at any position and in any direction. It is realized that liberties have been taken both with the reader and Mr. Anderson in discussing to such length the nature of the scale point, but its use is rather widespread and there still exists in the minds of many, doubt as to what it means—doubt as to its practicability. Perhaps Mr. Anderson will not be in total agreement with some of the expressed concepts. In any event a discussion was not deemed inadvisable since it is thought that more could have been said in the books about the scope of the scale point.

As mentioned previously, Taylor's theorem was used in "*Rigorous Analysis*" for deriving the tilt formula and scale point position formula. Shown in the book is the manner in which Brooks Taylor developed a relationship between planes that is comparable to the relationship existing between tilted and non-tilted lengths, relating to the photogrammetric problem. The use of Taylor's theorem is not essential to prove the formulas derived in "*Rigorous Analysis*, etc.," but Mr. Anderson found that it could be used advantageously. In the book it is pointed out that the fundamental tilt formula could be proved both by using the approximate formula for tilt displacement with the first two terms of the geometric progression and by using the correct tilt displacement formula and complete geometric progression. However, it is not of much significance that the tilt formula could be proven in the case of the former conditions, because the amount the displacement formula is approximated is exactly equal to the remaining terms of the geometric progression. The proof involving the latter conditions is quite effective.

Mr. Anderson's books do not cover all the photogrammetric formulas and intricacies that are used in photogrammetric work (what a book it would take!), but they do go a long way toward bringing within reach the inherent characteristics of the aerial photograph. Many of the good features appearing in the books have been left untouched in this discussion. There are many little things about the books that could be criticized but over and above this the books definitely have much to offer. Few persons are trying as intently to find out things about photogrammetry and decidedly fewer follow through by placing their findings before the public. Notwithstanding the fact that the books can be criticized for body and presentation of material, etc., any criticism offered in advance of an earnest effort to apprehend the subject matter is untimely. Considering the scope of the subject, consummation of the material in the books is not a matter of hours or days.

Mr. Anderson has suggested that the books could be used by students, and a number of problems are set up in *Applied Photogrammetry* with that thought in mind. Without a doubt one remembers much better what he learns from reason than what he learns from rhyme. Therefore, if the entire books are studied as a problem, it would be toward the aggrandizement of photogrammetrical knowledge.