THREE DERIVATIONS OF A TILT FORMULA

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THE three derivations of this article are for but one formula, by means of which, the tilt of the Aerial Photograph may be computed. They are derived separately as a function of: (1) Scale Variation, (2) Tilt Displacement, and (3) Focal Variation.

The Focal Variation derivation appears in a current publication,* but unfortunately, both the diagram and derivation are in error. Despite these errors, which are compensating, the resulting formula is correct. An attempt will be made in this article to show why these errors are compensating. This same tilt formula is also derived† rigorously as a function of the tilt displacement geometrical progression.

The incomprehensibility of the diagramming and derivation of the forementioned tilt formula which did, however, lead to the correct result, was called to the attention of the writer by Mr. Robert Kingsley.‡ Since the remedial procedure suggested by Mr. Kingsley and those envisaged by the writer differ somewhat, the following three derivations are offered with the hopes of drawing further discussion.

FIRST METHOD: *derived as a function of scale variation.*

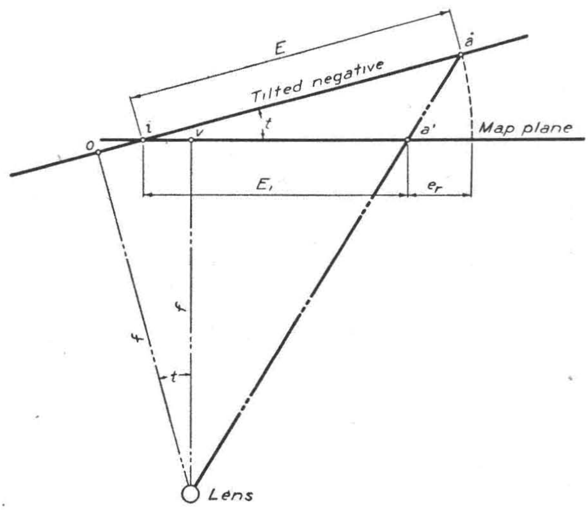

Fig. 1.

Figure 1 represents a partial cross section taken through a tilted negative at the instant of exposure. This cross section coincides with the principal line, which is normal to the tilt axis and passes through the principal point. The tilt axis is normal to the paper; the cross section of which is shown at *i*. The tilted photographic length *E* (the line under discussion) is represented by *ia*, and its corresponding map length *E*, by *ia'*. Relief is nil in this analogy.§

* *Applied Photogrammetry*, R. O. Anderson.

† *Rigorous Analysis of the Scale-Point and Tilt Formula*, R. O. Anderson.

‡ Associate Cartographic Engineer, Agricultural Adjustment Administration.

§ Methods are developed in *Applied Photogrammetry* covering the combined action at both tilt and relief.

The photographic scale of line *ia* is $S_p = D/E$, where D represents the corresponding horizontal ground distance of the photographic line *ia*. The map scale conforming to the photographic line *ia* is $S = D/E_{\prime}$. Had the negative been tilt-free at the instant of exposure, the photographic length would be $E_{\prime}$ and since $E_{\prime}$ is parallel to the measured ground distance D, the map scale would be $D/E_{\prime}$ assuming, of course, that the flying height at the instant of exposure is equal to the principal distance f multiplied by the map scale S. Seldom, if ever, are exposures made at the theoretical flying height, but it is a simple matter to compensate for this difference.

Obviously, the scale change induced by tilt is equal to $\Delta S = S_p - S$, and when, into which the foregoing scale values of S_p and S are substituted,

$$\Delta S = \frac{D}{E} - \frac{D}{E_{\prime}} = \frac{D(E_{\prime} - E)}{E_{\prime}E} \quad \text{(1)}$$

Referring to Figure 1, we find $E = E_{\prime} + e_r$. By means of the tilt displacement law previously established in *Applied Photogrammetry*

$$E = E_{\prime} + EE_{\prime}\tau \quad \text{(2)}$$

where τ equals sin t/f.
Substituting equation 2 in equation 1

$$\Delta S = \frac{D\left[E_{\prime} - (E_{\prime} + EE_{\prime}\tau)\right]}{E_{\prime}E} \quad \text{(3)}$$

Reducing equation 3

$$\Delta S = D\tau \quad \text{(4)}$$

Equation 4 has possibilities which have not been explored yet. Substituting, $D = SE_{\prime}$ in equation 4

$$\Delta S = SE_{\prime}\tau \quad \text{(5)}$$

Rewriting equation 5

$$\sin t = \frac{\Delta S}{SE_{\prime}} f \quad \text{(6)}$$

The term $\Delta S/S$ (actual photographic ratio change which is equal to e/E) represents the ratio change induced by tilt, effective over the map interval $E_{\prime}$. By definition, ratio is equal to the actual scale divided by the desired (map) scale, therefore, the scale change induced by tilt divided by the map scale is the ratio change induced by tilt. Denoting the actual ratio change $\Delta S/S$ which is effective over the map interval $E_{\prime}$, by $\Delta R'$ equation 6 becomes

$$\sin t = \frac{\Delta R'}{E_{\prime}} f \quad \text{(7)}$$

Since $E_{\prime}$ is the map length of photographic line *ia*,

$$\frac{\Delta R'}{E_{\prime}} = \Delta R \quad \text{(8)}$$

in which ΔR is the actual ratio change, induced by tilt normal to the tilt axis, per map unit.

Substituting equation 8 in equation 7

$$\sin t = \Delta R f. \qquad (9)$$

Equation 9 is identical to the equation developed in *Applied Photogrammetry* and also to the equation developed in *Rigorous Analysis of the Scale-Point and Tilt Formula*. The manner of obtaining ΔR in terms of three scale check lines is fully described in *Applied Photogrammetry*.

SECOND METHOD: *derived as a function of tilt displacement.*
The photographic length of line *ia*, Figure 1, is

$$E = E_{,} + e_{r}.$$

By means of the tilt displacement law previously established

$$E = E_{,} + \frac{E^2}{\frac{f}{\sin t} + E} \qquad (10)$$

Rewriting equation 10, letting $\sin t/f = \tau$

$$E = E_{,} + \frac{E^2\tau}{1 + E\tau} \qquad (11)$$

Expanding equation 11

$$E = \frac{E_{,} + E_{,}E\tau + E^2\tau}{1 + E\tau} \qquad (12)$$

Cross multiplying equation 12,

$$E + E^2\tau = E_{,} + E_{,}E\tau + E^2\tau \qquad (13)$$

Reducing and transposing equation 13,

$$E - E_{,} = E_{,}E\tau \qquad (14)$$

Substituting, $e_r = E - E_{,}$ in equation (14)

$$e_r = E_{,}E\tau \qquad (15)$$

Since $e/E = \Delta R'$, which is the ratio change induced in line *ia* by virtue of tilt, and since $e/E_{,}E = \Delta R$, which is the map unit ratio change induced by tilt, normal to the tilt axis, equation (15) becomes

$$\Delta R = \tau = \frac{\sin t}{f}.$$

Finally,

$$\sin t = f\Delta R. \qquad (16)$$

Equation (16) is identical to equation (9).

THIRD METHOD: *derived as a function at the focal change.*

As before stated, this method was derived in *Applied Photogrammetry* but two compensating errors were committed in the derivation, which did not interfere with obtaining the correct formula. An attempt will be made on the following pages to show why these errors are compensating and that the resulting expression is correct. No doubt, this derivation has puzzled many readers and it is quite safe to presume that a clarification of this derivation would be of interest. Those who ventured into this derivation and were forced to abandon the study were justified in becoming skeptic, but, perhaps they were unable to definitely locate the source of trouble—but still, arbitrary dismissal of what unquestionably has some foundation in fact should never be accepted supinely—and, fortunately for the world, it seldom is. The study of this confusing derivation persisted by many and finally the writer's attention was called to this matter by Mr. Kingsley. As a result, a more logical presentation of the original derivation is herewith offered together with two additional separate derivations for the same formula.

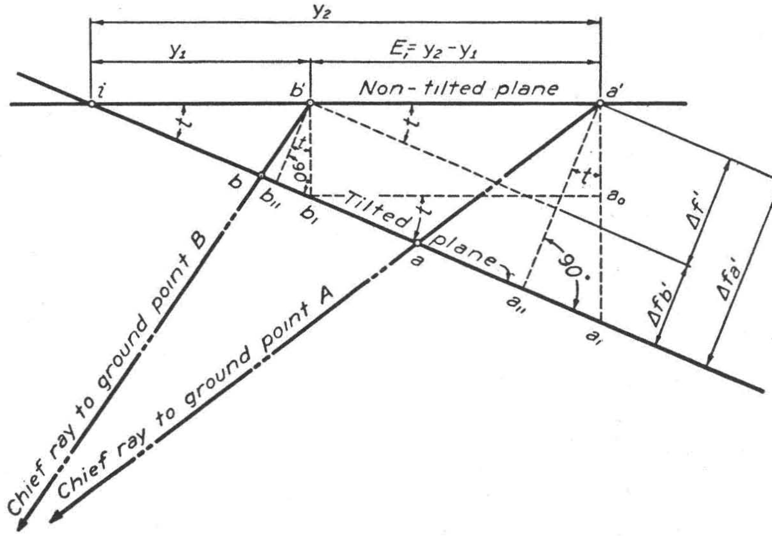

FIG. 2

This derivation (third method) is contingent upon the condition that the focal length change induced by tilt, of a given image point, is reckoned normal to the tilted plane. This condition should be evident as, by definition, the focal length, or more precisely, the principal distance, is the perpendicular distance from the plane of the negative, which is comparable with the tilted plane, to the rear nodal point of the lens.

Analytical Proof

The total ratio change induced by tilt, of the line *ia*, Figure 1, is

$$E_{,}\Delta R = \frac{\Delta S}{S} = \Delta R' \qquad (17)$$

where ΔR represents the ratio change per map unit and $E_{,}$ the map length ia'. Since

$$\Delta R' = \frac{\Delta f'}{f} = \frac{\Delta S}{S} \qquad (18)$$

it is evident that

$$\Delta f' = f\frac{\Delta S}{S} \quad (19)$$

Substituting equation (17) in equation (19)

$$\Delta f' = fE_{,}\Delta R \quad (20)$$

Substituting $\Delta R = \sin t/f$ (from equation (16)) in equation (20)

$$\Delta f' = fE_{,}\frac{\sin t}{f} \quad (21)$$

Rewriting equation (21)

$$\Delta f' = E_{,} \sin t \quad (22)$$

from which

$$\sin t = \frac{\Delta f'}{E_{,}} \quad (23)$$

By examining Figure 2, it will be seen that equation (23) is satisfied only when $\Delta f'$ is normal to the tilted plane, therefore, the proof is complete.

In reality the focal length of a given camera does not change by virtue of tilt, but the tilted plane rotates away from the non-tilted plane, thereby causing variable incremental focal length changes, for purposes of analogy only, which in turn cause scale changes. For instance, the incremental focal length change induced by tilt, at image point a (a' when converted to map position), on Fig. 2, is equal to $\Delta fa' = a'a_{,,}$ and the corresponding scale change is

$$\Delta S = \frac{\Delta fa'}{f} \cdot (S \text{ map})$$

Figure 2 shows all construction lines necessary to derive the tilt formula as a function of the focal length change induced by tilt. The focal change induced by tilt at image point b (b' when converted to map position) is

$$b'b_{,} = y_1 \tan t \quad (24)$$

when the focal change is considered parallel to the plumb line. However, it was shown that the focal change induced by tilt is effective normal to the tilted plane, therefore the correct focal change at image point b is

$$b'b_{,,} = \Delta f_{b'} = y_1 \tan t \cdot \cos t = y_1 \sin t \quad (25)$$

Likewise, the focal change at image point a, induced by tilt, is

$$a'a_{,,} = \Delta f_{a'} = y_2 \sin t \quad (26)$$

The focal change induced by tilt over the interval $E_{,} = y_2 - y_1$ is equal to

$$\sin t(y_2 - y_1) = \Delta f' \quad (27)$$

The focal change, induced by tilt, per map unit is then equal to

$$\Delta f = \frac{\Delta f'}{y_2 - y_1} = \sin t \quad (28)$$

Substituting $\Delta f = f\Delta R$ in equation (28)

$$\sin t = f\Delta R \quad (29)$$

Equation (27) is identical to both equations (9) and (16).

The diagrammatic error in the earlier derivation consists of stopping the chief rays at the tilted plane, points a and b on Figure 2, whereas they should have been extended to the non-tilted plane, points a' and b'. It is true that the chief rays are actually terminated by the negative, which is the tilted plane, but for purposes of analogy only, they must be extended to the non-tilted plane.

The two compensating errors committed in deriving this formula in *Applied Photogrammetry* consist of: (1) failure to multiply $a'a_{\prime}$ and $b'b_{\prime}$, of Figure 2, by cosine t, and (2) multiplying $a_{\prime}b_{\prime}$ by cosine t, which was wholly unnecessary. Error (2) was pointed out by Mr. Kingsley. A bit of retrospect on the part of the reader will make it quite obvious that these errors are compensating; the combined factorial error being (cos t)/(cos t), which is unity. Error (1) stated in words simply means that the focal change induced by tilt was considered as acting parallel to the plumb line when, as shown in this article, it should be reckoned normal to the tilted plane. Algebraically, $a'a_{\prime}$ was used as $\Delta fa'$, whereas $a'a_{\prime\prime}$ which is equal to $a'a_{\prime}\cdot\cos t$, should have been used. Error (2) consists of multiplying $b_{\prime}a_{\prime}$ by cosine t which was unnecessary, as the actual photographic ratio change (e/E) should be divided by its correspondent map length $(E_{\prime})$ in the course of obtaining ΔR.

From recent reports the study of photogrammetry is accelerating at a rapid pace. The scope of the science is enfolding to many and as a result, it seems that the average individual is gradually losing his traditional mathematical complex. The writer was recently informed by Professor Earl Church of Syracuse University that several eminent mathematicians are making excellent progress in photogrammetric research. Professor Church stated in his letter, "Some of their findings are of a more scholarly nature, more than anyting I have attempted to read in a long long time." And so it is beginning to look like the trail will be long and winding for those who have not been applying a concerted effort on the study of photogrammetry.

ANSWER TO QUIZ ON PAGE 184

FOR each correct answer give yourself 10 points. A score of 60% is considered passing and a score of 80% is considered excellent.

1. a, b, c, d (First three are identical)
2. a, b, c (Identical)
3. c
4. e
5. e
6. b
7. f
8. a, b, d, e
9. a and e
10. Either f, g, h, i, j or e, f, h, i, j