HEIGHT DETERMINATION WITH HIGH OBLIQUE PHOTOGRAPHS

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1. *Introduction*

THE purpose of this article is to describe a proposed method of photogrammetric mapping which would appear to be most suitable in the mountainous regions in the western part of China, where no triangulation points are available. Fully automatic stereoscopic plotting instruments made in Europe have, already, been extensively used in China both for detail plotting and control purposes, but these instruments, in projects where speed and economy are more important than the utmost precision, are impractical owing to their high initial cost. The Multiplex instruments on the other hand find their triumphal way in

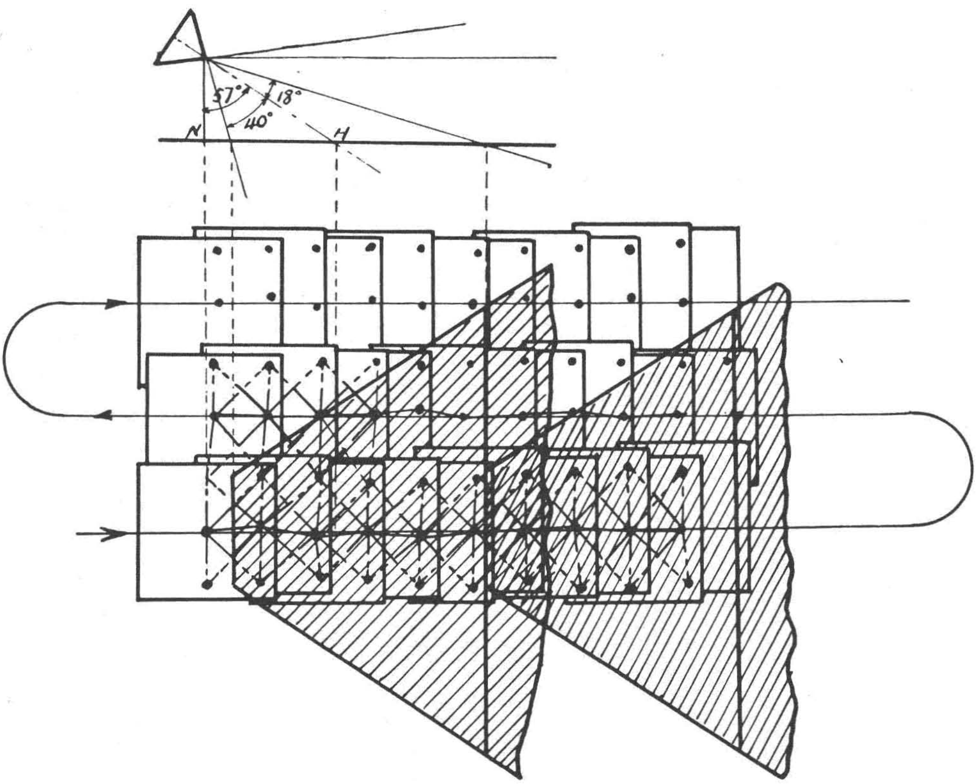

FIG. 1

many cartographical applications chiefly owing to their cheapness. Furthermore, the speed of working with them can always be increased by using more of them at the same time. However, a practical drawback arises in the detail plotting with such instruments as the minimum spacing between projectors does not allow a mapping scale smaller than 1:20,000. This from the economical point of view is too large, as a reproduction scale of 1:50,000 is that generally adopted for the topographical maps of China. Again, if control is extended by means of aerial triangulation with these instruments the height determinations will not be sufficiently precise for the purposes in view.

Geometrically the strongest position determinations are obtained if vertical photographs are used for planimetric plotting and high obliques for heights. Captain D. R. Crone of the Survey of India has already developed a method with this idea in mind, but his method, and other methods utilizing oblique photographs, require for the resection of each air station at least three ground control points whose planimetric positions and heights are known. In a region without sufficient triangulation points the planimetric positions of points can be obtained with sufficient accuracy from vertical photographs by radial-line methods. However, as far as height control is concerned, ground field parties are generally assumed to be necessary. Much of this field work could be eliminated if high oblique photographs are also taken in conjunction with the serial vertical ones, as it will be shown that when the planimetric positions of control points and air stations have been determined from the vertical photographs the heights of the control points can then be determined from the oblique photographs.

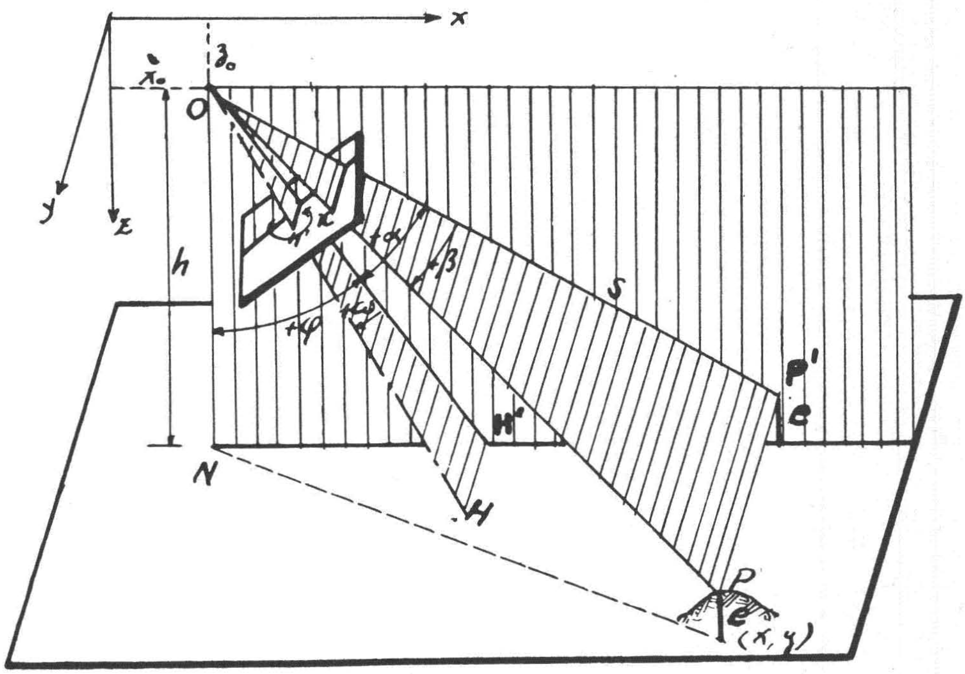

FIG. 2

Fig. 1 suggests as an example a suitable scheme for taking the oblique photographs. The camera taking them is mounted at a fixed angle with the vertical camera, and exposures with both cameras are taken simultaneously whenever the obliques are necessary for proper coverage. The oblique camera is tilted in the direction of flight by an amount which enables the apparent horizon to be imaged. In this example of flying height of 3000 meters is assumed and the focal lengths of both cameras is the same, namely 100 mms, The size of the negatives is 180×180 mms. but the part available for plotting has been taken, arbitrarily, to be 170×170 mms. Though theoretically oblique photographs will image points as far away as the horizon, in order that the image scale may not be too small, the angles of inclination from the vertical of perspective rays to points is limited to those of less than 75°.

2. *Notation* [See Fig. 2]

O	The camera station.
H	The point in the horizontal reference datum plane where the latter is cut by the camera axis.
N	The Nadir point in the horizontal reference datum plane of O.
$x_0y_0z_0$	Rectangular coordinates of the camera station.
h	Altitude of camera station above the horizontal reference datum plane.
P	A point on the ground or $P_1P_2P_i$ for specific points.
xyz	Rectangular coordinates of a point P or $x_1y_1z_1$, $x_2y_2z_2$, etc. for specific points.
e	Height of a point P above the horizontal reference datum plane or $e_1e_2e_i$ for specific points.
ϕ (Phi), ω (Omega), κ (Kappa	The primary, secondary and tertiary angular elements of the exterior orientation of the photograph about a primary axis parallel to the y axis.
ϕ	Corresponds to tip in the multiplex and is a rotational movement of the camera-axis about the y axis.
ω	Corresponds to list in the multiplex and is rotational about an axis perpendicular to the y axis and the camera axis.
κ	Corresponds to swing in the multiplex and is the rotation of the plane of the photograph about the camera axis.

If perpendiculars are dropped from P and H respectively to the XZ vertical plane containing O cutting it at P' and H' respectively then in Fig. 2

$$\begin{aligned} &\alpha \text{ is the angle } P'OH' \\ &\phi \text{ is the angle } H'ON \\ &\beta \text{ is the angle } POP' \\ \text{and } &\omega \text{ is the angle } HOH' \end{aligned}$$

3. *Analytical Method of Solution*

According to the scheme outlined in Section 1 the unknowns involved are the heights of the control points and the elements of exterior orientation of the oblique photograph. Since the horizontal position of each control point as determined from the vertical photographs gives two known quantities, namely the x and y rectangular coordinates, it is always possible by increasing the number of control points to derive as many conditional equations as there are unknowns. The analytical method of solutions is based on this principle. Now, experience[1] has shown that the true tilt of the camera axis at the moment of exposure can be deduced from the apparent horizon appearing on the photograph with a probable or mean square error of less than 3.25′. With a flying height of 3000 meters this introduces an uncertainty of only 3 meters in the horizontal position of the air station when this is deduced from this data. If an apparent horizon is not sufficiently well outlined on the photograph then x_0 and y_0 must be introduced as additional unknowns in the process, but, as already inferred and as is shown in Section 6, this is a possible procedure. For the moment, however, assume that the horizontal positions of the ground points and the air station are known, then the problem is to adjust the pencils of rays as a whole so that each perspective ray to a ground point passes through a vertical line erected over the latter's plotted map position.

[1] Gruber: Aeropolygonierung und Aeronivellement, Bildmessung und Luftbildwesen. 1935, p. 169.

This problem is better solved by an optical instrumental method as explained in Section 4, but a mathematical derivation is indispensable for the sake of establishing the working procedure and for studying the accuracy of the method.

From Fig. 2 we have

$$x - x_0 = (h - e) \tan(\phi + \alpha) \quad (1)$$

$$y - y_0 = s \cdot \tan\beta = [(h - e)/\cos(\phi + \alpha)] \tan\beta \quad (2)$$

We introduce approximate values h', e', ϕ', ω', and κ', which differ from their true values by Δh, Δe, $\Delta\phi$, $\Delta\omega$ and $\Delta\kappa$ and first derive approximate values α' and β' which differ from their true values by $\Delta\alpha$ and $\Delta\beta$.

Furthermore, it can be shown[2] that

$$\Delta\alpha = (\sin\alpha \cdot \tan\beta)\Delta\omega + (\cos\alpha \cdot \tan\beta \cdot \cos\omega - \sin\omega)\Delta\kappa \quad (3)$$

and

$$\Delta\beta = \cos\alpha \cdot \Delta\omega - (\cos\omega \cdot \sin\alpha) \cdot \Delta\kappa \quad (4)$$

and when we assume ω to be zero, which will nearly be the case when the oblique photograph has been taken in approximately the same horizontal direction as the line of flight, expressions (3) and (4) simplify to

$$\Delta\alpha = (\sin\alpha \cdot \tan\beta) \cdot \Delta\omega + (\cos\alpha \tan\beta) \cdot \Delta\kappa \quad (5)$$

and

$$\Delta\beta = \cos\alpha \cdot \Delta\omega - \sin\alpha \cdot \Delta\kappa \quad (6)$$

Then by substituting in (1) and (2) the approximate values of the unknowns and their differences and eliminating the obviously small terms in the transformations and also by assuming the approximations

$$h' - e' = (x - x_0) \cdot \cot(\phi' + \alpha') \quad \text{in (1)}$$

and

$$h' - e' = (y - y_0) \cdot \cos(\phi' + \alpha')/\tan\beta' \quad \text{in (2)}$$

we derive the following from (1).

$$F_1(\Delta h - \Delta e) + F_2 \cdot \Delta\phi + F_3 \cdot \Delta\omega + F_4 \cdot \Delta\kappa = (x - x_0) - (h' - e') \tan(\phi' + \alpha') \quad (7)$$

and from (2)

$$F_5(\Delta h - \Delta e) + F_6 \cdot \Delta\phi + F_7 \cdot \Delta\omega + F_8 \, \Delta\kappa = (y - y_0) - (h' - e') \cdot \tan\beta'/\cos(\phi' + \alpha') \quad (8)$$

in which $(\Delta h - \Delta e)$, $\Delta\phi$, $\Delta\omega$, amd $\Delta\kappa$ are now the unknown quantities, and the coefficients

$$F_1 = \tan(\phi' + \alpha')$$

$$F_2 = (x - x_0)/\cos(\phi' + \alpha') \sin(\phi' + \alpha')$$

$$F_3 = (x - x_0) \cdot \sin\alpha' \tan\beta'/\cos(\phi' + \alpha') \sin(\phi' + \alpha')$$

$$F_4 = (x - x_0) \cos\alpha' \tan\beta'/\cos(\phi' + \alpha') \sin(\phi' + \alpha')$$

$$F_5 = \tan\beta'/\cos(\phi' + \alpha')$$

$$F_6 = (y - y_0) \cdot \tan(\phi' + \alpha')$$

$$F_7 = (y - y_0) \cdot [\tan(\phi' + \alpha') \tan\beta' \cdot \sin\alpha' + (\cos\alpha'/\cos\beta' \cdot \sin\beta')]$$

$$F_8 = (y - y_0) \cdot [\tan(\phi' + \alpha') \tan\beta' \cdot \cos\alpha' + (\sin\alpha'/\cos\beta' \cdot \sin\beta')]$$

[2] Jordan-Eggert: Handbuch der Vermessungskunde, Vol. 2b, p. 474.

By multiplying all the coefficients in (7) by $\tan \beta'/\sin(\phi'+\alpha')$ and then subtracting the result from (8) we eliminate $(\Delta h - \Delta\epsilon)$ and obtain

$$F_9 \cdot \Delta\phi + F_{10} \cdot \Delta\omega + F_{11} \cdot \Delta\kappa = [(x - x_0) \cdot \tan \beta'/\sin(\phi' + \alpha')] - (y - y_0) \qquad (9)$$

where the coefficients

$$F_9 = (x - x_0) \cdot \cos(\phi' + \alpha')/\sin^2(\phi' + \alpha')$$

$$F_{10} = [(x - x_0)/\sin(\phi' + \alpha')]\,[\tan^2 \beta' \cdot \sin \alpha' \cdot \cot(\phi' + \alpha') - (\cos \alpha'/\cos^2 \beta')]$$

$$F_{11} = [(x - x_0)/\sin(\phi' + \alpha')]\,[\tan^2 \beta' \cdot \cos \alpha' \cdot \cot(\phi' + \alpha') - (\sin \alpha'/\cos^2 \beta')]$$

By computing the coefficients in (9) for three control points we derive three simultaneous equations which enable us to solve for $\Delta\phi$, $\Delta\omega$ and $\Delta\kappa$. Determination of the differences of height can then be made by substituting the calculated values of $\Delta\phi$, $\Delta\omega$, and $\Delta\kappa$ in equations (7) and (8).

In this analytical method of solution a photogoniometer is desirable for measuring the assumed values α' and β'.

4. *Optical Method of Solution*

Suppose that the horizontal positions of a number of control points and the position of the air station have been plotted on the control sheet. Also suppose the photograph has been correctly placed in a projecting camera having the same internal orientation and an objective lens with the same optical characteristics

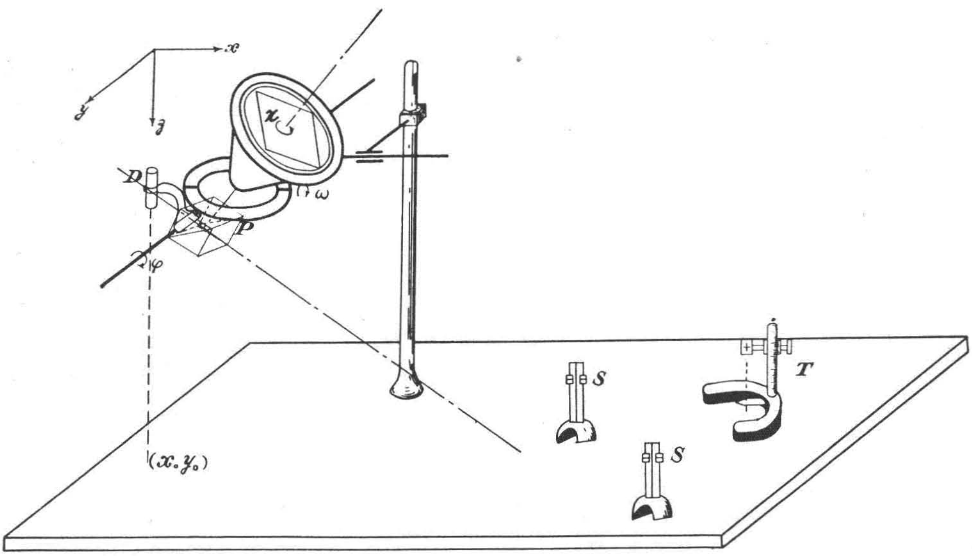

FIG. 3

as in the photographing camera, and that this projecting camera is oriented exteriorly over the map plot in the same relative position as the photographing camera was oriented over the ground when the exposure was made. Then, if vertical columns carrying fine vertical lines are erected so that the latter are accurately centered over the horizontal position of each control point, it should be possible to measure the heights of the control points above the datum reference plane, by means of optical intersections of the perspective rays with the vertical lines.

To carry out this idea an instrument is suggested as shown schematically in Fig. 3. Two triangular prisms are placed together to form a rectangular piece P, and this allows simultaneous observation of the photograph and objects in the map space. A projecting camera is placed in such a relationship to P that perspective rays from the object space passing through P without reflection will converge to the same exterior perspective center as the reflected rays from the photograph. Means must therefore be provided to orient the projecting camera for tip, ϕ list, ω and swing κ. A diopter, which can be removed when necessary in order to allow the observer to place his eye in the correct position for observation, is placed at the exterior perspective center pointing vertically down. The prisms, diopter, and projection camera are all mounted together as a unit on a

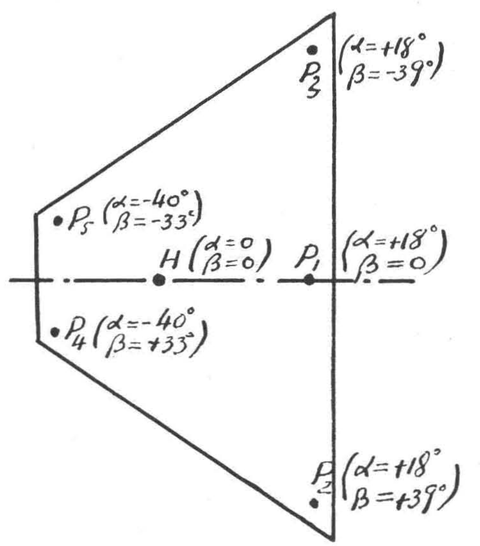

FIG. 4

framework and means are provided to move this unit bodily in the three linear directions. Above each plotted point may be placed a vertical column S, one of whose sides is a plane surface. On this surface is centered a fine vertical line. Means are also provided for sliding a horizontal line up and down the column. This horizontal line is used to determine the approximate height of the visual intersection of a perspective ray with the vertical line, but, once the photograph has been correctly oriented, the S columns may be replaced by a measuring column T to which a vernier scale is attached. This enables readings of height above the plotting plane to be made to the nearest 0.1 mm.

The photograph unit is first placed in its correct position by sighting down the diopter and moving the unit until the diopter appears centered over the map position of the air station. The photograph is then oriented by a systematic elimination of the y-parallaxes. By y-parallax is meant the displacement in the y direction of the intersection of a perspective ray with the vertical plane of a column, when the vertical plane is parallel to the y axis. A minimum of three points is necessary to accomplish the orientation of the photograph and the choice of points to be used is judged by computing the coefficients in equation (9) in Section 3, in which the constant term represents the y-parallax.

As an example, if an oblique photograph has been taken as shown in Fig. 1 and the points shown in Fig. 4 are used, then the computed coefficients from (9) are

Point	$\Delta\phi$	$\Delta\kappa$	$\Delta\omega$
H	—	—	−5.48
P_1	—	3.05	−9.45
P_2	2.15	6.75	−15.18
P_3	−2.15	6.75	−15.18
P_4	6.53	0.43	−6.05
P_5	−6.53	0.43	−6.05

From these we see that the parallex at H, which is a point imaged near the principal point is only dependent on ω and can be eliminated by adjusting the photograph for $\Delta\omega$. We then proceed to the point P_1 whose image is in the background of the photograph approximately in the principal plane and adjust the photograph for $\Delta\kappa$. The third unknown $\Delta\phi$ can then be found by parallax observation at any of the remaining points; preference being given to the point with the largest coefficient, which will be either P_4 or P_5 in the foreground.

Having correctly oriented the photograph, its mount can then be shifted vertically to afford a convenient position for measuring the heights of the control points and then by means of the measuring column these can be determined. Corrections for curvature and refraction must of course be made in the usual manner.

5. *Accuracy of the Method*

From Fig. 1 it is apparent that within the area of each oblique photograph there will always be a large selection of points whose horizontal positions have been determined by radial-line intersections from the vertical photographs. Though three points are theoretically sufficient to solve the problem under consideration, the accuracy can be increased by increasing the number of points used. If this were done a least square adjustment of the residual errors would be necessary.

The method of least squares may also be utilized to determine the m.s.e. (mean square error) of the determination of $\Delta\phi$, $\Delta\kappa$ and $\Delta\omega$ in any particular case.[3] Assuming the m.s.e. of the y-parallax to be $m = \pm 2.5$ meters, which corresponds to an error of ± 0.1 mm. in a plotting scale of 1:25,000, then in the example shown in Fig. 4 we get by this method for $\Delta\phi$, $\Delta\kappa$ and $\Delta\omega$ m.s.e. of $\pm 0.9'$, $\pm 2.4'$ and $\pm 0.9'$ respectively. The m.s.e. of Δe which is approximately the same as the m.s.e. of $(\Delta h + \Delta e)$ can be obtained either from equation (7) or equation (8). In the former the result is the effect due to placing the measuring column parallel to the $z-y$ plane in the height measurements, while the latter assumes the column to have been placed parallel to the $z-x$ plane.

In the example of Fig. 4 the m.s.e. of the height determinations for each point were found by this procedure to be

Point	H	P_1	P_2	P_3	P_4	P_5
m.s.e. (meters)	±1.7	±2.7	±5.2	±5.2	±3.3	±3.3

However, the assumption that the y-parallax can be kept to within ± 0.1 mm. could hardly be fulfilled if the planimetry had been fixed by principal point radial-line intersections from the vertical photographs. Hence if greater ac-

[3] Wright and Hayford: The Adjustment of Observations, 1906, p. 127.

curacy is required a more precise method of radial triangulation should be resorted to and the plotting scale should also be enlarged. R. Finsterwalder[4] has computed the m.s.e. of height determinations for radial triangulation with vertical photographs assuming a m.s.e. of parallax elimination in the picture plane of +0.03 mm. Assuming the field of view of the camera was normal, the m.s.e. of the height determinations for each point were found to be at a flying height of 3000 meters as below.

Point	M	N	O_A	O_B	P	Q
m.s.e. (meters)	± 18	± 18	± 11	± 11	± 18	± 18

Comparing the two sets of figures, it will be seen that the average m.s.e. in height determinations in the vertical example is about four times as large as in the oblique example. Hence in these examples if it were possible to maintain an accuracy in the planimetry derived from the vertical photographs of 0.03 mm. the accuracy of the height determinations by the oblique method should be four times that of those determined from the verticals alone.

6. *Exterior Orientation When the Position of the Camera Station is Unknown*

The natural horizon sometimes fails to give the position of the camera station as in in the cases when the horizon line is obscured by the presence of mountains or clouds. Furthermore it may be more convenient sometimes not take the oblique photographs simultaneously with the verticals, and hence the direct determination of the horizontal position of the air station from the verticals is impossible. In either case in order to obtain the required position two new variables Δy_0 and Δx_0 have to be included in the solution in equation (9) and this involves the use of five control points instead of three. From equations (1) and (2) the additional coefficients for Δy_0 and Δx_0 in (9) may be deduced to be 1 and $\tan \beta'/\sin(\phi'+\alpha')$ respectively.

7. *Conclusion*

Although the attainable accuracy of the method described above remains to be tested it is easy to see that stronger determinations of heights can be obtained by it than can be expected if vertical photographs were alone used. Moreover, additional advantages are

(1) The density of height control determined on the ground can be greatly reduced.

(2) The plotting instrument is simple in construction and includes no parts of high mechanical precision.

(3) The instrument will be inexpensive, thus making it possible to increase the speed of mapping by increasing the number used.

(4) The method both in theory and practice is simple and easy.

COMMENT ON DR. CHIH-CHO WANG'S PAPER

By O. M. Miller

Dr. Chih-Cho Wang is to be congratulated on suggesting a practical technique for utilizing high oblique photographs in conjunction with vertical photographs. The use of verticals for planimetry and obliques for height determinations is a very sound principle as Dr. Wang clearly demonstrates, and though

[4] R. Finsterwalder: Der unregelmässige Fehler der räumlichen Doppelpunkteinschaltung. Allgemeine Vermessungs-Nachrichten. 1932. Nr. 41, 42, 43.

the analytical solution may seem somewhat cumbersome it enables a clear picture to be obtained of the probable accuracy obtainable by this procedure. On the other hand the practical application of the method becomes very simple once an instrument is available. The proposed instrument is in many respects similar to the American Geographical Society's single eyepiece plotter except that in the latter the primary axis of exterior orientation is the Z axis and a single measuring column is utilized, this being made to move over the plotting board by a polar coordinate mechanism centered at the position of the air station.

It is not made quite clear in the paper whether it is intended to include a lens system in the proposed instrument, but it is assumed that this is so as otherwise practical difficulties of eye accommodation may arise. In the American Geographical Society's instrument this difficulty is avoided by means of the pinhole mirror device and a viewing telescope.

On one point it is felt that the author is perhaps unduly optimistic in his statement that his proposed instrument requires no parts of high mechanical precision. Though the precision requirements are possibly not so high as for a precise stereoscopic plotting instrument, nevertheless to set the exterior orientation of the photograph to within the tolerances suggested by the author requires standards for construction and adjustment that are by no means light.