ANALYTICAL COMPUTATIONS IN AERIAL PHOTOGRAMMETRY

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MATHEMATICAL analysis in aerial photogrammetry or the computation of survey data from measurements on aerial photographs, serves several important purposes. It provides an effective means of teaching the fundamental theory of photogrammetry in a thorough, coherent, and systematic manner. It provides a powerful method of attack for research problems in photogrammetry. It exposes completely all possibilities of applying photogrammetry, not exclusively to topographic mapping, but to many other types of surveying. It can be combined effectively with some of the simpler and more practical applications of photogrammetry to planimetric and topographic mapping.

Although analytical methods are susceptible to wide variations, there are in reality four fundamental problems, namely: (1) Space resection; (2) Orientation of the photograph in space; (3) Space intersection; and (4) Photogrammetric extension of surveys without ground control. This paper presents brief explanations of these four problems in order, together with a brief discussion of an additional practical method for computing approximate areas, distances, and horizontal positions of ground points.

Space Resection

In surveying work it is convenient to define the positions of ground points by three rectangular coordinates in space, referred usually to a north-and-south y-axis, an east-and-west x-axis, and a vertical z-axis, with the horizontal xy-plane taken at the zero elevation for the system of levels used, usually sea level. The problem of space resection consists of determining the three rectangular coordinates of the point in the air from which a photograph was exposed, with these coordinates referred to the same axes as those used in the ground surveying. This point where a photograph was taken, generally called the "exposure station" of the photograph and designated by L in all of the accompanying geometric figures, is technically the space position of the incident nodal point of the camera lens at the instant of exposure of the photographic film. The required space coordinates of L are called X, Y, Z.

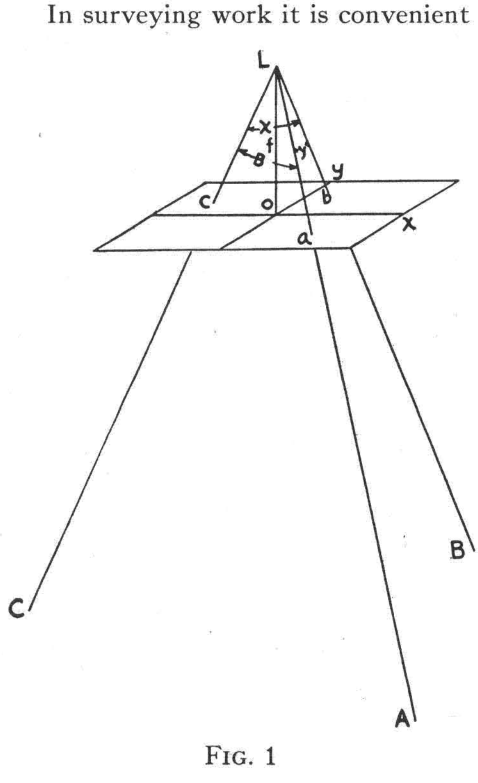

Fig. 1

The data for the problem consist of: (1) The survey space coordinates of three ground control points whose images are sharply defined in the photograph being used; and (2) the plane coordinates on the photograph itself, of the images of these three points, referred to the two rectangular axes of the photograph indicated by the camera marks. The survey coordinates of the ground points must have been previously determined upon the ground by usual geodetic procedure, such as triangulation, traverse, and levelling. The photographic coordinates of the images may be measured by any ordinary means such as an engineer's scale, but the accuracy of the results of the problem can be increased considerably if they are measured with a comparator, a precise photogrammetric instrument for this purpose.

The three ground control points are designated by A, B, and C, and their photographic images by a, b, and c, respectively. The survey coordinates of A, B, and C, are designated by (X_A, Y_A, Z_A), (X_B, Y_B, Z_B), and (X_C, Y_C, Z_C), and the photographic coordinates of a, b, and c, by (x_a, y_a), (x_b, y_b), and (x_c, y_c).

In Figure 1, abc represents the plane of the film; L represents the emergent nodal point of the camera lens; a, b, and c are the images designated in the previous paragraph; f is the effective focal length of the camera; o is the principal point of the photograph; and ox and oy are the rectangular geometric axes of the photograph defined by the camera marks. The angle aLb is designated by γ, bLc by α, and cLa by β.

It is easily shown that these three angles can be found from the expressions

$$\cos\gamma = \frac{x_a x_b + y_a y_b + f^2}{(La)(Lb)}$$
$$\cos\alpha = \frac{x_b x_c + y_b y_c + f^2}{(Lb)(Lc)} \qquad [2]$$
$$\cos\beta = \frac{x_c x_a + y_c y_a + f^2}{(Lc)(La)}$$

where

$$La = \sqrt{x_a^2 + y_a^2 + f^2}$$
$$Lb = \sqrt{x_b^2 + y_b^2 + f^2} \qquad [1]$$
$$Lc = \sqrt{x_c^2 + y_c^2 + f^2}$$

These three angles are identical with those subtended at the exposure station by the three ground points A, B, and C. The problem of space resection now consists of finding the survey coordinates (X, Y, Z) of a point L in space, such that the three angles subtended at L by A, B, and C, or the angles ALB, BLC, and CLA, will be exactly equal to γ, α, and β, respectively.

This is accomplished by stating the cosines of the angles ALB, BLC, and CLA in terms of the unknown coordinates X, Y, and Z, equating these to the known numerical values for $\cos\gamma$, $\cos\alpha$, and $\cos\beta$, respectively, and solving these equations for the three unknown quantities X, Y, and Z.

It can be proved that

$$\cos ALB = \frac{(X - X_A)(X - X_B) + (Y - Y_A)(Y - Y_B) + (Z - Z_A)(Z - Z_B)}{(LA)(LB)}$$
$$\cos BLC = \frac{(X - X_B)(X - X_C) + (Y - Y_B)(Y - Y_C) + (Z - Z_B)(Z - Z_C)}{(LB)(LC)} \qquad [3]$$
$$\cos CLA = \frac{(X - X_C)(X - X_A) + (Y - Y_C)(Y - Y_A) + (Z - Z_C)(Z - Z_A)}{(LC)(LA)}$$

where

$$
\begin{aligned}
LA &= \sqrt{(X - X_A)^2 + (Y - Y_A)^2 + (Z - Z_A)^2} \\
LB &= \sqrt{(X - X_B)^2 + (Y - Y_B)^2 + (Z - Z_B)^2} \qquad [4] \\
LC &= \sqrt{(X - X_C)^2 + (Y - Y_C)^2 + (Z - Z_C)^2}
\end{aligned}
$$

The equations, therefore, which are to be solved for the desired values X, Y, and Z, are

$$
\begin{aligned}
&(X-X_A)(X-X_B)+(Y-Y_A)(Y-Y_B)+(Z-Z_A)(Z-Z_B)-(LA)(LB)\cos\gamma=0 \\
&(X-X_B)(X-X_C)+(Y-Y_B)(Y-Y_C)+(Z-Z_B)(Z-Z_C)-(LB)(LC)\cos\alpha=0 \qquad [5] \\
&(X-X_C)(X-X_A)+(Y-Y_C)(Y-Y_A)+(Z-Z_C)(Z-Z_A)-(LC)(LA)\cos\beta=0
\end{aligned}
$$

in which $\cos\gamma$, $\cos\alpha$, and $\cos\beta$, as well as the survey coordinates of the control points, are known numerical values, but in which LA, LB, and LC are radical expressions involving the unknown quantities. The solution of these equations in this form is virtually impossible.

To effect a solution of these equations there are first found approximate values for the coordinates X, Y and Z, designated by (X), (Y), and (Z). These are found by making a quick graphical solution of the three-point problem to find (X) and (Y), and by a quick approximate determination of (Z) by the fundamental scale and altitude relation for aerial photographs, assuming for the moment true verticality of the camera axis. Then these quantities are substituted in [5] to find the numerical values of the left-hand members, called v_1, v_2, and v_3, respectively. These quantities of course show the amounts by which conditions [5] fail to be satisfied by the approximate values (X), (Y), and (Z). In order to find the correct values of X Y, and Z, it is necessary to determine the corrections ΔX, ΔY, and ΔZ to be applied to the approximate values such that the changes produced by these corrections in the left members of [5] will be respectively equal to $-v_1$, $-v_2$, and $-v_3$. The changes produced in the left members of [5] by small changes $\Delta X, \Delta Y$, and ΔZ in the three variables, are found by partial differentiation; and by equating these total increments to $-v_1$, $-v_2$, and $-v_3$ there are obtained three very simple linear equations which can be easily solved for X, Y, and Z. These equations are

$$
\begin{aligned}
U\,\Delta X + V\,\Delta Y + W\,\Delta Z + v_1 &= 0 \\
U'\,\Delta X + V'\,\Delta Y + W'\,\Delta Z + v_2 &= 0 \qquad [6] \\
U''\Delta X + V''\Delta Y + W''\Delta Z + v_3 &= 0
\end{aligned}
$$

where the coefficients are given by the following expressions

$$
\begin{aligned}
U &= [1 - (LA/LB)\cos\gamma](X - X_B) + [1 - (LB/LA)\cos\gamma](X - X_A) \\
V &= [1 - (LA/LB)\cos\gamma](Y - Y_B) + [1 - (LB/LA)\cos\gamma](Y - Y_A) \\
W &= [1 - (LA/LB)\cos\gamma](Z - Z_B) + [1 - (LB/LA)\cos\gamma](Z - Z_A) \\
U' &= [1 - (LB/LC)\cos\alpha](X - X_C) + [1 - (LC/LB)\cos\alpha](X - X_B) \\
V' &= [1 - (LB/LC)\cos\alpha](Y - Y_C) + [1 - (LC/LB)\cos\alpha](Y - Y_B) \qquad [7] \\
W' &= [1 - (LB/LC)\cos\alpha](Z - Z_C) + [1 - (LC/LB)\cos\alpha](Z - Z_B) \\
U'' &= [1 - (LC/LA)\cos\beta](X - X_A) + [1 - (LA/LC)\cos\beta](X - X_C) \\
V'' &= [1 - (LC/LA)\cos\beta](Y - Y_A) + [1 - (LA/LC)\cos\beta](Y - Y_C) \\
W'' &= [1 - (LC/LA)\cos\beta](Z - Z_A) + [1 - (LA/LC)\cos\beta](Z - Z_C)
\end{aligned}
$$

The coefficients of ΔX, ΔY, and ΔZ, are computed in [7] using the approximate values of the coordinates of the exposure station.

SPACE RESECTION

Photograph No. f				a x_a y_a b x_b y_b c x_c y_c			La [1] Lb Lc		cosine γ [2] cosine α cosine β
L	(X)	(Y)	(Z)						
A	X_A	Y_A	Z_A	$(X)-X_A$	$(Y)-Y_A$	$(Z)-Z_A$	(LA)[3]	$\frac{LB}{LA}\cos\gamma$	$\frac{LC}{LA}\cos\beta$
B	X_B	Y_B	Z_B	$(X)-X_B$	$(Y)-Y_B$	$(Z)-Z_B$	(LB)	$\frac{LA}{LB}\cos\gamma$ $\frac{LC}{LB}\cos\alpha$	
C	X_C	Y_C	Z_C	$(X)-X_C$	$(Y)-Y_C$	$(Z)-Z_C$	(LC)	$\frac{LB}{LC}\cos\alpha$	$\frac{LA}{LC}\cos\beta$

1st 3 terms 4th term [4]	1st terms of U, V, W 2nd terms of U, V, W [5]	U X V Y W Z v_1 U' X V' Y W' Z v_2	
v_1 1st 3 terms	1st terms of U', V', W' 2nd terms of U', V', W' [5]	U'' X V'' Y W'' Z v_3	
4th term [4]	1st terms of U'', Y'', W''		
v_2	2nd terms of U'', V'', W'' [5]	Solutions of equations	
1st 3 terms	(X) (Y) (Z)		
4th term [4]	$\frac{\Delta X}{X}$ $\frac{\Delta Y}{Y}$ $\frac{\Delta Z}{Z}$		
v_3			

	Second Solution		
	(X) (Y) (Z) $\frac{\Delta X}{X}$ $\frac{\Delta Y}{Y}$ $\frac{\Delta Z}{Z}$ Exposure station		

$X-X_A$ $Y-Y_A$ $Z-Z_C$ $X-X_B$ $Y-Y_B$ $Z-Z_B$ $X-X_C$ $Y-Y_C$ $Z-Z_C$	LA [3] LB LC	1st 3 terms 4th term [4] v_1	1st 3 terms 4th term [4] v_2	1st 3 terms 4th term [4] v_3

[1] See formulas [1].
[2] See formulas [2].
[3] See formulas [4].
[4] From equations [5].
[5] See formulas [7].

After the linear equations [6] are solved for ΔX, ΔY, and ΔZ, the application of these corrections to the approximate values (X), (Y), and (Z) gives the corrected coordinates X, Y, and Z of the exposure station. The corrected values for these coordinates should be substituted again in [5] to find whether residuals

SPACE RESECTION

Photograph I				q	3.68	−71.56	166.236	.907084
f 150.00 mm.				b	82.29	−74.88	186.758	.648273
				a	83.56	83.56	190.957	.530115
L	4600	34500	19785					
Q	5000	25000	400	−400	9500	19385	21591.39	.986		.591
B	15000	25000	1000	−10400	9500	18785	23479.49	.834	.664	
A	15000	45000	800	−10400	−10500	18985	24059.10		.633	.476

458557225	−6	133	271	−7311	−662	13279	−1163209
−459850611	−1726	1577	3118	−5614	−1616	17876	−2944302
−1293386	−3494	3192	6312	−1732	1710	3389	−1293386
365043225	−3817	−3854	6967	−1866	−169	3389	−296868
−366206434				−1064	−306	3389	−558192
	−164	3886	7928				
−1163209	−5450	−5502	9948	134	1879		−996518
272434225				802	−137		−261324
−275378527	4600	34500	19785				
	411	501	339	10	137		−72657
−2944302							
	5011	35001	20124	812			−333981

L	5011	35001	20124	Second Solution					
Q	5000	25000	400						
B	15000	25000	1000	11	10001	19724	22114.62	.975		.574
A	15000	45000	800	−9989	10001	19124	23780.82	.844	.653	
				−9989	−9999	19324	23941.12		.644	.490

477111898	0	250	493	−7022	−90	13515	244895
−477038920	−1558	1560	2983	−5089	−839	18257	367963
72978	−3466	3470	6636	−1558	1810	3476	72978
369332298	−3556	−3560	6879	−1806	−23	3476	62986
−369087403				−969	−160	3476	70057
	5	4260	8402				
244895	−5094	−5099	9855	248	1833		9992
281036698				837	−137		7071
−280668735	5011	35001	20124				
	−9	−4	−23	19	137		747
367963							
	5002	34997	20101	856			7818
	Exposure station						

2	9997	19701	22092.29	476228814	368628414	280229014
−9998	9997	19101	23764.43	−476228787	−368630745	−280231562
−9998	−10003	19301	23927.99			
				27	−2331	−2548

still remain; and if these residuals are still not negligible, a second solution must be made. In such cases it is seldom necessary to re-compute the coefficients of ΔX, ΔY, and ΔZ in equations [6].

Although the symbols used, together with the mathematics underlying this explanation, make this exact method appear somewhat involved, there is after all little difficulty in the computation itself. In fact the form for the computation covers but half a sheet of paper, and two complete solutions require only about an hour with a calculating machine. This method will not fail under extreme conditions of large tilts and extensive topographic relief.

PHOTOGRAPH ORIENTATION

The orientation of a tilted aerial photograph in space can be resolved into three component elements in several slightly different ways. A convenient resolution is into the "tilt," the "swing," and the "azimuth." The last is often called the "azimuth of the principal plane." These are shown in Figure 2, and are defined, together with some other terms, as follows: The vertical plane *vLo* containing the camera axis is called the principal plane; the line *ov* in which the principal plane intersects the plane of the photograph is called the principal line; the point *o* where the camera axis meets the plane of the photograph is called the principal point; the point *v* where a vertical line through the emergent node of the camera lens intersects the plane of the photograph is called the nadir point; the angle between the plane of the photograph and a horizontal plane, shown by the equivalent angle *vLo* since *Lv* is vertical and *Lo* is perpendicular to the plane of the photograph, is called the "tilt," and is designated by *t;* the direction on the photograph of the principal line *ov*, referred to the geometric axes of the photograph and measured conventionally like an azimuth clockwise from the positive *y*-axis to *ov*, is called the "swing," and is designated by *s;* the usual survey azimuth of the principal plane *VLO*, measured clockwise from north to the direction *VO*, is called the "azimuth" or the "azimuth of the principal plane." This last element is actually the azimuth of the camera axis at the instant of exposure, in accordance with the usual definition of azimuth in ordinary surveying. It is designated by α_{VO}, the subscript showing that the symbol designates the azimuth of *VO*, but not of *OV*. It should be noted by the reader that these three elements, the tilt, the swing, and the azimuth of the principal plane, actually *orient* the photograph in space, and that these three angles together with the survey coordinates of the exposure station, actually fix the photograph in space.

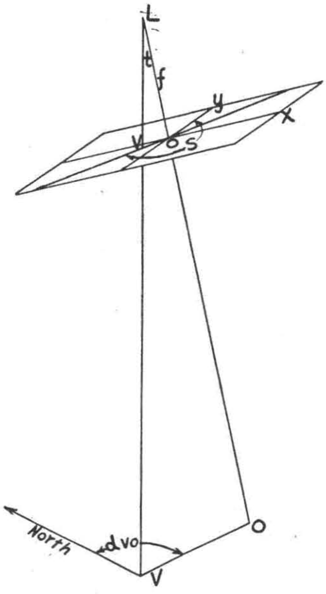

FIG. 2

The problem in hand is to determine *t*, *s*, and α_{VO}, after the coordinates of the exposure station have been found.

In Figure 3, *L* represents the exposure station; *A*, *B*, and *C* represent the three ground control points at different elevations; *a*, *b*, and *c* represent the images of these three points on a tilted aerial photograph; *o* represents the principal point and *v* the nadir point of the photograph. The three vertical angles *VLA*, *VLB*, and *VLC*, called m_A, m_B, and m_C, respectively, measured outward from the vertical line *LV* to the lines *LA*, *LB*, and *LC*, respectively, are obviously given by

$$\begin{aligned} \cos m_A &= (Z - Z_A)/LA \\ \cos m_B &= (Z - Z_B)/LB \\ \cos m_C &= (Z - Z_C)/LC \end{aligned} \qquad [8]$$

in which all numerators and denominators are values which have already been calculated.

It now becomes necessary to locate the nadir point *v* on the photograph in

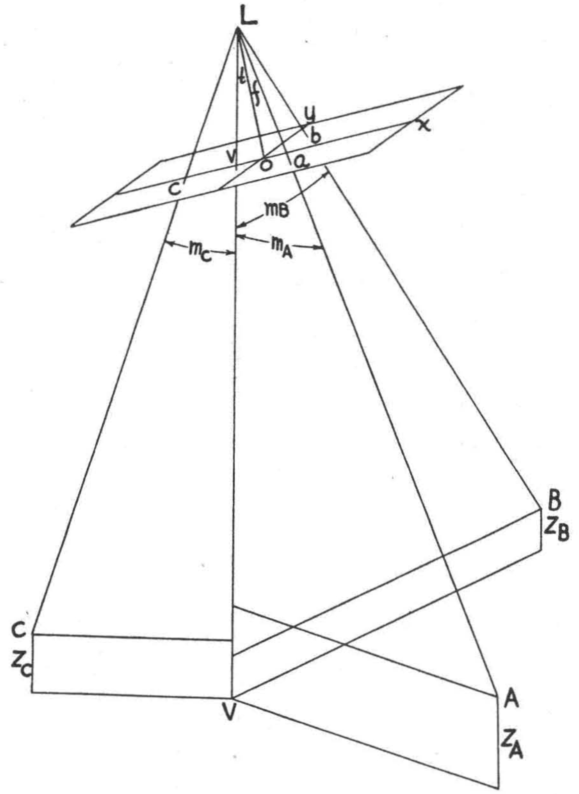

FIG. 3

such a manner that angle vLa will be equal to m_A, angle vLb to m_B, and angle vLc to m_C. If the unknown photographic coordinates of the nadir point v are designated by x, y, it can easily be proved that

$$\begin{aligned} \cos vLa &= (x_a x + y_a y + f^2)/(La)(\sqrt{x^2 + y^2 + f^2}) \\ \cos vLb &= (x_b x + y_b y + f^2)/(Lb)(\sqrt{x^2 + y^2 + f^2}) \\ \cos vLc &= (x_c x + y_c y + f^2)/(Lc)(\sqrt{x^2 + y^2 + f^2}) \end{aligned} \qquad [10]$$

in which

$$\begin{aligned} La &= \sqrt{x^{\ 2} + y_a^2 + f^2} \\ Lb &= \sqrt{x_b^2 + y_b^2 + f^2} \\ Lc &= \sqrt{x_c^2 + y_c^2 + f^2} \end{aligned} \qquad [9]$$

If the right-hand members of equations [10] are set equal respectively to the known numerical values $\cos m_A$, $\cos m_B$, and $\cos m_C$ found by [8], we have three equations with but two unknown quantities x and y. Then if the radicals are eliminated from these three equations, simple algebraic transformation leads to the two simple linear equations

$$\begin{aligned} ux + vy + w &= 0 \\ u'x + v'y + w' &= 0 \end{aligned} \qquad [11]$$

in which the coefficients are given by

$$\begin{aligned} u &= x_a/(La \cos m_A) - x_b/(Lb \cos m_B) \\ v &= y_a/(La \cos m_A) - y_b/(Lb \cos m_B) \\ w &= f^2/(La \cos m_A) - f^2/(Lb \cos m_B) \\ u' &= x_b/(Lb \cos m_B) - x_c/(Lc \cos m_C) \\ v' &= y_b/(Lb \cos m_B) - y_c/(Lc \cos m_C) \\ w' &= f^2/(Lb \cos m_B) - f^2/(Lc \cos m_C) \end{aligned} \qquad [12]$$

These coefficients of x and y are known definite numerical values. The solution of the simple equations [11] gives the coordinates x and y of the nadir point v on the photograph.

Then the following relations follow at once:

$$ov = \sqrt{x^2 + y^2} \qquad [13]$$

$$\tan t = ov/f \qquad [14]$$

$$\tan s = x/y \qquad [15]$$

with the swing s placed in the proper quadrant in accordance with the algebraic signs of x and y. These give two of the three elements of the space orientation of the photograph.

To continue with finding the azimuth of the principal plane, it is easiest to

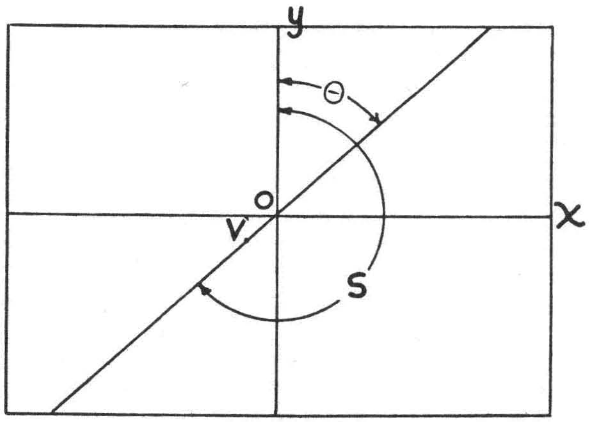

FIG. 4

transform first the photographic coordinates of the images a, b, and c, by rotating the rectangular axes through the angle θ which is $180° - s$, and then by translating the axes by the algebraic addition of the value of ov to all of the new ordinates. It can be seen from Figure 4 that these transformations place the origin at the nadir point v, and the positive y-axis along the principal line in the direction vo. The analytical geometry formulas for this rotation and translation are

$$\begin{aligned} \text{New } x &= x \cos \theta + y \sin \theta \\ \text{New } y &= -x \sin \theta + y \cos \theta + ov \end{aligned} \qquad [16]$$

To avoid confusion, the photographic coordinates of the images a, b, and c after these transformations, will henceforth be designated by the same symbols as those heretofore used for the original coordinates, namely, (x_a, y_a), (x_b, y_b), and (x_c, y_c).

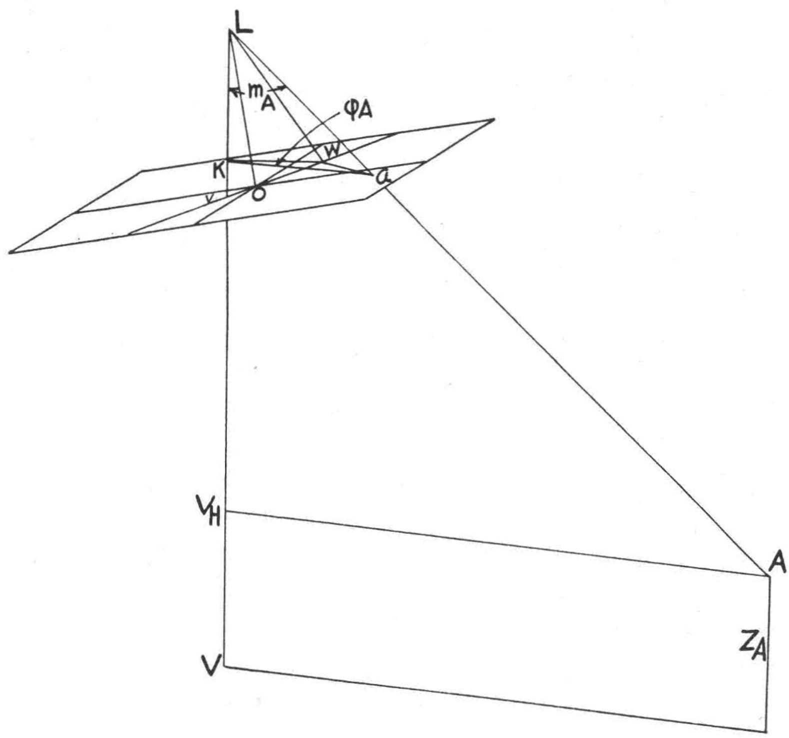

FIG. 5

Figure 5 shows a tilted photograph with the image a of one of the control points A. If aw is drawn perpendicular to the principal line, it lies wholly within the plane of the photograph and is equal to the new x_a. Also vw equals y_a. If wk is drawn perpendicular to Lv and k is joined to a, then kw equals $y_a \cos t$. The angle wka, designated by ϕ_A, is a horizontal angle, and is actually the survey angle between the principal plane and the vertical plane VLA. From the figure it is seen that

$$\tan \phi_A = x_a/(y_a \cos t)$$

and similarly

$$\begin{aligned} \tan \phi_B &= x_b/(y_b \cos t) \\ \tan \phi_C &= x_c/(y_c \cos t) \end{aligned} \quad [17]$$

Moreover the survey azimuths of the vertical planes VLA, VLB, and VLC, or the survey azimuths of LA, LB, and LC, or of VA, VB, and VC, according to the usual plane surveying practice, are given by

$$\begin{aligned} \tan \alpha_A &= (X - X_A)/(Y - Y_A) \\ \tan \alpha_B &= (X - X_B)/(Y - Y_B) \\ \tan \alpha_C &= (X - X_C)/(Y - Y_C) \end{aligned} \quad [18]$$

with these azimuths placed in the proper quadrants in accordance with the relative positions of L and the points A, B, and C, respectively. Then the azimuth of the principal plane is given by direct subtraction as follows:

$$\begin{aligned} \alpha_{VO} &= \alpha_A - \phi_A \\ \alpha_{VO} &= \alpha_B - \phi_B \\ \alpha_{VO} &= \alpha_C - \phi_C \end{aligned} \quad [19]$$

The agreement between these three values furnishes an excellent check on the computations.

This completes the determination of the three elements of space orientation of the aerial photograph. This entire computation requires about a half-hour with a calculating machine.

Space Orientation

a	$\cos m_A$ [1]	$\tan m_A$ [2]	$La \cos m_A$ [3]	1st terms of u, v, and w [4]
b	$\cos m_B$	$\tan m_B$	$Lb \cos m_B$	2nd terms of u, v, w; 1st of u', v', w'
c	$\cos m_C$	$\tan m_C$	$Lc \cos m_C$	2nd terms of u', v', and w'

$\tan t$ [6]	$\tan s$ [7]	u	v	w [4]
t	s	u'	v'	w'
$\sin t$	θ [8]			
$\cos t$	$\sin \theta$	Solution of equations [11]		
$f \sec t$	$\cos \theta$			
		x	y	ov [5]

$x_a \cos \theta$	$y_a \sin \theta$	$-x_a \sin \theta$	$y_a \cos \theta$ [9]	New x_a	New y_a [9]	$y_a \cos t$	$\tan \phi_A$ [10]
$x_b \cos \theta$	$y_b \sin \theta$	$-x_b \sin \theta$	$y_b \cos \theta$	New x_b	New y_b	$y_b \cos t$	$\tan \phi_B$
$x_c \cos \theta$	$y_c \sin \theta$	$-x_c \sin \theta$	$y_c \cos \theta$	New x_c	New y_c	$y_c \cos t$	$\tan \phi_C$

ϕ_A	$\tan \alpha_A$ [11]	α_A	α_{VO} [12]	$y_a \sin t$	13	16	19
ϕ_B	$\tan \alpha_B$	α_B	α_{VO}	$y_b \sin t$	14	17	20
ϕ_C	$\tan \alpha_C$	α_C	α_{VO}	$y_c \sin t$	15	18	21
			α_{VO} [22]				

[1] By formula [8].
[2] Found from cosines.
[3] Use La, Lb, Lc, from resection computation.
[4] See formulas [12].
[5] By formula [13].
[6] By formula [14].
[7] By formula [15].
[8] $\theta = 180° - s$.
[9] Four of the terms for formulas [16]. New x and new y found by formulas [16].
[10] By formulas [17].
[11] By formulas [18].
[12] By formulas [19].
[13] $f \sec t - y_a \sin t$.
[14] $f \sec t - y_b \sin t$.
[15] $f \sec t - y_c \sin t$.
[16] $\sqrt{x_a^2 + y_a^2 \cos^2 t}$.
[17] $\sqrt{x_b^2 + y_b^2 \cos^2 t}$.
[18] $\sqrt{x_c^2 + y_c^2 \cos^2 t}$.
[19] Use formula [23] of Space Intersection, inverted, to find $\tan m_A$.
[20] Same for finding $\tan m_B$.
[21] Same for finding $\tan m_C$.

Checks:
[22] The three values for α_{VO} should be identical. In case of small discrepancies in the three values, enter in this space the mean.
[19,20,21] These tangents of the m's should be identical with their values found at the top of the computation in the second column.

Space Intersection

Every surveyor understands the problem of intersection in plane surveying. If from two known points, horizontal directions and vertical angles are measured to some undetermined point, the horizontal position and the elevation of the new point can be calculated. In fact, after the horizontal position is determined, the identity of the elevations computed with each of the two vertical angles furnishes a complete check on the whole intersection problem.

The space intersection problem in aerial photogrammetry is exactly analogous to this. If an undetermined point has its image appearing in each of two photographs for which the foregoing calculations of exposure station coordinates

SPACE ORIENTATION Photograph I

q	.89176	.50744	148.243	+.0248	−.4827	+151.7778
b	.80376	.74023	150.109	+.5482	−.4988	+149.8911
a	.80663	.73274	154.032	+.5425	+.5425	+146.0735

$\tan t = .03491$	$\tan s = 1.0084$
$t = 2°00.0'$	$s = 45°14.3'$
$\sin t = .03490$	$\theta = 134°45.7$
$\cos t = .99939$	$\sin\theta = +.71004$
$f \sec t = 150.092$	$\cos\theta = -.70416$

− 5234	+ 161	+ 18867
+ 57	−10413	+ 38176
+ 1	− 161	+ 590
− 5233		+ 19457
$x = +3.718$	$y = +3.687$	$ov =$ 5.236

q	− 2.591	−50.810	− 2.613	+50.390	− 53.401	+ 53.013	+ 52.981	1.0079
b	−57.945	−53.168	−58.429	+52.728	−111.113	− 0.465	− 0.465	238.95
a	−58.840	+59.331	−59.331	−58.840	+ 0.491	−112.935	−112.866	.00435

314°46.5′	.00020	180°00.7′	225°14.2′	+1.850	148.242	75.224	.50744
269 45.6	1.0001	134 59.7	225 14.1	−0.016	150.108	111.114	.74023
179 45.0	.99950	44 59.1	225 14.1	−3.941	154.033	112.867	.73275
			$\alpha_{VO} = 225\ 14.1$				

and orientation elements have been completed, then by means of horizontal directions and vertical angles measured by the photographs from their respective exposure stations, the horizontal position and the elevation of the new point can be computed.

In this explanation of photogrammetric intersection in space, the same symbolism as that used heretofore will be preserved, except that primes will be added to designate data for the second one of the two photographs.

First the photographic coordinates are measured (with the comparator if possible) on each of the two photographs, of the image p or p' of the point P whose survey position is to be determined.

Then the coordinate axes of each photograph are rotated and translated exactly as described in the discussion of space orientation, giving coordinates of the image of the undetermined point on each photograph referred to the nadir point for the origin and the principal line ov or $o'v'$ for the positive y-axis. These transformed coordinates on one photograph will be designated by x_p, y_p and on the other by x_p', y_p'.

At this point the reader should refer to Figure 5 and imagine the control point A replaced by the undetermined point P and the image a by the image p. In accordance with the explanation of the space orientation, the horizontal angles taken clockwise from the principal planes are given by

$$\tan\phi = x_p/(y_p \cos t) \qquad \tan\phi' = x_p'/(y_p' \cos t')$$

and the survey azimuths from the exposure stations L and L' respectively to the undetermined point are

$$\alpha_P = \alpha_{VO} + \phi \qquad \alpha_P' = \alpha_{VO}' + \phi' \qquad [20]$$

These two survey azimuths from two known points L and L' to P are sufficient for determining the horizontal position of P. Although plane trigonometry suffices for the calculation, it is perhaps preferable to use the analytics method, namely, that of finding the equations of the horizontal projections of LP and $L'P$ and solving them simultaneously for X_P and Y_P. Following this method, if we call s and s' the analytical geometry slopes (positive upward to right, negative upward to left) of the horizontal projections of LP and $L'P$, respectively, then

$$s = \cot \alpha_P \quad \text{and} \quad s' = \cot \alpha_P' \tag{21}$$

with the proper signs assigned. The equations of these horizontal projections of LP and $L'P$ are

$$\begin{aligned} Y_P - Y_L &= s(X_P - X_L) \\ Y_P - Y_L' &= s'(X_P - X_L') \end{aligned} \tag{22}$$

with X_P and Y_P the only unknown quantities. These equations are quickly solved simultaneously for the survey coordinates X_P and Y_P, giving the horizontal position of the new point P.

Again reference to Figure 5 in which control point A has been replaced by the undetermined point P and image a by image p, shows that

$$kp = \sqrt{kw^2 + wp^2} = \sqrt{x_p^2 + y_p^2 \cos^2 t}$$

$$kv = y_p \sin t \qquad Lv = f \sec t$$

$$Lk = f \sec t - y_p \sin t$$

SPACE INTERSECTION

Point	Photo	Point	Photo
x_p	y_p	x_p'	y_p'
$x_p \cos \theta$ $y_p \sin \theta$	$-x_p \sin \theta$[1] $y_p \cos \theta$	$x_p' \cos \theta'$ $y_p' \sin \theta'$	$-x_p' \sin\theta'$ $y_p' \cos \theta'$
New x_p	New y_p [2] $y_p \cos t$	New x_p'	New y_p' $y_p' \cos t'$
$\tan \phi$ [3] α_P [4]	ϕ Slope [5]	$\tan \phi'$ [3] α_P' [4]	ϕ' Slope [5]
Equations in X and Y[6]			
X		Y	
$y_p \sin t$ [9]	[7] $\cot m_P$ [11]	$y_p' \sin t'$ [10]	[8] $\cot m_P'$ [11]
V_HP [12]	Z_L $(Z_L - Z_P)$ [13]	$V_H'P$ [12]	Z_L' $(Z_L' - Z_P)$ [13]
	Z_P		Z_P
Point	X_P	Y_P	Z_P

[1] First four terms of formulas [16].
[2] By formulas [16].
[3] By formulas [17].
[4] By formulas [20]
[5] By formulas [21].
[6] Equations [22], solved for X and Y.
[7] $f \sec t - y_p \sin t$.
[8] $f \sec t' - y_p' \sin t'$.
[9] $\sqrt{x_p^2 + y_p^2 \cos^2 t}$.
[10] $\sqrt{x_p'^2 + y_p'^2 \cos^2 t'}$.
[11] By formulas [23].
[12] By formulas [24].
[13] By formulas [25].

Check:

The two values for Z_P should be identical. In case of a small discrepancy attributable to small errors in measurements, the mean is entered at the bottom of the form.

Space Intersection

Point $B3$ Computed from Measurements on Photographs II′ & II

Resection and Orientation Data for Photographs II′ & II

Photo II′		Photo II	
Exposure Station	Orientation	Exposure Station	Orientation
$X=14997$ ft.	$t=0°59.7'$	$X=15003$ ft.	$t=1°29.3'$
$Y=15002$	$s=180°29.0'$	$Y=34995$	$s=0°20.3'$
$Z=20201$	$\alpha_{VO}=0°29.0'$	$Z=20000$	$\alpha_{VO}=180°20.3'$
	$\sin t=.01737$		$\sin t=.02599$
	$\cos t=.99985$		$\cos t=.99966$
	$f \sec t=150.022$		$f \sec t=150.051$
	$ov=2.606$		$ov=3.899$

Space Intersection

$b3$	Photo II′	$b3$	Photo II
+80.77	+78.16	+81.28	+77.38
+80.767 − 0.660	+ 0.682 +78.157	−81.278 − 0.457	− 0.480 +77.378
+80.107	+81.445 +81.433	−81.735	+80.797 +80.770
.98372 45°00.8′	44°31.8′ + .99954	1.0119 134°59.9′	314°39.6′ − .99994
$Y-15002=+.99954(X-14997)$ $Y-34995=-.99994(X-15003)$			
$X=24999$ ft.		$Y=25000$ ft.	
+ 1.415 114.230	148.607 1.300945	+ 2.100 114.910	147.951 1.287538
14142.1	20201 18398	14135.8	20000 18200
	1803		1800
$B3$ $X=24999$		$Y=25000$	$Z=1801$

and that

$$\cot m_P = (f \sec t - y_p \sin t)/\sqrt{x_p^2 + y_p^2 \cos^2 t}$$
$$\cot m_P' = (f \sec t' - y_p' \sin t')/\sqrt{x_p'^2 + y_p'^2 \cos^2 t'} \qquad [23]$$

These give the two vertical angles needed to find the elevation of P. Following the same figure with the same changes,

$$V_HP = \sqrt{(X_P - X_L)^2 + (Y_P - Y_L)^2}$$
$$V_H'P = \sqrt{(X_P - X_L')^2 + (Y_P - Y_L')^2} \qquad [24]$$

Then

$$Z_L - Z_P = V_HP \cot m_P \qquad [25]$$
$$Z_L' - Z_P = V_H'P \cot m_P'$$

Subtracting these from the known values Z_L and Z_L', respectively, gives two values of Z_P. Their equality furnishes a complete check on the computations.

The survey coordinates can therefore be found for any point whose image appears on each of any two overlapping aerial photographs whose exposure stations and space orientations have been calculated. With a calculating machine, this space intersection computation requires about twenty minutes for each point.

ANALYTICAL BRIDGING OR PHOTOGRAMMETRIC EXTENSION OF SURVEYS WITHOUT GROUND CONTROL

The ultimate problem in the analytical geometry of aerial photogrammetry consists of extending the exposure station and orientation computations from one photograph to a succeeding one without additional ground control data. This is theoretically possible. If an initial photograph of a flight strip has its exposure station and its space orientation determined from three ground control points appearing in this photograph, then, even though the second photograph which overlaps the first may not contain any of the original control points, its exposure station and space orientation can theoretically be determined analytically if the photographic rectangular coordinates are measured for the images of *five* common points on the two photographs. The theory underlying this five-point procedure is not difficult, but the equations encountered are extremely difficult to solve.

However, a slight variation of this five-point procedure produces a much easier process which has been called the "four-point method." For example, suppose in Figure 6 that the two initial photographs contain three ground con-

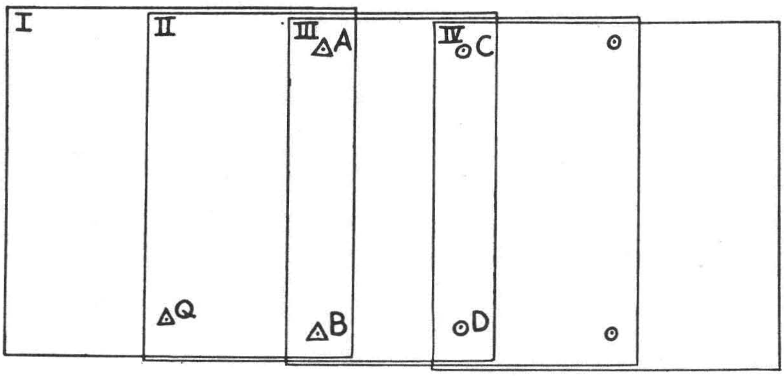

FIG. 6

trol points A, B, and Q, disposed somewhat as shown, with A and B approximately opposite the center of photograph II. It is always possible to have the three indispensable ground control points control the first two photographs as they do in this figure. There are to be used no additional control points. Suppose C and D are merely two points chosen approximately opposite the center of photograph III. Now, the four-point method consists of calculating the exposure station and the complete space orientation for photograph III, from the known survey coordinates of A and B, together with the measurements of the photographic coordinates of the images of C and D on photograph II and of the images of A, B, C, and D on photograph III. Incidentally this calculation yields as a by-product the complete space coordinates of C and D on the survey datum.

Then when the extension to photograph IV is undertaken, points *C* and *D* whose survey coordinates will then be fully known, become regular control points. That is, they become the "*A*" and "*B*" points for computing photograph IV. Two new points are then chosen approximately opposite the center of photograph IV for the "*C*" and "*D*" points and the photographic plane coordinates of their images are measured on photographs III and IV. Then these are the complete data for carrying out the four-point computation for photograph IV, and the position of its exposure station and its space orientation can be completely computed. Again positions become available for the new *C* and *D* points and the computations can be extended in a similar manner throughout the flight strip.

All operators of stereoscopic instruments such as the multiplex projector, acquire confidence in stereoscopic orientation and stereoscopic bridging, because they actually *see* the orientations perfected by the elimination of so-called *y*-parallaxes in the field of overlap of successive photographs. Similarly, the ultimate test of the precision of the analytical determination of the exposure stations and orientation data for two overlapping photographs is the subsequent determination of positions of ground points by intersections. The check in these intersections, which consists of the identity of the two values of each elevation found by equations [25], as many readers have doubtless already discovered, is directly analogous to the instrumental elimination of the *y*-parallaxes. This check establishes the same confidence in analytical computations as that which stereoscopic operators feel in the instrumental orientation of aerial photographs and in stereoscopic bridging.

Now the four-point method to be explained at this time, as the introductory discussion has already suggested, actually removes *y*-parallaxes analytically at the *A*, *B*, *C* and *D* points, or at the four corners of the field of overlap of the two photographs. This feature adds considerable interest to this analytical process.

Furthermore if difficulty is encountered in obtaining satisfactory checks in intersection computations using two photographs which are fixed in space independently by individual groups of three control points, the difficulty is attributable to inconsistency between the geodetic determinations of the control data, or to errors in image identification, or to inaccuracies in the photographic measurements. The first of these difficulties does not enter into computations of intersections using two photographs where one has been computed from the preceding one by the four-point method. Although the four-point calculations will always give satisfactory results in subsequent intersection computations, there is considerable danger in extending bridging calculations through too large a number of photographs without additional ground control data.

These remarks of the last three paragraphs might well be placed at the end of this discussion, but they are inserted here with the hope of adding interest to the explanation to follow of the process of analytical bridging.

In this discussion, the method will be explained for photograph III shown in Figure 6. The explanation given above of the use of the *A*, *B*, *C*, and *D* points shows that the four-point method for any other photograph is identical with that for photograph III, except that the *A* and *B* points for any subsequent photograph will have their survey coordinates determined photogrammetrically from the preceding bridging calculations, whereas for photograph III the *A* and *B* points are control points. The four-point method itself will not be affected by this difference.

It is assumed that the survey coordinates of the exposure station for photograph II, called X_{II}, Y_{II}, Z_{II}, have already been computed, together with the space orientation elements, the tilt called t_{II}, the swing called s_{II}, and the azimuth of the principal plane called α_{II}. The survey coordinates for the A and B points are known, as they will always be in the bridging method, as explained before. These coordinates will be called (X_A, Y_A, Z_A) and (X_B, Y_B, Z_B). The photographic rectangular coordinates of the images a, b, c, and d on photograph III which is being computed, are called (x_a, y_a), (x_b, y_b), (x_c, y_c), and (x_d, y_d); and those of c and d on photograph II just preceding are called (u_c, v_c) and (u_d, v_d).

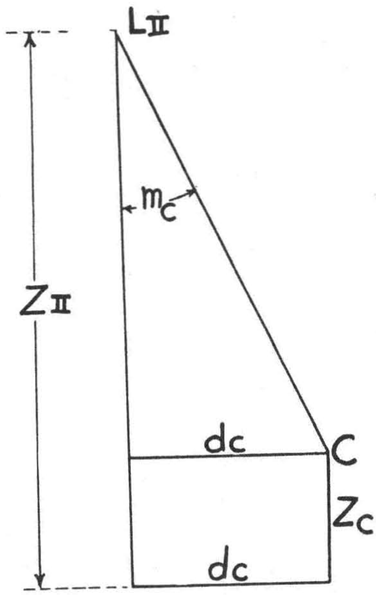

FIG. 7

The space intersection method already presented shows how the survey azimuths from the exposure station L_{II} of photograph II to the ground points C and D can be computed. These azimuths are called α_C and α_D respectively. The formulas used for obtaining them are [16], [17], and [20]. The space intersection method likewise shows how to find the vertical angles at L_{II} measured from a vertical line down from L_{II} outward to the lines $L_{II}C$ and $L_{II}D$ in space. These angles are called m_C and m_D. The formula for obtaining them is [23].

Formulas [16], [17], [20], and [23] may be restated for this case as follows. For transforming the coordinate system on photograph II to the nadir point for the origin and to the principal line for the y-axis,

$$\theta = 180° - s_{II} \qquad [26]$$

$$\begin{aligned} \text{new } u_c &= u_c \cos\theta + v_c \sin\theta \\ \text{new } v_c &= -u_c \sin\theta + v_c \cos\theta + ov \\ \text{new } u_d &= u_d \cos\theta + v_d \sin\theta \\ \text{new } v_d &= -u_d \sin\theta + v_d \cos\theta + ov \end{aligned} \qquad [27]$$

(New coordinates are henceforth designated by the same symbols as before without ambiguity)

$$\begin{aligned} \phi_C &= \tan^{-1}(u_c/v_c \cos t_{II}) \\ \phi_D &= \tan^{-1}(u_d/v_d \cos t_{II}) \end{aligned} \qquad [28]$$

$$\begin{aligned} \alpha_C &= \alpha_{II} + \phi_C \\ \alpha_D &= \alpha_{II} + \phi_D \end{aligned} \qquad [29]$$

$$\begin{aligned} \tan m_C &= \sqrt{u_c^2 + v_c^2 \cos^2 t_{II}}/(f \sec t_{II} - v_c \sin t_{II}) \\ \tan m_D &= \sqrt{u_d^2 + v_d^2 \cos^2 t_{II}}/(f \sec t_{II} - v_d \sin t_{II}) \end{aligned} \qquad [30]$$

It might be noted that, if the elevations of the ground points C and D were known, it would be possible to find the horizontal survey coordinates of these two ground points. For, calling d_C the horizontal projection V_HC of $L_{II}C$ and

d_D that of $L_{II}D$, it is obvious that

$$\begin{aligned} d_C &= (Z_{II} - Z_C) \tan m_C \\ d_D &= (Z_{II} - Z_D) \tan m_D \end{aligned} \qquad [31]$$

and then from the usual plane surveying method, that the differences in X and Y between L_{II} and the ground points C and D, called ΔX_C, ΔY_C and ΔX_D, ΔY_D, respectively, are

$$\begin{aligned} \Delta X_C &= d_C \sin \alpha_C & \Delta X_D &= d_D \sin \alpha_D \\ \Delta Y_C &= d_C \cos \alpha_C & \Delta Y_D &= d_D \cos \alpha_D \end{aligned} \qquad [32]$$

and

$$\begin{aligned} X_C &= X_{II} + \Delta X_C & X_D &= X_{II} + \Delta X_D \\ Y_C &= Y_{II} + \Delta Y_C & Y_D &= Y_{II} + \Delta Y_D \end{aligned} \qquad [33]$$

But of course from the very nature of the problem, the ground elevations of C and D are not known.

However at this point in the computation it becomes necessary to find roughly approximate values for d_C and d_D. The best way to find them is to make a very rough estimate of the elevations of C and D. In practice it is in most cases satisfactory to take an average of the two known elevations of A and B, and use this as the approximate value for each of the elevations of C and D. It might be preferred to place photographs II and III under a viewing stereoscope and to estimate the elevations of C and D from the known elevations of A and B; the rough approximation desired does not warrant taking any further trouble in estimating these elevations. With parentheses used throughout this discussion to designate approximate values, the assumed approximate elevations of C and D are (Z_C) and (Z_D), approximate values of d_C and d_D are

$$\begin{aligned} (d_C) &= [Z_{II} - (Z_C)] \tan m_C \\ (d_D) &= [Z_{II} - (Z_D)] \tan m_D \end{aligned} \qquad [34]$$

and the approximate coördinates (X_C), (Y_C), (X_D), and (Y_D) can be found from formulas [32] and [33]. Although the Z's, the d's, and the coordinates of the C and D points are all approximations, it is to be noted that they are all rigidly consistent between themselves and with the known fixed values of the α's and the m's.

If the solution of the four-point method gave as part of the results the corrections necessary to be applied to (Z_C) and (Z_D) to obtain the true elevations, then the complete ground positions of C and D would be determined. However it simplifies the method enormously to determine instead of these, the corrections necessary to be applied to (d_C) and (d_D) to obtain their true values. In this way the correct ground survey coordinates of C and D will likewise be determined. Using Δ's to signify the corrections to all approximate values to obtain the correct ones,

$$\begin{aligned} d_C &= (d_C) + \Delta d_C \\ d_D &= (d_D) + \Delta d_D \end{aligned} \qquad [35]$$

with Δd_C and Δd_D to be determined by the solution of the problem, and

$$\left.\begin{aligned} Z_C &= (Z_C) + \Delta Z_C \quad \text{where} \quad \Delta Z_C = -\Delta d_C \cot m_C \\ Z_D &= (Z_D) + \Delta Z_D \quad \text{where} \quad \Delta Z_D = -\Delta d_D \cot m_D \end{aligned}\right\} \qquad [36]$$

$$\left.\begin{aligned} X_C &= (X_C) + \Delta X_C \quad \text{where} \quad \Delta X_C = + \Delta d_C \sin \alpha_C \\ Y_C &= (Y_C) + \Delta Y_C \quad \text{where} \quad \Delta Y_C = + \Delta d_C \cos \alpha_C \\ \\ X_D &= (X_D) + \Delta X_D \quad \text{where} \quad \Delta X_D = + \Delta d_D \sin \alpha_D \\ Y_D &= (Y_D) + \Delta Y_D \quad \text{where} \quad \Delta Y_D = + \Delta d_D \cos \alpha_D \end{aligned}\right\} \qquad [37]$$

It is likewise necessary at the beginning of this four-point problem to find approximate values for the unknown survey coordinates of the exposure station of photograph III, as in the space resection method already explained. These coordinates of the exposure station are called (X, Y, Z) and their approximate values are designated by $[(X), (Y), (Z)]$. The (X) and (Y) can be found by a rough graphical solution of the three-point problem, using the A and B points and the approximate position of either C or D. The (Z) can be found roughly from the ground length of AB and the photographic length of ab, using the

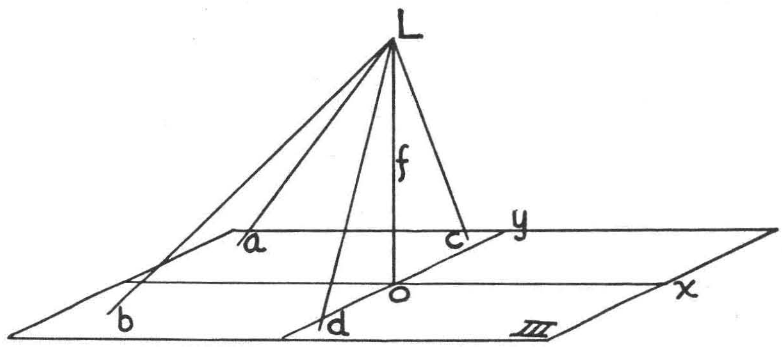

FIG. 8

ordinary scale relationship for a truly vertical photograph, assuming verticality for this approximation.

Now, to summarize to this point, we have the following symbols for coordinates of which we have actual numerical values:

X_A, Y_A, Z_A, space coordinates of A on the survey datum,
X_B, Y_B, Z_B, space coordinates of B on the survey datum,
(X_C), (Y_C), (Z_C), approximate survey coordinates of C, consistent between themselves and the angles α_C and m_C from photograph II,
(X_D), (Y_D), (Z_D), approximate survey coordinates of D, consistent between themselves and the angles α_D and m_D from photograph II,
(x_a, y_a), (x_b, y_b), (x_c, y_c), (x_d, y_d), rectangular coordinates for the four images a, b, c, and d on photograph III;

and the following symbols for unknown coordinates:

$$\left.\begin{aligned} &X_{II}+d_C \sin \alpha_C \\ &Y_{II}+d_C \cos \alpha_C \\ &Z_{II}-d_C \cot m_C \end{aligned}\right\},$$ survey coordinates of C, expressed in terms of a single unknown quantity d_C, which is to be determined,

$$\left.\begin{aligned} &X_{II}+d_D \sin \alpha_D \\ &Y_{II}+d_D \cos \alpha_D \\ &Z_{II}-d_D \cot m_D \end{aligned}\right\},$$ survey coordinates of D, expressed in terms of a single unknown quantity d_D, which is to be determined,

X, Y, Z, unknown survey coordinates of the exposure station of photograph III, to be determined.

As in the space resection problem, the angles subtended at the emergent node of the camera lens by pairs of the images a, b, c, and d on photograph III, are given by

$$\begin{aligned} \cos aLb &= (x_a x_b + y_a y_b + f^2)/(La)(Lb) \\ \cos aLc &= (x_a x_c + y_a y_c + f^2)/(La)(Lc) \\ \cos aLd &= (x_a x_d + y_a y_d + f^2)/(La)(Ld) \\ \cos bLc &= (x_b x_c + y_b y_c + f^2)/(Lb)(Lc) \\ \cos bLd &= (x_b x_d + y_b y_d + f^2)/(Lb)(Ld) \end{aligned} \quad [38]$$

where $La=\sqrt{x_a^2+y_a^2+f_2}$, $Lb=\sqrt{x_b^2+y_b^2+f^2}$, etc. There are six of these angles, but only the above five are used in this problem.

Inasmuch as these angles are identical with the angles *ALB*, *ALC*, *ALD*, *BLC*, and *BLD*, equation [5] in Space Resection gives rise to six condition equations, five of which are stated for this problem as follows:

$$\begin{aligned} &(X - X_A)(X - X_B) + (Y - Y_A)(Y - Y_B) + (Z - Z_A)(Z - Z_B) \\ &\qquad - (LA)(LB) \cos aLb = 0 \\ &[X - X_A][X \overset{*}{-} (X_{II} + d_C \sin \alpha_C)] + [Y - Y_A][Y - (Y_{II} + d_C \cos \alpha_C)] \\ &\qquad + [Z - Z_A][Z - (Z_{II} - d_C \cot m_C)] - (LA)(LC) \cos aLc = 0 \\ &[X - X_A][X - (X_{II} + d_D \sin \alpha_D)] + [Y - Y_A][Y - (Y_{II} + d_D \cos \alpha_D)] \\ &\qquad + [Z - Z_A][Z - (Z_{II} - d_D \cot m_D)] - (LA)(LD) \cos aLd = 0 \\ &[X - X_B][X - (X_{II} + d_C \sin \alpha_C)] + [Y - Y_B][Y - (Y_{II} + d_C \cos \alpha_C)] \\ &\qquad + [Z - Z_B][Z - (Z_{II} - d_C \cot m_C)] - (LB)(LC) \cos bLc = 0 \\ &[X - X_B][X - (X_{II} + d_D \sin \alpha_D)] + [Y - Y_B][Y - (Y_{II} + d_D \cos \alpha_D)] \\ &\qquad + [Z - Z_B][Z - (Z_{II} - d_D \cot m_D)] - (LB)(LD) \cos bLd = 0, \end{aligned} \quad [39]$$

in which

$$\begin{aligned} LA &= \sqrt{(X-X_A)^2+(Y-Y_A)^2+(Z-Z_A)^2} \\ LB &= \sqrt{(X-X_B)^2+(Y-Y_B)^2+(Z-Z_B)^2} \\ LC &= \sqrt{[X-(X_{II}+d_C \sin \alpha_C)]^2+[Y-(Y_{II}+d_C \cos \alpha_C)]^2+[Z-(Z_{II}-d_C \cot m_C)]^2} \\ LD &= \sqrt{[X-(X_{II}+d_D \sin \alpha_D)]^2+[Y-(Y_{II}+d_D \cos \alpha_D)]^2+[Z-(Z_{II}-d_D \cot m_D)]^2} \end{aligned} \quad [40]$$

It is to be noted the five equations [39] contain but five unknown quantities, namely, X, Y, Z, d_C, and d_D. Theoretically these equations suffice to determine these quantities, giving the space position of the exposure station and indirectly the survey coordinates of C and D. The solution of these equations in this form is of course impossible.

The actual method of solving them is analogous to that used in space resection. First, approximate values are found for LA, LB, LC, and LD, by substituting in equations [40] the approximate values already found for X, Y, Z, d_C, and d_D. Then all of the approximate values are substituted in [39] to find the amounts by which these equations fail to be satisfied. Let us call the numerical values of the left members of equations [39], when the approximate values are substituted therein, v_1, v_2, v_3, v_4, and v_5, respectively.

Now the amounts by which the left-hand members of equations [39] change with small changes ΔX, ΔY, ΔZ, Δd_C, and Δd_D, in the five unknowns, can be found by partial differentiation of the left-hand members of [39]. If these total increments are equated to $-v_1$, $-v_2$, $-v_3$, $-v_4$, and $-v_5$, respectively, we have five linear equations in ΔX, ΔY, ΔZ, Δd_C, and Δd_D, the necessary corrections to the assumed approximate values to cause equations [39] to be satisfied. These equations are

$$
\begin{aligned}
&U' \,\Delta X + V' \,\Delta Y + W' \,\Delta Z && + v_1 = 0 \\
&U'' \,\Delta X + V'' \,\Delta Y + W'' \,\Delta Z + M'' \,\Delta d_C && + v_2 = 0 \\
&U''' \Delta X + V''' \Delta Y + W''' \Delta Z && + N''' \Delta d_D + v_3 = 0 \\
&U^{\mathrm{IV}} \,\Delta X + V^{\mathrm{IV}} \,\Delta Y + W^{\mathrm{IV}} \,\Delta Z + M^{\mathrm{IV}} \,\Delta d_C && + v_4 = 0 \\
&U^{\mathrm{V}} \,\Delta X + V^{\mathrm{V}} \,\Delta Y + W^{\mathrm{V}} \,\Delta Z && + N^{\mathrm{V}} \,\Delta d_D + v_5 = 0
\end{aligned} \qquad [41]
$$

in which

$$
\begin{aligned}
U' &= [1 - (LA/LB) \cos aLb](X - X_B) + [1 - (LB/LA) \cos aLb](X - X_A) \\
V' &= [1 - (LA/LB) \cos aLb](Y - Y_B) + [1 - (LB/LA) \cos aLb](Y - Y_A) \\
W' &= [1 - (LA/LB) \cos aLb](Z - Z_B) + [1 - (LB/LA) \cos aLb](Z - Z_A) \\
U'' &= [1 - (LA/LC) \cos aLc](X - X_C) + [1 - (LC/LA) \cos aLc](X - X_A) \\
V'' &= [1 - (LA/LC) \cos aLc](Y - Y_C) + [1 - (LC/LA) \cos aLc](Y - Y_A) \\
W'' &= [1 - (LA/LC) \cos aLc](Z - Z_C) + [1 - (LC/LA) \cos aLc](Z - Z_A) \\
M'' &= -(X - X_A) \sin \alpha_C - (Y - Y_A) \cos \alpha_C + (Z - Z_A) \cot m_C \\
&\quad + [(LA/LC) \cos aLc][(X - X_C) \sin \alpha_C + (Y - Y_C) \cos \alpha_C \\
&\quad \overset{*}{-} (Z - Z_C) \cot m_C] \\
U''' &= [1 - (LA/LD) \cos aLd](X - X_D) + [1 - (LD/LA) \cos aLd](X - X_A) \\
V''' &= [1 - (LA/LD) \cos aLd](Y - Y_D) + [1 - (LD/LA) \cos aLd](Y - Y_A) \\
W''' &= [1 - (LA/LD) \cos aLd](Z - Z_D) + [1 - (LD/LA) \cos aLd](Z - Z_A) \\
N''' &= -(X - X_A) \sin \alpha_D - (Y - Y_A) \cos \alpha_D + (Z - Z_A) \cot m_D \\
&\quad + [(LA/LD) \cos aLd][(X - X_D) \sin \alpha_D + (Y - Y_D) \cos \alpha_D \\
&\quad - (Z - Z_D) \cot m_D] \\
U^{\mathrm{IV}} &= [1 - (LB/LC) \cos bLc](X - X_C) + [1 - (LC/LB) \cos bLc](X - X_B) \\
V^{\mathrm{IV}} &= [1 - (LB/LC) \cos bLc](Y - Y_C) + [1 - (LC/LB) \cos bLc](Y - Y_B) \\
W^{\mathrm{IV}} &= [1 - (LB/LC) \cos bLc](Z - Z_C) + [1 - (LC/LB) \cos bLc](Z - Z_B) \\
M^{\mathrm{IV}} &= \overset{*}{-} (X - X_B) \sin \alpha_C - (Y - Y_B) \cos \alpha_C + (Z - Z_B) \cot m_C \\
&\quad + [(LB/LC) \cos bLc][(X - X_C) \sin \alpha_C + (Y - Y_C) \cos \alpha_C \\
&\quad - (Z - Z_C) \cot m_C] \\
U^{\mathrm{V}} &= [1 - (LB/LD) \cos bLd](X - X_D) + [1 - (LD/LB) \cos bLd](X - X_B) \\
V^{\mathrm{V}} &= [1 - (LB/LD) \cos bLd](Y - Y_D) + [1 - (LD/LB) \cos bLd](Y - Y_B) \\
W^{\mathrm{V}} &= [1 - (LB/LD) \cos bLd](Z - Z_D) + [1 - (LD/LB) \cos bLd](Z - Z_B) \\
N^{\mathrm{V}} &= \overset{*}{-} (X - X_B) \sin \alpha_D - (Y - Y_B) \cos \alpha_D + (Z - Z_B) \cot m_D \\
&\quad + [(LB/LD) \cos bLd][(X - X_D) \sin \alpha_D + (Y - Y_D) \cos \alpha_D \\
&\quad - (Z - Z_D) \cot m_D]
\end{aligned} \qquad [42]
$$

The computation of the coefficients in [41] is made by substituting in [42] the approximate values of all of the quantities, the work following a cyclical routine and being much easier than it first appears. Equations [41] are very easily solved simultaneously for ΔX, ΔY, ΔZ, Δd_C, and Δd_D, by simple algebraic elimination. Then formulas [36] and [37] give the desired corrected survey coordinates of the exposure station and of the C and D points.

A substitution of the corrected values in equations [39] may show that the residuals still remaining are not negligible. In this case a second solution may be necessary.

The form for the entire computation by the four-point method covers but one sheet of paper. The work with a calculating machine resolves itself into mere routine operations, and the entire procedure requires only about two hours.

After the four-point computation of the coordinates of an exposure station, the elements of space orientation of the photograph, namely, the tilt, the swing, and the azimuth of the principal plane, are calculated by exactly the same procedure as that previously explained. There are, however, additional checks with four points now available instead of three as before.

FOUR POINT COMPUTATION

PRELIMINARY COMPUTATION FROM PRECEDING PHOTOGRAPH

Photograph No.			Points:		$C=$		$D=$	
c	u_c	v_c	$u_c \cos\theta$	$v_c \sin\theta$	$-u_c \sin\theta$	$v_c \cos\theta$ [1]	New u_c	New v_c [2]
d	u_d	v_d	$u_d \cos\theta$	$v_d \sin\theta$	$-u_d \sin\theta$	$v_d \cos\theta$	New u_d	New v_d
$v_c \cos t$	$\tan\phi_C$ [3]	ϕ_C	α_C [4]	$v_c \sin t$	[5]	[7]		
$v_d \cos t$	$\tan\phi_D$	ϕ_D	α_D	$v_d \sin t$	[6]	[8]		
$\tan m_C$ [9]	(Z_C)[10]	$Z_L-(Z_C)$	(d_C) [11]	$(d_C)\sin\alpha_C$	$(d_C)\cos\alpha_C$	(X_C) [12]	(Y_C) [13]	(Z_C) [14]
$\tan m_D$	(Z_D)	$Z_L-(Z_D)$	(d_D)	$(d_D)\sin\alpha_D$	$(d_D)\cos\alpha_D$	(X_D)	(Y_D)	(Z_D)
$\sin\alpha_C$	$\cos\alpha_C$	$\cot m_C$ [15]	To be used for					
$\sin\alpha_D$	$\cos\alpha_D$	$\cot m_D$	Photograph No.					

[1] See formulas [26] and [27].
[2] By formulas [27].
[3] By formulas [28].
[4] By formulas [29].
[5] f sec $t-v_c \sin t$.
[6] f sec $t-v_d \sin t$.
[7] $\sqrt{u_c^2+v_c^2 \cos^2 t}$.
[8] $\sqrt{u_d^2+v_d^2 \cos^2 t}$.
[9] By formulas [30].
[10] Assumed values.
[11] By formulas [31].
[12] See formulas [32] and [33].
[13] See formulas [32] and [33].
[14] Same assume values as before.
[15] By formulas [30].

COMPUTATION OF LENGTHS, AREAS, AND HORIZONTAL POSITIONS BY SIMPLE PARALLAX METHODS

The analytical method to be explained at this time provides a simple means for calculating lengths between any desired ground points and for calculating farm areas. An absolute minimum of control is used, consisting of only the length of one line on the overlap of the first pair of photographs, without even requiring the elevations of the extremities of this line. From this single line the computation of lengths throughout a strip of photographs follows very easily, without any additional control data, and without requiring or even determining any elevations of any ground points whatever, yet giving results for both lengths and areas which are practically corrected for topographic relief. This is all accomplished by using the photographs in pairs instead of singly.

The method utilizes the well known parallax formulas which are first derived for the ideal case of two overlapping vertical phtographs exposed from exactly equal altitudes. In Figure 9, I and II represent two vertical overlapping photographs; H is the common elevation of the exposure stations above the ground datum plane; P is a ground point with elevation h, having its images at p and p' respectively, the point P being taken, for convenience in drawing the figure, in the vertical plane containing the exposure stations; x and x' are the two abscissas of the images measured with respect to the rectangular axes in the photographs having their x-axes in the vertical plane of the line of flight; X is the ground abscissa of the point P, measured with respect to ground rectangular

FOUR POINT METHOD

Photograph No.								
				cos aLb [17]	L	(X)	(Y)	(Z)
				cos aLc	A	X_A	Y_A	Z_A
a	x_a	y_a	La [16]	cos aLd	B	X_B	Y_B	Z_B
b	x_b	y_b	Lb	cos bLc	C	(X_C)	(Y_C)	(Z_C) [18]
c	x_c	y_c	Lc	cos bLd	D	(X_D)	(Y_D)	(Z_D)
d	x_d	y_d	Ld					

$(X)-X_A$	$(Y)-Y_A$	$(Z)-Z_A$	(LA) [19]	[20]	[22]	[24]		
$(X)-X_B$	$(Y)-Y_B$	$(Z)-Z_B$	(LB)	[21]			[26]	[28]
$(X)-(X_C)$	$(Y)-(Y_C)$	$(Z)-(Z_C)$	(LC)		[23]		[27]	
$(X)-(X_D)$	$(Y)-(Y_D)$	$(Z)-(Z_D)$	(LD)			[25]		[29]

1st 3 terms [30]	1st 3 terms	1st 3 terms	1st 3 terms	1st 3 terms
4th term [30]	4th term	4th term	4th term	4th term
v_1	v_2	v_3	v_4	v_5

1st terms of U', V', W' [31]	sin α_C		cos α_C		cot m_C
2nd terms of U', V', W'	sin α_D		cos α_D		cot m_D
1st terms of U'', V'', W''	U''	V''	W''	M''	v_2 [32]
2nd terms of U'', V'', W''	U^{IV}	V^{IV}	W^{IV}	M^{IV}	v_4
1st terms of U''', V''', W'''	U'''	V'''	W'''	N'''	v_3
2nd terms of U''', V''', W'''	U^{V}	V^{V}	W^{V}	N^{V}	v_5
1st terms of U^{IV}, V^{IV}, W^{IV}	Elimination of Δd_C				
2nd terms of U^{IV}, V^{IV}, W^{IV}					
1st terms of U^{V}, V^{V}, W^{V}	Elimination of Δd_D				
2nd terms of U^{V}, V^{V}, W^{V}					

1st three terms	2nd three terms [31]	M''	U'	V'	W'	v_1 [32]	$\Delta X=$		ft.
			Solution of equations [41]				$\Delta Y=$		
							$\Delta Z=$		
							$\Delta d_C=$		
1st three terms	2nd three terms	N'''					$\Delta d_D=$		
							(X_C)	(Y_C)	(Z_C) [34]
							ΔX_C [33]	ΔY_C [33]	ΔZ_C
1st three terms	2nd three terms	M^{IV}					X_C	Y_C	Z_C
1st three terms	2nd three terms	N^{V}					(X_D)	(Y_D)	(Z_D)
			(X)	(Y)	(Z)		ΔX_D [33]	ΔY_D [33]	ΔZ_D [34]
			ΔX	ΔY	ΔZ		X_D	Y_D	Z_D
			X	Y	Z				

[16] By formulas [1].
[17] By formulas [38].
[18] From preliminary computation above.
[19] By formulas [40].
[20] (LB/LA) cos aLb.
[21] (LA/LB) cos aLb.
[22] (LC/LA) cos aLc.
[23] (LA/LC) cos aLc.
[24] (LD/LA) cos aLd.
[25] (LA/LD) cos aLd.
[26] (LC/LB) cos bLc.
[27] (LB/LC) cos bLc.
[28] (LD/LB) cos bLd.
[29] (LB/LD) cos bLd.
[30] From equations [39].
[31] See formulas [42].
[32] Equations [41].
[33] See formulas [37].
[34] See formulas [36].

axes in the datum plane with the X-axis in the direction of the line of flight and the origin at the ground plumb point for the left photograph; and B is the air-base or the distance LL' between exposure stations.

From similar triangles we have $X/x=(H-h)/f$. If the triangle $L'o'p'$ is placed adjacent to the triangle Lop, with $L'o'$ coinciding with Lo, the side pp' is then equal to the algebraic difference between the abscissas x and x'. This difference is called the parallax for the point P and is denoted by p. Then by similar triangles it is seen that $B/p=(H-h)/f$. If the ground point P is not situated in the vertical plane containing the exposure stations and therefore has both X and Y ground coordinates, the first equation above of course still holds; and furthermore, by similar reasoning, $Y/y=(H-h)/f$, where y is the ordinate of the image p on photograph I. Hence the well known parallax formulas follow at once:

$$X = (B/p)x \qquad Y = (B/p)y \qquad H - h = (B/p)f \tag{43}$$

The method to be explained also utilizes the principle that if (x, y) and (x', y') are respectively the rectangular coordinates in photographs I and II of the images p and p' of the ground point P outside the vertical plane containing the line of flight, if the x-axes of the photographs are in the vertical plane containing the line of flight, then the ordinates y and y' are exactly equal. This can be proved very easily by similar triangles.

Photograph I contains the images a and b and photograph II contains the images a' and b', of the extremities A and B of a ground line whose horizontal

FOUR POINT COMPUTATION

PHOTOGRAPH III′

Data used from preceding computation of Photograph II′:

Exposure Station	Photo II′ Orientation	Constants
$X=14997$ ft.	$t=$ 0°59.7′	$\sin t=$ .01737
$Y=15002$	$s=$180°29.0′	$\cos t=$ .99985
$Z=20201$	$\alpha_{VO}=$ 0°29.0′	f $\sec t=$150.022 mm.
		$ov=$ 2.606 mm.

Survey Coordinates of A and B Points:

	X	Y	Z
$A=C1$	15000 ft.	5000 ft.	800 ft.
$B=C5$	15000	25000	1000

PRELIMINARY COMPUTATION FROM PRECEDING PHOTOGRAPH

Photo II′			$C=E1$				$D=E5$		
$e1$	+78.85	−81.48	+78.847	+0.688	+0.665	−81.477	+79.535	−78.206	
$e5$	+80.77	+78.16	+80.767	−0.660	+0.682	+78.157	+80.107	+81.445	
−78.194	1.0171	134°30.8′	134°59.8′	−1.358	151.380	111.535			
+81.433	.98372	44 31.8	45 00.8	+1.415	148.607	114.230			
.73679	900	19301	14220.8	+10056	−10055	25054	4952	900	
.76867	900	19301	14836.1	+10493	+10488	25491	25495	900	
+.70715	−.70707	1.357242	To be used for Photograph III′						
+.70727	+.70694	1.300945							

Four Point Method

	Photograph III′							
				.645877	*L*	26000	14200	19550
				.907815	*C*1	15000	5000	800
*c*1	−85.22	−74.23	187.810	.503747	*C*5	15000	25000	1000
*c*5	+88.78	+86.82	194.730	.513208	*E*1	25054	4952	900
*e*1	− 6.39	−73.35	167.096	.906355	*E*5	25491	25495	900
*e*5	− 6.49	+88.46	174.262					

*C*1	+11000	+ 9200	+18750	23605.14	.660	.801	.465		
*C*5	+11000	−10800	+18550	24119.34	.632			.443	.820
*E*1	+ 946	+ 9248	+18650	20838.50		1.028		.594	
*E*5	+ 509	−11295	+18650	21809.60			.545		1.002

+369452500	+445175100	+251372500	+256485100	+473542500
−367723868	−446550304	−259338356	−257943918	−476772783
+ 1728632	− 1375204	− 7965856	− 1458818	− 3230283

+ 3740	+ 3128	+ 6375	+.70715		−.70707		1.357242	
+ 4048	− 3974	+ 6826	+.70727		+.70694		1.300945	
+ 2189	+ 1831	+ 3731	+2163	+1572	+ 3209	−7882		−1375204
− 26	− 259	− 522	+6511	−2261	+17904	−8761		−1458818
+ 5885	+ 4922	+10031	+6117	− 217	+18517		−7270	−7965856
+ 232	− 5139	+ 8486	+1979	−1921	+ 3302		−7983	−3230283
+ 6127	− 6016	+10332	+2163	+1572	+ 3209	−7882		−1375204
+ 384	+ 3755	+ 7572	+5858	−2034	+16108	−7882		−1312453
+ 1980	− 1944	+ 3339	+6117	− 217	+18517		−7270	−7965856
− 1	+ 23	− 37	+1802	−1749	+ 3007		−7270	−2941771

− 7779	+ 687	
+ 6505	− 6722	− 7882
+25448	−26021	
− 7780	+ 196	
− 6504	− 4352	− 7270
+24393	−13223	
− 7779	+ 397	
− 7636	− 3884	− 8761
+25177	−15036	
− 7780	+ 361	
+ 7635	− 8001	− 7983
+24133	−24311	

+7788	− 846	+13201	+1728632
+3695	−3606	+12899	+ 62751
+4315	+1532	+15510	−5024085
+7610	− 827	+12899	+1689086
+3695	−3606	+12899	+ 62751
+3589	+1274	+12899	−4178315
+3915	+2779		+1626335
+ 106	−4880		+4241066
+ 60	−2779		+2415148
+3975			+4041483

26000	14200	19550
−1017	+847	+523
24983	15047	20073

$\Delta X = -1017$ ft.
$\Delta Y = +847$
$\Delta Z = +523$
$\Delta d_C = -71.59$
$\Delta d_D = -644.05$

25054	4952	900
− 51	+ 51	+ 97
25003	5003	997
25491	25495	900
−456	−455	+838
25035	25040	1738

FOUR POINT METHOD

Photograph III′ (2nd solution)									
				.645877	L	24983	15047	20073	
				.907815	$C1$	15000	5000	800	
				.503747	$C5$	15000	25000	1000	
				.513208	$E1$	25003	5003	997	
				.906355	$E5$	25035	25040	1738	
$C1$	+9983	+10047	+19273	23917.59	.640	.818	.440		
$C5$	+9983	− 9953	+19073	23717.12	.651			.467	.798
$E1$	− 20	+10044	+19076	21558.67		1.007		.565	
$E5$	− 52	− 9993	+18335	20881.45			.577		1.029

+367256427	+468364156	+252451668	+263668956	+448644668
−366377831	−468097947	−251588359	−262408158	−448870370
+ 878596	+ 266209	+ 863309	+ 1260798	− 225702

+ 3594	+ 3617	+ 6938		+.70715		−.70707	1.357242	
+ 3484	− 3474	+ 6656		+.70727		+.70694	1.300945	
+ 1817	+ 1829	+ 3508	+1817	+1759	+ 3374	−7035		+ 266209
0	− 70	− 134	+5312	− 936	+18464	−6858		+1260798
+ 5590	+ 5626	+10793	+5568	+1399	+18549		−6951	+ 863309
− 22	− 4227	+ 7756	+2019	−1721	+ 3321		−7064	− 225702
+ 5321	− 5305	+10166	+1771	+1715	+ 3289	−6858		+ 259511
− 9	+ 4369	+ 8298	+5312	− 936	+18464	−6858		+1260798
+ 2017	− 2011	+ 3853	+5568	+1399	+18549		−6951	+ 863309
+ 2	+ 290	− 532	+1987	−1693	+ 3268		−6951	− 222092

− 7059	− 14		+7078	+ 143	+13594	+ 878596	$\Delta X = +13$ ft.		
+ 7104	− 7151	− 7035	+3541	−2651	+15175	+1001287	$\Delta Y = -13$		
+26158	−26073		+3581	+3092	+15281	+1085401	$\Delta Z = -71$		
							$\Delta d_C = +3.67$		
− 7061	− 21		+7078	+ 143	+13594	+ 878596	$\Delta d_D = -58.45$		
− 7103	− 4076	− 6951	+3172	−2375	+13594	+ 896968			
+25073	−13763		+3186	+2751	+13594	+ 965574	25003	5003	997
							+ 3	− 3	− 5
− 7059	− 8		+3906	+2518		− 18372			
− 7037	− 4013	− 6858	+ 14	+5126		+ 68606	25006	5000	992
+25887	−14628							$C=E1$	
			+ 7	+2518		+ 33701	25035	25040	1738
							− 41	− 41	+ 76
− 7061	− 38		+3899			− 52073			
+ 7036	− 7269	− 7064					24994	24999	1814
+24813	−24545		24983	15047	20073			$D=E5$	
			+ 13	− 13	− 71				
Exposure Station									
			24996	15034	20002				

FOUR POINT METHOD

				.645877	*L*	24996	15034	20002
				.907815	*C*1	15000	5000	800
				.503747	*C*5	15000	25000	1000
				.513208	*E*1	25006	5000	992
				.906355	*E*5	24994	24999	1814
*C*1	+9996	+10034	+19202	23860.39				
*C*5	+9996	− 9966	+19002	23671.02				
*E*1	− 10	+10034	+19010	21495.61				
*E*5	+ 2	− 9965	+18188	20738.96				

+364797576	+465611216	+249277158	+261129216	+444939558
−364791180	−465612538	−249274001	−261132041	−444940851
+ 6396	− 1322	+ 3157	− 2825	− 1293

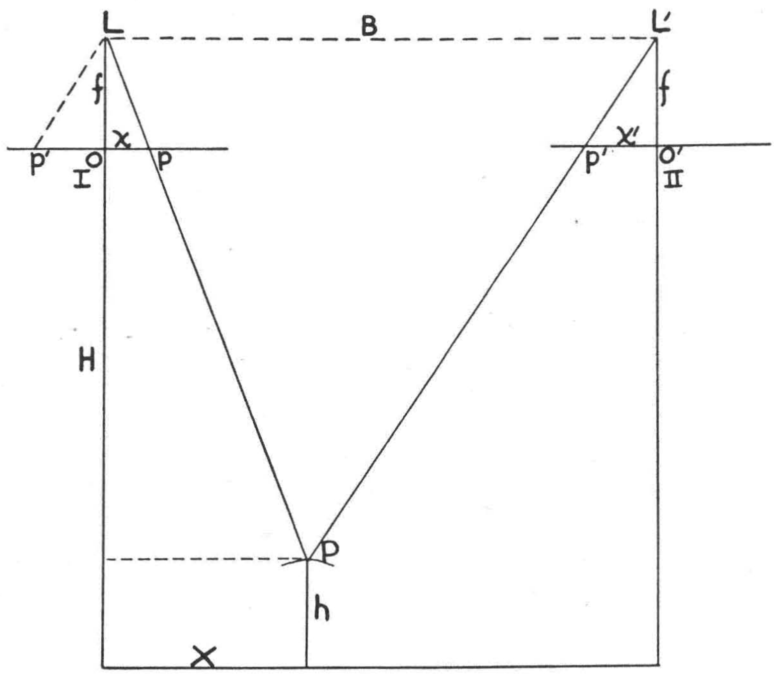

FIG. 9

length has been determined by ground surveying. It has been mentioned that this single distance constitutes all the ground control work needed except for subsequent checks, and that even the elevations of the points *A* and *B* are not required.

Sharply defined pass points are chosen somewhat as shown in Figure 10. That is, *ç* and *d* are chosen about opposite the center of photograph II, near the lateral margins of the photographs, *e* and *f* are chosen opposite the center of photograph III, etc. Photographic rectangular coordinates are then measured for all of these points upon all of the photographs in which they appear, these coordinates being referred to the geometric axes of the photographs as indicated by the camera marks, and the measurements being made with a comparator if possible. On photograph I the coordinates are measured for the images *a*, *b*, *c*, and *d*; on photograph II for *a*, *b*, *c*, *d*, *e*, and *f*; on photograph III for *c*, *d*, *e*, *f*, *g*, and *h*; etc. At the same time the coordinates are measured of the images of the extremities of any line whose length is desired, or of the corners of any property whose area is to be determined, on both of the two photographs in which they appear.

At the beginning it is necessary to determine for each pair of overlapping photographs the direction on each one of the principal point ray toward the principal point of the other. To do this, for the first pair, for example, there is found the angle θ through which the axes of photograph I must be rotated in order that the positive x-axis may contain the principal point of photograph II, and also the angle θ' through which the axes of photograph II must be rotated

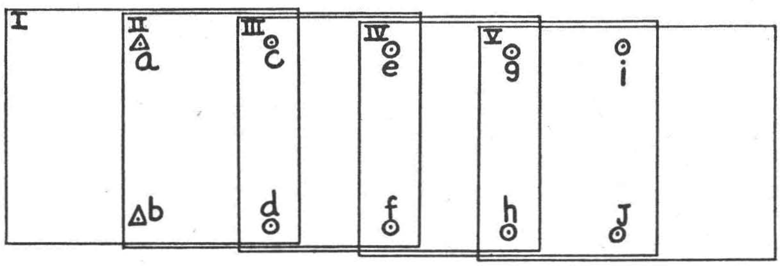

FIG. 10

in order that the negative x-axis may contain the principal point of photograph I.

In each pair of photographs there have been measured the coordinates of four common points, those in the first pair being *a*, *b*, *c*, and *d*, those in the second pair being *c*, *d*, *e*, and *f*, etc. For the four points common to the first pair, let us call the coordinates on photograph I (x_a, y_a), (x_b, y_b), (x_c, y_c), and (x_d, y_d), and on photograph II the same symbols with primes. Since the rotation of the axes through the as yet undetermined angles θ and θ' places the x-axes into the vertical plane of the line of flight, for the ideal case of two vertical photographs from equal altitudes the ordinates of corresponding points become identical, as stated above. That is, we would have

$$\begin{aligned} -x_a \sin\theta + y_a \cos\theta &= -x_a' \sin\theta' + y_a' \cos\theta' \\ -x_b \sin\theta + y_b \cos\theta &= -x_b' \sin\theta' + y_b' \cos\theta' \\ -x_c \sin\theta + y_c \cos\theta &= -x_c' \sin\theta' + y_c' \cos\theta' \\ -x_d \sin\theta + y_d \cos\theta &= -x_d' \sin\theta' + y_d' \cos\theta' \end{aligned} \qquad [44]$$

Theoretically any two of these equations would be sufficient to solve for θ and θ'. However, owing to the fact that the two photographs are probably taken from slightly different altitudes, and to the fact that either or both may be slightly tilted, the four equations are never quite consistent. In practice the best results will be obtained by weighting the first two equations inversely proportionally to the ordinates of *a* and *b* respectively on photograph I, combining the resulting equations; by weighting the second pair of equations inversely proportionally to the ordinates of *c* and *d* respectively on photograph II, combining the resulting equations; and then solving simultaneously for θ and θ' the two combined equations. This is equivalent to solving for θ and θ' the two equations obtained by eliminating cos θ between the first two equations, and by eliminating cos θ' between the last two. Thus the weighting merely simplifies the solution of the equations. Since the ordinates of a and b on photograph I and those of c and d on photograph II will ordinarily change very little with the rotation of the axes, the values of θ and θ' thus determined will so place the axes that the two pairs of residuals in the four inconsistent equations will be approximately proportional to the ordinates of the respective points after the rotation of the axes. This places the axes in the proper positions to pass through the conjugate principal points regardless of any difference in altitude between the exposure stations.

The two trigonometric equations resulting from the combination just described may be solved very easily for θ and θ'. Since θ and θ' are always small angles, both cosines can first be taken as unity and an approximate solution made for sin θ and sin θ'. Then another solution can be made with better values for cos θ and cos θ'. This is continued until values of θ and θ' are found which satisfy the equations. However two solutions usually suffice and require but a few minutes.

Corresponding angles are determined for the pair of photographs II and III in a similar manner, using the four common points c, d, e, and f. This process is carried out for every pair of photographs in the strip.

The next step in the computation is to rotate the axes of the photographs through these calculated angles. That is, for the first pair of photographs we find the coordinates of the four common points a, b, c, and d on photograph I with the axes rotated through the angle θ, and on photograph II with the axes rotated through the angle θ'; for the second pair of photographs II and III, we find the coordinates of the four common points c, d, e, and f on photograph II with the axes rotated through the angle θ and on photograph III with the axes rotated through θ'; and so on throughout the strip. These coordinates after the rotation of the axes are found very easily by means of a calculating machine, using the well known analytical geometry formulas

$$\begin{aligned} \text{New } x &= x \cos\theta + y \sin\theta \\ \text{New } y &= -x \sin\theta + y \cos\theta \end{aligned} \qquad [45]$$

To avoid complication no new symbols are introduced for the coordinates of the various points after the rotation of the axes. There will be no ambiguity in using the same symbols as before to designate now the coordinates after rotation.

The next step for each pair of photographs is to make the ordinates of each of the four common points on one photograph *exactly* equal to the corresponding ordinates on the other photograph. This is brought about by a proportional numerical increase or decrease in both the abscissa and ordinate of each point on one of the photographs of the pair. For instance, in the case of photographs I and II, we find the factor by which y_a' after the rotation of the axes must be multiplied to give y_a, and we multiply both x_a' and y_a' by this factor; the same operation is carried out for the other three of the common points.

Which one of the two photographs of the pair is used to transform the coordinates of the four common points is of no consequence as far as the results are concerned. Usually the coordinates of the four common points on photograph II are transformed to make the ordinates equal to the corresponding ordinates on photograph I; similarly in the second pair of photographs the coordinates of the four common points on photograph III are transformed to make the ordinates equal to the corresponding values on photograph II; etc.

Again no new symbols will be introduced in this step for the corrected values

PARALLAX METHOD

Photo No. a x_a y_a b x_b y_b c x_c y_c d x_d y_d	Photo No. x_a' y_a' x_b' y_b' x_c' y_c' x_d' y_d'	$\sin\theta$ $\cos\theta$ $\sin\theta'$ $\cos\theta'$ Coefficients for Equations [44]
Photo No. $x_a\cos\theta$ $y_a\sin\theta$ $-x_a\sin\theta$ $y_a\cos\theta$ $x_b\cos\theta$ $y_b\sin\theta$ $-x_b\sin\theta$ $y_b\cos\theta$ $x_c\cos\theta$ $y_c\sin\theta$ $-x_c\sin\theta$ $y_c\cos\theta$ $x_d\cos\theta$ $y_d\sin\theta$ $-x_d\sin\theta$ $y_d\cos\theta$		Elimination of $\cos\theta$ from 1st pair Elimination of $\cos\theta'$ from 2nd pair
Coordinates after rotation by formulas [45]	p_a Formulas p_b [46] p_c p_d	 Elimination of $\sin\theta'$ from new pair
Photo No. $x_a'\cos\theta'$ $y_a'\sin\theta'$ $-x_a'\sin\theta'$ $y_a'\cos\theta'$ $x_b'\cos\theta'$ $y_b'\sin\theta'$ $-x_b'\sin\theta'$ $y_b'\cos\theta'$ $x_c'\cos\theta'$ $y_c'\sin\theta'$ $-x_c'\sin\theta'$ $y_c'\cos\theta'$ $x_d'\cos\theta'$ $y_d'\sin\theta'$ $-x_d'\sin\theta'$ $y_d'\cos\theta'$		Elimination of $\sin\theta$ from new pair $\cos\theta$ $\cos\theta'$ $\sin\theta=$ In terms of the cosines of θ and θ' $\sin\theta'=$
Coordinates after rotation by formulas [45]	Coordinates after equalization of ordinates	$\theta=$ $\theta'=$ $\sin\theta=$ $\sin\theta'=$ $\cos\theta=$ $\cos\theta'=$

$10000/p_a=$ $10000/p_b=$ See formula [43]	(X_A) (X_B) ——— Diff.	(Y_A) (Y_B) ——— Diff.	$(D)=$ Square root of sum of squares of differences
$B=[D/(D)]10000$			See formula [47]
$B/p_a=$ $B/p_b=$ See formula [43]	X_A X_B ——— Diff.	Y_A Y_B ——— Diff.	$D=$ Should check given value
$B/p_c=$ $B/p_d=$	X_C X_D ——— Diff.	Y_C Y_D ——— Diff.	$CD=$

of the coordinates on the one photograph in each pair throughout the strip. The original symbols will now indicate the coordinates after rotation of the axes and after the correction for scale difference in the last step.

Next there are found the values of the parallaxes mentioned above in connection with the parallax formulas. In the first pair of photographs we find the parallaxes for the four points a, b, c, and d from the expressions

$$p_a = x_a - x_a' \qquad p_c = x_c - x_c' \qquad [46]$$
$$p_b = x_b - x_b' \qquad p_d = x_d - x_d'$$

using the abscissas obtained after the three preceding steps. Then we continue

PARALLAX METHOD

	Photograph IV		V	
e	+ 2.36	+34.67	−62.90	+52.38
f	−10.86	−40.20	−76.72	−21.74
g	+83.31	+23.07	+12.56	+40.18
h	+65.47	−54.73	− 4.32	−39.24

	Photograph IV			
e	+ 2.30	− 7.90	+ 0.54	+33.76
f	−10.57	+ 9.16	− 2.47	−39.14
g	+81.12	− 5.26	+18.98	+22.46
h	+63.75	+12.47	+14.91	−53.29

e	− 5.60	+34.30	p=69.68
f	− 1.41	−41.61	67.54
g	+75.86	+41.44	74.37
h	+76.22	−38.38	70.13

	Photograph V			
e	−60.67	−13.81	−16.59	+50.53
f	−74.00	+ 5.73	−20.23	−20.97
g	+12.12	−10.60	+ 3.31	+38.76
h	− 4.17	+10.35	− 1.14	−37.85

e	−74.48	+33.94	−75.28	+34.30
f	−68.27	−41.20	−68.96	−41.61
g	+ 1.52	+42.07	+ 1.50	+41.44
h	+ 6.18	−38.99	+ 6.08	−38.38

sin θ	cos θ	sin θ′	cos θ′
− 2.36	+34.67 =	+62.90	+52.38
+10.86	−40.20 =	+76.72	−21.74
−83.31	+23.07 =	−12.56	+40.18
−65.47	−54.73 =	+ 4.32	−39.24
− 2.74	+40.20 =	+ 72.93	+60.74
+10.86	−40.20 =	+ 76.72	−21.74
+ 8.12	=	+149.65	+39.00
−83.31	+23.07 =	− 12.56	+40.18
−67.04	−56.04 =	+ 4.42	−40.18
−150.35	−32.97 =	− 8.14	
+ 0.44	=	+ 8.14	+ 2.12
−150.35	−32.97 =	− 8.14	
−149.91	−32.97 =		+ 2.12
+ 8.12	=	+149.65	+39.00
− 8.12	− 1.78 =	− 0.44	
	− 1.78 =	+149.21	+39.00

	cos θ	cos θ′
sin θ =	−.21994	−.01414
sin θ′ =	−.01194	−.26134

θ = −13°10.1′	θ′ = −15°17.5′
sin θ = −.22780	sin θ′ = −.26372
cos θ = +.97370	cos θ′ = +.96459

10000/69.68 = 143.5235	− 803.73	+4922.28	
10000/67.54 = 148.0604	− 209.65	−6161.83	
	594.08	11084.11	(*EF*) 11100.02
B (10031/11100)10000	9037 ft.		
9037/69.68 = 129.7005	− 726.32	+4448.21	
9037/67.54 = 133.8004	− 189.46	−5568.37	
	536.86	10016.58	*EF* 10031 ft.
9037/74.37 = 121.5157	+9218.67	+5035.73	
9037/70.13 = 128.8571	+9820.85	−4945.15	
	602.18	9980.88	*GH* 9999 ft.

by finding the parallaxes for *c*, *d*, *e*, and *f* in the second pair of photographs, and so on throughout the strip.

The next operation is to use the known ground length of *AB*, the control line appearing in the first pair of photographs, to determine the air base *B* for this pair of photographs, together with the horizontal ground distance *CD*; then from the known length of *CD* to determine the air base for the second pair of photographs, together with the ground distance *EF*; etc. Thus the air bases will be determined for all the pairs of photographs in the entire strip.

DETERMINATION OF THE AREA OF A FOUR-SIDED FIGURE

Measured coordinates of vertices	Photo IV		Photo V		
g4	+ 2.36	+34.67	−62.90	+52.38	
h4	+40.67	+28.48	−26.38	+45.70	$B=9037$ ft.
i5	+85.19	+59.65	+20.39	+77.05	
h5	+45.56	+64.17	−17.55	+81.48	

Photo IV: $\sin\theta = -.22780$ $\cos\theta = +.97370$

$x\cos\theta$	$y\sin\theta$	$-x\sin\theta$	$y\cos\theta$	New x	New y	Parallax
+ 2.30	+ 7.90	+ 0.54	+33.76	− 5.60	+34.30	69.68
+39.60	− 6.49	+ 9.27	+27.73	+33.11	+37.00	70.48
+82.95	−13.59	+19.41	+58.08	+69.36	+77.49	70.00
+44.36	−14.62	+10.38	+62.48	+29.74	+72.86	67.59

Photo V: $\sin\theta' = -.26372$ $\cos\theta' = +.96459$

$x\cos\theta'$	$y\sin\theta'$	$-x\sin\theta'$	$y\cos\theta'$	New x	New y	Coordinates with ordinates equalized	
−60.67	−13.81	−16.59	+50.53	−74.48	+33.94	−75.28	+34.30
−25.45	−12.05	− 6.96	+44.08	−37.50	+37.12	−37.37	+37.00
+19.67	−20.32	+ 5.38	+74.32	− 0.65	+79.70	− 0.63	+77.49
−16.93	−21.49	− 4.63	+78.60	−38.42	+73.97	−37.84	+72.86

		X	Y	Area:
9037/69.68=129.7005	G4	− 726.3	+ 4448.2	143957807.12
9037/70.48=128.2191	H4	+4245.6	+ 4743.6	94074509.32
9037/70.00=129.1057	I5	+8955.0	+10004.0	
9037/67.59=133.7074	H5	+3977.0	+ 9742.0	2) 49883297.80
	G4	− 726.3	+ 4448.2	
				24941649 sq. ft.
				572.581 acres

For the first pair let us designate by D the known ground length AB. First assume temporarily for the air base B some arbitrary value such as unity, or as in the illustrative computations to follow, 10,000 feet. Corresponding ground coordinates are then found for A and B by means of the parallax formulas [43],

$$X_A = (10000/p_a)x_a \qquad X_B = (10000/p_b)x_b$$
$$Y_A = (10000/p_a)y_a \qquad Y_B = (10000/p_b)y_b$$

Then we find the ground distance (D) between these points corresponding to these coordinates, by taking the square root of the sum of the squares of the differences in X and Y. Then the correct air base will be equal to $D/(D)$ multiplied by the arbitrary value taken for the air base; that is, in this case,

$$B = [D/(D)]10000 \qquad [47]$$

Then with air base B known, correct ground coordinates for the points C and D can be found from the parallax formulas [43],

$$X_C = (B/p_c)x_c \qquad X_D = (B/p_d)x_d$$
$$Y_C = (B/p_c)y_c \qquad Y_D = (B/p_d)y_d$$

Then the ground length CD is found by taking the square root of the sum of the squares of the differences in X and Y.

The length of CD is then used as the known length in the second pair of photographs, from which the second air base is computed together with the ground distance *EF*. This process is continued throughout the entire flight strip.

Finally, with the air base and the angles θ and θ' determined for every overlapping pair of photographs, it becomes a simple matter to find the ground lengths of any desired lines whatever. On a pair of photographs where there appears a line *ST*, for example, whose length is desired, the coordinates are measured as before for the extremities of the line on each photograph. The values of these coordinates after the rotation of the photographic axes through the angles θ and θ' and after the equalization of the corresponding ordinates, are found as described above. Then parallaxes are found for the points *S* and *T*, as shown in formula [46]. The parallax formulas [43] then give ground coordinates of the points *S* and *T*, from which the length can be found at once.

If a property survey is desired, it is simple to find lengths of all the property boundaries and also the angles at the property corners, from the ground coordinates of the corners found in this photogrammetric process.

The area of a piece of property is found very easily from the ground coordinates of the corners, using a calculating machine, by the usual analytical geometry method for finding the area of a polygon from the coordinates of the vertices. The process requires no more time than finding the area by planimeter, and it gives far more accurate results than the planimeter. Furthermore, the planimeter method requires information regarding the elevation of the tract or the elevations of the corners, whereas this parallax method requires no information whatever regarding the elevations of the property corners.

Test of the Parallax Method on a Strip of Five Photographs

Computations by the parallax method, which has been described, were carried through a strip of five photographs as illustrated in Figure 10. The five photographs used were from a set of synthetic photographs calculated by Co. B, 30th Engineers, at Fort Belvoir, for test purposes.

The photographs in this strip had tilts ranging from 0° to 3°. The topographic relief for the extremities *A* and *B* of the control line, for the pass points used as shown in Figure 10, and for the vertices of the area determined in the preceding illustrative computation, comprised the following widely different elevations:

	Point	Elevation		Point	Elevation
Control line	*A*	150 feet	Vertices of area	*G*4	500 feet
	B	1800	determined	*H*4	900
				*I*5	1050
Pass points	*C*	2200		*H*5	300
	D	1000			
	E	2000			
	F	750			
	G	500			
	H	0			
	I	2000			
	J	800			

Although neither the tilts nor these elevations are used at all in the computations of the ground distances or areas by the parallax method, nevertheless the wide variation in the elevations, coupled with the large tilts, severely tests the method, which is supposed to correct the distances and areas determined for topographic relief without actually using the elevations. How effectively the method does this can be seen from the tabulation of the results of the computation.

RESULTS OF THE COMPUTATION OF THE STRIP OF FIVE PHOTOGRAPHS

Photo pair	θ	θ'	Starting length	B	Length determined	Correct value†	Error
I– II	2°02′	2°00′	AB=10000*	10035	CD=10029	10000	0.29%
II–III	− 5 03	− 4 59	CD=10029	10892	EF=10062	10000	0.62%
III–IV	6 51	− 3 32	EF=10062	9747	GH=10031	10000	0.31%
‡IV– V	−13 10	−15 18	GH=10031	9037	IJ= 9999	10000	0.01%

* Ground control.
† Correct lengths are known in this case since the photographs are synthetic ones.
‡ The complete computation of this pair is previously shown as the illustration.

Area determined in the foregoing illustrative computation,	572.581 acres
Correct value of this area	573.921 acres
Error	1.340 acres or 0.23%

RADIAL TRIANGULATION

Several variations of radial plotting have been used in photogrammetry for the graphical determination of horizontal positions of ground points. The differences between them lie principally in the different ray centers used, in the method of transferring the ray centers to adjacent photographs, and in the manner of making and using templates.

The various ray centers which have been used are the principal points, the nadir points, the isocenters, and substitute center points. The geometric principles governing the choice between the first three can be briefly stated as follows: (1) With truly vertical photographs, image displacements caused by topographic relief radiate from the principal points, and angles between the principal point rays are exactly equal to the corresponding horizontal angles on the ground; (2) with tilted photographs, image displacements caused by topographic relief radiate from the nadir points, but angles between nadir point rays are not exactly equal to the corresponding angles on the ground; (3) with tilted photographs taken of absolutely flat terrain, regardless of the magnitudes of the tilts, angles measured on the photographs between rays from the isocenters are *exactly* equal to the corresponding ground angles. Consequently, with truly vertical photographs, radial plotting using principal point rays will give horizontal positions of ground points which are absolutely free from errors, regardless of the magnitude of topographic variations. If the photographs have tilts, no matter how large the tilts may be, and if the ground is perfectly flat, radial plotting using isocenter rays will give horizontal positions of ground points absolutely free from errors. If the problem involves both tilts and topographic relief, no ray center will give results free from errors. If the tilts are as small as those usually encountered, and if the ground has considerable topographic relief, radial plotting with nadir point rays will give by far the best results, and will ordinarily give horizontal positions in which the errors are negligible.

Obviously neither the nadir points nor the isocenters can be used unless the individual tilts of the photographs are known approximately. Also none of the three points, the principal point, the nadir point, or the isocenter, can be used without some scheme for marking the conjugate points on adjacent photographs. The stereoscopic method for transferring these points to adjacent photographs has proved very satisfactory in practice. The substitute center point has a distinct advantage in that conjugate points can be marked on adjacent photographs by mere identification of the images of the points chosen. Of course small errors in the horizontal positions determined arise from the use of the substitute center points, but these errors are frequently negligible in practical work.

At the outset of the previous discussion of the calculation of distances and areas by the parallax method, a very simple and satisfactory method was shown for calculating on each photograph the directions of the principal point rays toward the principal points of the adjacent overlapping photographs. If the directions of the principal point rays on each photograph to the various images arranged approximately as shown in Figure 10, together with the directions found from the θ's to the conjugate principal points, are considered as horizontal directions read from a direction theodolite at each exposure station, it becomes possible to compute a net work of triangulation consisting of the two control points, the pass points of Figure 10, and the principal points, as vertices. In this manner the X and Y survey coordinates can be calculated for all of the points in this net work and for any other points desired. Thus complete planimetric survey data become available for the flight strip as well as data for computing any desired lengths or areas. Inasmuch as the methods of computing this triangulation are identical with those universally used for any triangulation in either plane or geodetic surveying, the explanation of the procedure is scarcely required here.

Of course it is obvious that this calculation of radial triangulation is not confined to principal point triangulation. If the approximate tilts are known so that approximate nadir points and isocenters become available, then the entire scheme of triangulation can likewise be calculated using nadir points or isocenters in place of the principal points.

In fact the problem of radial triangulation, like all other problems in aerial photogrammetry, is susceptible to many variations.

Scale Determination

The complete determination of the scale of a single *vertical* photograph consists of only the determination of the altitude of the exposure station. This can be done by using two ground points A and B whose images a and b appear in the photograph, with the distance AB measured on the ground and with the elevations of A and B determined on the ground. Let the photographic coordinates of the images a and b be called (x_a, y_a) and (x_b, y_b), and let the horizontal ground length AB be called D and the elevations of A and B be called Z_A and Z_B. A very rough value for the altitude H of the exposure station can be found from the scale relation

$$(H - h)/f = D/d$$

in which

$$d = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}$$

and in which h is taken as the mean of Z_A and Z_B. Then consider that the images are displaced inward radially toward the center of the photograph to compensate for the elevations Z_A and Z_B, giving the positions of the images for corresponding points at zero elevation. To do this, multiply x_a and y_a by $(H-Z_A)/H$ and x_b and y_b by $(H-Z_B)/H$, using the approximate value of H. Now find d again from these revised photographic coordinates, and then find a better value of H from the sea-level scale relation.

$$H/f = D/d,$$

using in this the new d. With this revised H, displace the images again, and repeat this until the value of H will no longer affect the photographic coordinates used to find d. Two approximations almost invariably suffice to find the correct H.

Then if a length is desired for a line joining any two ground points of unequal elevations, the usual expressions for scale relationship

$$X = [(H - h)/f]x \qquad Y = [(H - h)/f]y \tag{48}$$

in which H is the exposure station altitude and h is the elevation of one of the ground points, give ground coordinates X, Y directly from the photographic coordinates x, y, for each of the two points. Then the true ground length is at once obtained by finding the square root of the sum of the squares of the differences in X and Y between the two ground points.

The complete scale determination for a *tilted* photograph consists of finding the elevation of the exposure station, and also the tilt and the swing. The space resection problem and the space orientation problem for a single photograph supply these data completely, together with additional data, not a part of the scale determination, for uniquely fixing the photograph in space.

In this case, if the correct length is desired for a line joining any two ground points having different elevations, the exact expressions for the scale relationship for tilted photographs

$$\begin{aligned} X &= [(H - h)/(f \sec t - y \sin t)]x \\ Y &= [(H - h)/(f \sec t - y \sin t)]y \cos t \end{aligned} \tag{49}$$

in which t is the tilt, and x, y, are the photographic coordinates after the transformations by formulas [16], will give the *exact* X and Y ground coordinates for each point. Then the correct length can be found as before from the horizontal ground coordinates.

Absolutely correct values can be found for areas in either of the above two cases, if the elevations of the vertices on the ground are known. The photographic coordinates substituted in [48] for a vertical photograph, or in [16] and then [49] for a tilted photograph, give the correct X and Y ground coordinates of the vertices, and the area can then be found using the usual analytical geometry method for finding the area of a polygon from the coordinates of the vertices.

Of course the ground X and Y coordinates given by [48] from a vertical photograph, are referred to the ground plumb point as the origin and to vertical coordinate planes including the rectangular axes of the photograph; and likewise the ground X and Y coordinates given by [49] from a tilted photograph, are referred to the ground plumb point as the origin and to vertical coordinate planes perpendicular to each other with the YZ-plane coinciding with the principal plane of the photograph. But this has no bearing on the lengths of lines or the areas of polygons found from the scale determination of the photograph.

However, if it should be desired to find the exact space coordinates on the survey datum for a ground point whose elevation is known, using measurements on a tilted photograph for which the resection and space orientation problems have been solved, the method is shown in the discussion of the Four-Point Method, and consists of applying formulas [16], [17], [20], [23], [31], [32], and [33]. But the horizontal coordinates X, Y, of the exposure station, and α_{VO}, the azimuth of the principal plane, required for this work, are not strictly a part of the scale determination problem.

Some analytical solutions of the complete scale determination problem other than space resection and space orientation previously described, furnishing as they do only the Z coordinate of the exposure station together with t and s, but not α_{VO}, will merely give lengths and areas, but not actual ground survey coordinates, from measurements on the photographs. Some of these analytical solutions of the scale determination problem may be found extremely useful in certain practical work. None of these has been explained here on account of the fact that the analytical solutions explained for space resection and space orientation of the photograph include the complete scale determination.

Scale Determination

Case 1, assuming that the photograph is truly vertical.

Point	Measured photo coordinates x	y	Given ground elevations		
$d2$	+39.25	−41.87	$D2$	1000 ft.	
$d4$	+42.90	+40.28	$D4$	2800	
	3.65	82.15			

$f = 150.00$ mm.

Given distance $D2$ to $D4 = 10000$ ft.

$\sqrt{3.65^2 + 82.15} = 82.23$

$(H-h)/150.00 = 10000/82.23$ $\qquad H-h = 18242$ ft.

Approx. $(H) = 18242 + 1900 = 20142$ ft.

Point		Coordinates corrected for topographic relief	
$d2$	$19142/21042 = .950$	+37.29	−39.78
$d4$	$17342/20142 = .861$	+36.94	+34.68
		0.35	74.46

$\sqrt{0.35^2 + 74.46} = 74.46$

$H/150.00 = 10000/74.46$ $\qquad H = 20145$ Final

To find the distance from $B2$ to $B4$, given the following data:

Point	Measured photo coordinates x	y	Ground elevations	
$b2$	−42.35	−44.97	$B2$	2400 ft.
$b4$	−38.49	+35.88	$B4$	800

Point	$H-h$	$(H-h)/f$	Ground Coordinates X	Y
$B2$	17745	118.300	−5008	−5320
$B4$	19345	128.967	−4964	4627
			44	9947

$\sqrt{44^2 + 9947^2} = 9947$ ft. Desired distance.

Case 2, using the photograph as tilted.

The space resection and the space orientation for this photograph have already been calculated. In fact this is the same photograph for which these computations have already been shown under those topics. The part of the results which constitute the scale determination are as follows:

$$H \text{ or } Z_L = 20{,}201 \text{ ft.} \qquad t = 0°59.7' \qquad s = 180°29.0'$$

To find the distance from $B2$ to $B4$, given the same data as before, namely,

Point	Measured photo coordinates x	y	Ground elevations	
$b2$	−42.35	−44.97	$B2$	2400 ft.
$b4$	−38.49	+35.88	$B4$	800

Using the following constants

$\theta = 180° - s = -0°29.0'$ $\sin\theta = -.00844$ $\cos\theta = +.99996$

$\sin t = .01737$ $\cos t = .99985$ $f \sec t = 150.02$ $ov = 2.61$

from the space orientation data, we first substitute the given coordinates in formulas [16] thus

	$x\cos\theta$	$y\sin\theta$	$-x\sin\theta$	$y\cos\theta$	New x	New y
$b2$	-42.35	$+0.38$	-0.36	-44.97	-41.97	-42.72
$b4$	-38.49	-0.30	-0.32	$+35.88$	-38.79	$+38.17$

Then we find

	$y\cos t$	$y\sin t$	$(f\sec t - y\sin t)$	$H-h$	$\dfrac{(H-h)}{(f\sec t - y\sin t)}$
$b2$	-42.71	-0.74	150.76	17801	118.075
$b4$	$+38.16$	$+0.66$	149.36	19401	129.894

and by formulas [49] we have the ground coordinates

	X	Y
$B2$	-4957	$-$ 5043
$B4$	-5039	$+$ 4957
	82	10000

Required distance $B2$ to $B4 = \sqrt{82^2 + 10{,}000^2} = 10{,}000$ ft.

The method for Case 1 is correct for truly vertical photographs. The results for the exposure station altitude and for the distance from $B2$ to $B4$ are in error, however, because the photograph, instead of being a truly vertical one, actually has a tilt of 0°59.7′. The method for Case 2 is rigorously correct.

The photograph used is one of the Fort Belvoir synthetic photographs, and the correct length of $B2$–$B4$ is therefore known. It is 10,000 ft. When the tilt is neglected the length found for this line has an error of 53 ft. or 0.53%. When the tilt is considered, the length determination is exactly correct.

A Simple Method for the Photogrammetric Extension of Surveys without Ground Control Using Two Parallel Overlapping Strips of Photographs

After some experience with analytical computations it will be found that the space resection, the space orientation, and the space intersection problems can be solved easily and rapidly. The four-point method, however, for extending a photogrammetric survey through a strip of photographs without using ground control after the initial triangle, does require considerable time even for an experienced computer. To avoid the four-point calculations, therefore, a simple means has been devised for extending the photogrammetric survey through two overlapping parallel strips of photographs, using onlv the resection, orientation, and intersection methods.

Figure 11 shows two overlapping parallel strips of photographs. Let us suppose that there are four initial control points A, B, C, and Q situated as shown in the figure, points whose ground coordinates on the survey datum have been determined by the usual geodetic methods. Then let sharply identifiable pass points be chosen situated as shown by the figure, with a_3, b_3, and c_3 approximately opposite the centers of photographs III and III′, a_4, b_4, and c_4 approximately opposite the centers of photographs IV and IV′, etc. Measurements are first made with the comparator on all of the photographs of the rectangular coordinates of the images of all these points, with these coordinates referred to the geometric axes of the respective photographs as usual. That is, on photograph I, x and y coordinates are measured for the images a, b, and q; on photograph I′ for b, c, and q; on photograph II for a, b, q, a_3, and b_3; on photograph II′ for b, c, q, b_3, and c_3; on photograph III for a, b, a_3, b_3, a_4, and b_4; on photograph III′ for b, c, b_3, c_3, b_4, and c_4; etc.

Then the space coordinates on the survey datum for the exposure stations, and the elements of space orientation, can be computed for photographs *I* and *II*, using the control points A, B, and Q; and the same calculations for photographs I′ and II′ can be made using control points B, C, and Q, by the space resection and space orientation methods previously explained.

Then the ground coordinates on the survey datum for the point B_3 can be computed by the space intersection method previously explained, using measurements on photographs II and II′.

Next, the exposure station and elements of space orientation are computed

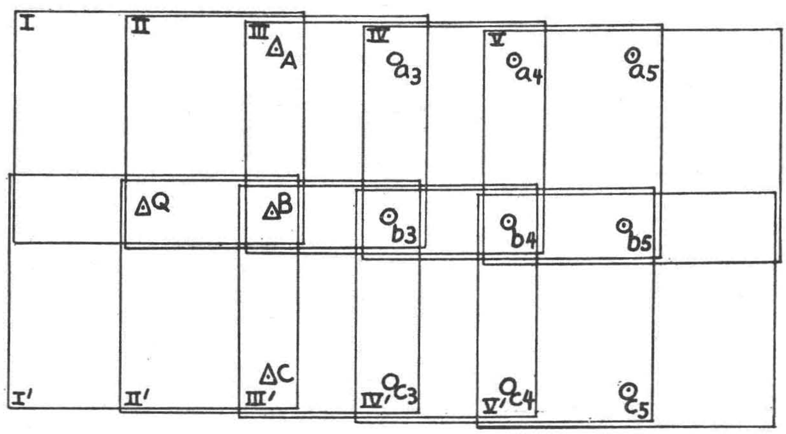

FIG. 11

for photograph III using as control the three points A, B, and B_3, and the same calculations are made for photograph III′ using the points B, C, and B_3.

Then by the space intersection method again, the survey space coordinates are computed for A_3 using measurements on photographs II and III, those for C_3 using measurements on photographs II′ and III′, and those for B_4 using measurements on photographs III and III′. After this, points A_3, B_3, and B_4 are used for control for exposure station and orientation calculations for photograph IV, and the points B_3, C_3, and B_4 are used for photograph IV′. Then in turn photographs III and IV are used for space intersection to compute the survey coordinates of A_4, III′ and IV′ are used to determine C_4, and IV and IV′ to determine B_5.

By the continuation of this procedure both of the entire strips of photographs can be computed and the space positions of all of the pass points can be determined, all with no additional control after the four initial points, and all by using only the simple space resection, orientation, and intersection methods.

Subsequently of course the space intersection method will determine the space coordinates on the survey datum for any desired points situated anywhere within the area covered by the two strips of photographs.

It might be thought at first that this method would fail unless the photographs in the two strips were situated exactly opposite each other laterally, as they are in Figure 11. This is not true however. If the photographs in each strip have the customary sixty percent overlap, even if the photographs of the two

RESULTS OF COMPUTATIONS FOR THE TWO PARALLEL STRIPS OF PHOTOGRAPHS SHOWN IN FIGURE 11

‖	Photo	Point	Control points used	Photos used	Results: Exp. sta. X, Y, Z	Results: Orientation t, s, αvo	Results: Ground coordinates X, Y, Z	Correct values* X, Y, Z′
R, O‡	I		A† B† Q†		5002 34997 20101	2°00′ 45 14 225 14		
R, O	I′		B C† Q		5002 15003 20001	0°01′ 32 28 212 28		
R, O	II		A B Q		15003 34995 20000	1°29′ 0 20 180 20		
R, O	II′		B C Q		14997 15002 20201	1°00′ 180 29 0 29		
I§		B3		II II′			24999 25000 1801	25000 25000 1800
R, O	III		A B B3		25000 34999 20399	3°00′ 305 00 125 00		
R, O	III′		B C B3		24997 15005 20001	3°01′ 305 05 125 05		
I		A3		II III			25000 44998 801	25000 45000 800
I		C3		II′ III′			25000 5000 1001	25000 5000 1000
I		B4		III III′			34997 25001 253	35000 25000 250
R, O	IV		A3 B3 B4		35002 34994 19998	0°01′ 133 58 313 58		
R, O	IV′		B3 C3 B4		35001 15014 20001	1°59′ 235 26 45 25		

RESULTS OF COMPUTATIONS FOR THE TWO PARALLEL STRIPS OF PHOTOGRAPHS SHOWN IN FIGURE 11—*Continued*

‖	Photo	Point	Control points used	Photos used	Results			Correct values* X, Y, Z
					Exp. sta. X, Y, Z	Orien-tation t, s, αvo	Ground coordi-nates X, Y, Z	
I		*A*4		III IV			35000 44998 205	35000 45000 200
I		*C*4		III′ IV′			35004 4998 145	35000 5000 150
I		*B*5		IV IV′			44991 25003 1058	45000 25000 1050
R, *O*	V		*A*4 *B*4 *B*5		45021 34965 19788	1°56′ 268 04 88 06		
R, *O*	V′		*B*4 *C*4 *B*5		45029 15043 19887	1°35′ 99 28 267 25		
I		*A*5		IV V			45003 45001 681	45000 45000 700
I		*C*5		IV′ V′			45013 4998 22	45000 5000 50
I		*B*6		V V′			54965 25006 611	55000 25000 600

* These correct values are known because these photographs are synthetic ones.
† *A*, *B*, *C*, and *Q* are the only ground control points used for the entire group of photographs.
‡ See illustrative computations for Space Resection and Space Orientation.
§ See illustrative computation for Space Intersection.
‖ Resection and Orientation, or Intersection.

strips are staggered in any possible manner, the procedure described will always apply. The slight variations in the procedure which may be necessary in any particular case are so obvious when a sketch is made to show the overlap, that no further explanation is required here.

It should be noted that if two parallel strips of photographs are available, this method really offers a much easier means than the four-point method for extending photogrammetric surveys into inaccessible territory.

Test of the Method for the Photogrammetric Extension of Surveys through Two Parallel Overlapping Strips of Photographs

Computations by this resection and intersection method of extending photogrammetric surveys without ground control, using two parallel overlapping strips of photographs, were made for the group of photographs shown in Figure 11. These are a part of the series of synthetic photographs computed by Co. B, 30th Engineers, at Fort Belvoir, for test purposes.

It will be noticed that, although the tilts were large enough and the variations in topographic relief great enough to test the method rather severely, the results are very satisfactory.

The computations involve only the resection, orientation, and intersection methods, for all of which illustrative examples have already been shown. Therefore there is given here only a tabulation of the results of these computations for this group of photographs. However the illustrative computations for resection and for space orientation are actually those for photograph I in this group shown in Figure 11; and the illustrative computation for space intersection is actually the computation of the survey coordinates of *B*3 using photographs II and II′, as it was done in this series of computations.

Conclusion

In conclusion two points should be mentioned regarding this paper. First, the demands of recent research work have brought analytical methods in photogrammetry to the attention of several very eminent mathematicians, and it is entirely possible that their findings may partially supersede some of the methods presented here.

Secondly, it should be emphasized that the methods presented here can be extended to countless surveying problems other than those specifically shown, such as the determination of the bearings and lengths of property boundaries, earth-work calculations, determination of tree heights and timber estimating in forestry surveys, etc. It is of course impossible to present anything more than the most fundamental problems in this brief discussion of analytical methods in aerial photogrammetry.