THE CONTRIBUTION TO MAPPING ERRORS OF VARIOUS IMPERFECTIONS IN THE CONSTRUCTION AND MANIPULATION OF MULTIPLEX EQUIPMENT

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IN THE series of instruments used in the Multiplex method of aerial mapping there are some errors which have a significant effect on the accuracy of the maps produced, but which are small enough to be within the realm of good instrument design and manufacture and careful manipulation. It is this class of errors to which this paper is devoted.

There are relatively few mechanical requirements that are not adequately covered by ordinarily good design and workmanship. A study of the optical

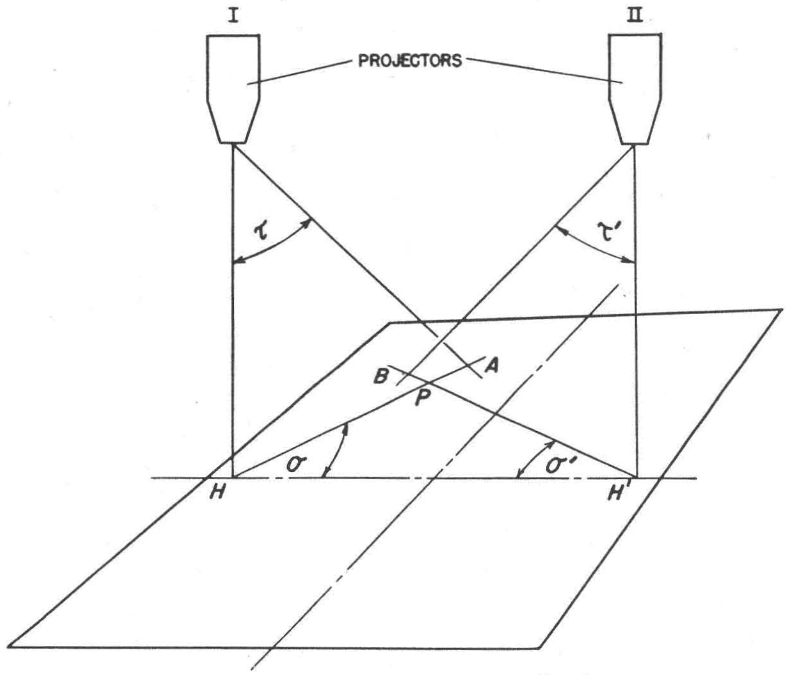

FIG. 1

requirements shows that the recognized standards of perfection for optical design are adequate with one, and possibly two exceptions. The ordinary conception of good definition for the center of the field of view is certainly adequate. A certain amount of deterioration of the image quality for large filed angles is usually expected. A point near the ege of the field of one projector may lie near the center of the field of the adjacent projector. If there is a marked difference in the definition of the two overlapped points, trouble may be experienced in matching. Whether there is any systematic relation between this condition of unequal definition and errors in reading the model is not fully known, but danger of a haphazard error of observation is certainly present. This is a question that may be worth investigating in the future.

The one optical aberration that causes errors in mapping that far exceeds all other errors due to optical causes is distortion. Therefore, the major part of this paper is devoted to a study of the effect of distortion of the optical systems on errors of mapping. The errors of polarimetry are small in comparison to those in the third dimension and are given little attention here. In most cases they are negligible.

In the work that follows, no attempt has been made to designate which unit of the optical systems is being considered. The results can be assumed to represent the total of all the optical systems involved. The examples given show distortion data of a 6 inch Metrogon lens, and it is assumed that these data transferred to the model without any modification by the printer or projector lenses. In actual practice this is not the case, since there is a compensation of the distortion of the camera lens by the printer lens. The equipment is, therefore, subject to design errors that are appreciably less than those shown in this analysis.

Fig. 1 shows two projectors forming a model of a plane surface by means of two overlapped images, at the nominal image plane $HH'P$. P is any point in this plane and the lines HP and $H'P$ are in the plane. The distances PA and PB are the linear distortions t and t' in this plane. As planes above or below the nominal image and parallel to it are taken successively, the points A and B travel along the projections of the lines HP and $H'P$ on these planes. Fusion occurs at a height such that these points determine a line at right angles to the line $H-H'$.

If the center of coordinates is chosen in the nominal image plane, the X component of the motion of the Point A is

$$X = Z \tan \tau \cos \sigma$$

and

$$X' = Z \tan \tau' \cos \sigma'.$$

The X component of the distortion in the nominal image plane is

$$X = t \cos \sigma$$

$$X' = t' \cos \sigma'.$$

The point of fusion, as defined, is given by

1. $$Z (\tan \tau \cos \sigma + \tan \tau' \cos \sigma') = t \cos \sigma + t' \cos \sigma'.$$

If the value of the distortion in the nominal image plane is changed, the value of Z is changed in accordance with the equation

2. $$dZ (\tan \tau \cos \sigma + \tan \tau' \cos \sigma') = dt \cos \sigma + dt' \cos \sigma'.$$

The value of the distortion can be changed in accordance with the conception of calibrated focal length, in which case it is changed in agreement with the equation

$$F_A \tan \tau + t_A = F_c \tan \tau + tc$$

where the subscript A denotes axial values and c calibrated values. This can be written:

$$dF \tan \tau = dt.$$

Similarly,

$$dF \tan \tau' = dt'.$$

Putting these values in 2

$$dZ = dF.$$

The introduction of a calibrated focal length produces a constant change of level for all points of the model. This does not change the faithfulness of reproduction of the object by the model in any way except to change its scale. Even this would not occur if the magnification of the printer remains unchanged. It seems that the calibrated focal length is not a valid criterion for judging the appropriateness of any distortion correction of optical systems used in Multiplex equipment. An investigation of the effect of actual changes of distortion correction is necessary to guide such judgment.

Let the center of coordinates in the nominal image plane be also in the geometric center of the model.

Let S be the width of a single projected image.

Let ρ be the fractional part of the image that is overlapped.

Then the width of the overlapped area is $S\rho$.

Let A be the fractional part of one-half of the overlapped area that is the X coordinate of the point P.

Let l, l' be the distances from H and H' to P.

Let h be the projection distance.

The X coordinate of the point P is $.5S\rho A$.

The X coordinate of H is $-.5S(1-\rho)$.

The X coordinate of H' is $-.5S(1-\rho)$.

Then:

$$\cos\sigma = \frac{.5S(1-P) + .5S\rho A}{l}$$

$$\cos\sigma' = \frac{.5S(1-P) - .5S\rho A}{l'}$$

$$\tan\tau = \frac{l}{h-\bar{z}}$$

$$\tan\tau' = \frac{l'}{h-\bar{z}}$$

Where $\bar{z}$ is elevation above the nominal image plane of the point in question.

$$\tan\tau\cos\sigma + \tan\tau'\cos\sigma' = \frac{.5S}{h-\bar{z}}\left[(1-\rho)+\rho A+(1-\rho)-\rho A\right] = \frac{S}{h-\bar{z}}(1-\rho)$$

and

$$3. \quad Z = \left\{\frac{t}{\tan\tau}\left[(1-\rho)+\rho A\right] + \frac{t'}{\tan\tau'}\left[(1-\rho)-\rho A\right]\right\}\frac{.5}{(1-\rho)}$$

It is seen that if $t/\tan\tau$ is a constant, this reduces to

$$Z = .5\,\frac{t}{\tan\tau} = K$$

Since Z is a constant there is no warping of the model from its correct shape due to a distortion correction that satisfies the condition that $t/\tan\tau$ shall be a constant. This indicates that when the residual distortion errors cannot be reduced further, the balancing of these errors should be chosen to approximate the above condition as closely as possible. Fig. 2 shows three different possible

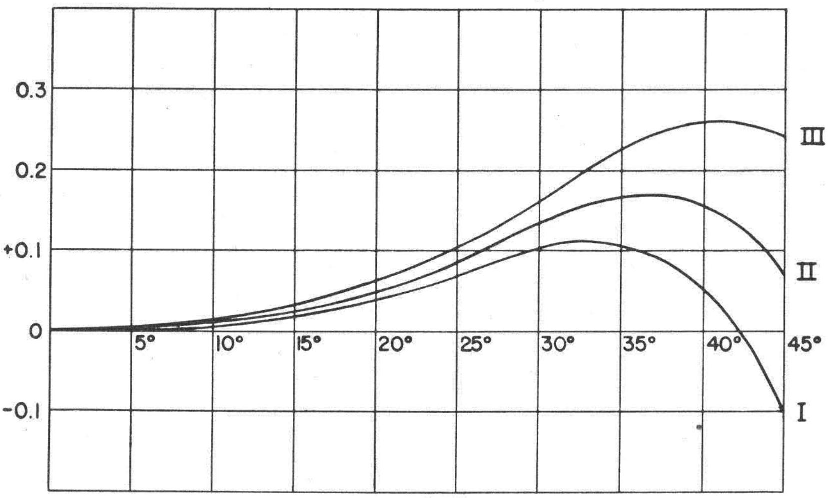

FIG. 2

distortion corrections of the same lens. Threse curves are plotted in the usual way, the linear distortion against the field angle.

Fig. 3 shows the same curves, but plotted with tan τ as the abscissa and $t/\tan\,\tau$ as the ordinate. It is evident that curve no. II more nearly approaches

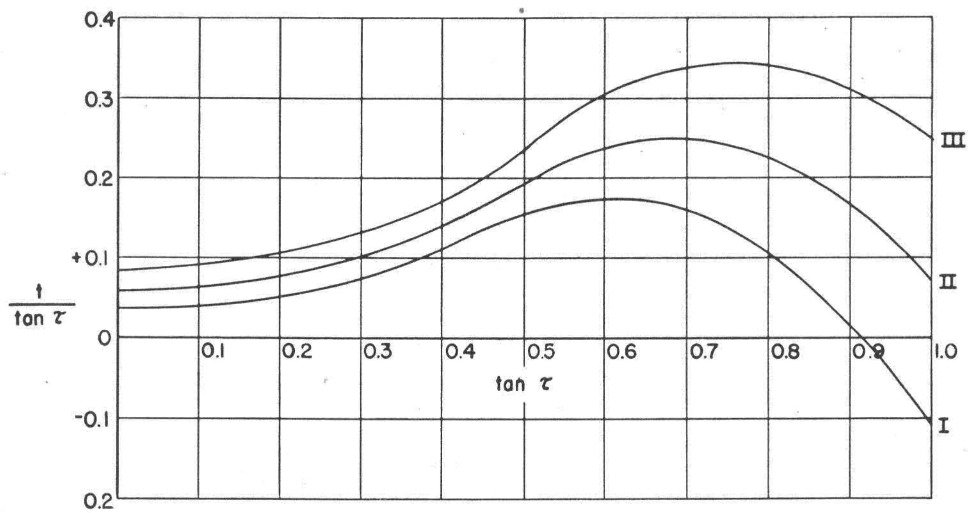

FIG. 3

the correct condition than the other two. Numbers I and III are about equal in this regard. It is noteworthy that their steepest slopes occur at different places along their lengths.

So far it has been assumed that the two projectors are correctly adjusted as they would be if there were no distortion. In the presence of distortion it is customary to remove the resulting of parallax by one of several routines of adjustment. Since this adjustment affects the shape of the model, it is necessary to add it to the analysis before investigating the effect on the model of using different types of distortion correction.

It is easily seen by similar triangles that a translation along the X axis produces

$$dZ = dX \frac{h}{s(1-\rho)}$$

A translation a'ong the Y axis produces no effect on the model.

The effect of a translation along the Z axis is shown by

$$dl = dh \tan \tau = \frac{dX}{\cos \sigma}$$

$$SO\ dZ = \frac{hdh}{s(1-\rho)} \tan \tau \cos \sigma$$

$$= \frac{.5}{(1-\rho)} [(1-\rho) + \rho A] dh$$

This is zero when

$$\rho A = -(1-\rho)$$

which is the coordinate of the point H. The line at right angles to the X axis passing through H is the axis of rotation of the model.

If K is the angle of swing,

$$dK = d\sigma$$

$$h \tan \tau \cos \sigma = .5S(1-\rho) + .5S\rho A$$

$$-h \tan \tau \sin \sigma\, dK = .5S\rho dA - dX$$

$$dZ = -\frac{h^2}{S(1-\rho)} \tan \tau \sin \sigma\, dK$$

$$dZ = -\frac{h}{S(1-\rho)} y\, dK$$

This is a rotation of the model about the X axis.

When a projector is rotated around the y axis, let the angle of tip be ω. Let μ be the projection of τ on the XZ plane. Then

$$\tan \tau \cos \sigma = \tan \mu$$

$$X = h \tan \mu$$

$$dX = h \sec^2 \mu\, d\mu = h \sec^2 \mu\, d\omega$$

$$dX = h(1 + \tan^2 \mu)\, d\omega$$

$$dX = h(1 + \tan^2 \tau \cos^2 \sigma)\, d\omega$$

Let m be the projection of the principal ray on a plane parallel to the XZ plane and passing through the point P. From the triangle formed by the principal ray, its projection and the line parallel to the y axis and passing through the center of perspective of the projector it is seen that

$$\frac{dy}{y} = \frac{dm}{m}$$

where

$$m = \frac{X}{\sin \mu}$$

From the triangle formed by the projected principal ray, h and X it is seen that

$$dm = dX \sin \mu$$

$$dy = \frac{y}{x} \sin^2 \mu \, dx$$

$$dy = \tan \sigma \sin^2 \mu \, dx$$

$$dy = h \tan \sigma \tan^2 \tau \cos^2 \sigma \, d\omega$$

$$dy = h \tan^2 \tau \sin \sigma \cos \sigma \, d\omega$$

$$dZ = \frac{h^2}{S(1 - \rho)} (1 + \tan^2 \tau \cos^2 \sigma) \, d\omega$$

The effect of tilt of the model is obtained by reversing the X and y of the tip equations. Thus

$$dX = h \tan^2 \tau \sin \sigma \cos \sigma \, d\nu$$

$$dy = h(1 + \tan^2 \tau \cos^2 \sigma) \, d\nu$$

$$dZ = \frac{h^2}{S(1 - \rho)} \tan^2 \tau \sin \sigma \cos \sigma \, d\nu$$

where ν is the projection of τ on the yz plane, and $d\nu$ is the angle of tilt. When the y parallax is removed from any point, c, by a tip of the projector opposite this point,

$$d\omega = \frac{dy_c}{h \tan^2 \tau_c \sin \sigma_c \cos \sigma_c}$$

$$dy_c = t_c' \sin \sigma_c' - t_c \sin \sigma_c$$

where the primed quantities refer to the projector that is not tipped.

$$d\omega = \frac{t_c' \sin \sigma_c'}{h \tan^2 \tau_c \sin \sigma_c \cos \sigma_c} = \frac{t_c}{h \tan^2 \tau_c \cos \sigma_c}$$

$$= \frac{1}{h \tan \tau_c \cos \sigma_c} \left[\frac{t_c' \sin \sigma_c'}{\tan \tau_c \sin \sigma_c} - \frac{t_c}{\tan \tau_c} \right]$$

$$= \frac{1}{h \tan \tau_c \cos \sigma_c} \left[\frac{t_c'}{\tan \tau_c'} - \frac{t_c}{\tan \tau_c} \right]$$

$$dZ = \frac{h}{S(1 - \rho)} \, \frac{1 + \tan^2 \tau \cos^2 \sigma}{\tan \tau_c \cos \sigma_c} \left[\frac{t_c'}{\tan \tau_c'} - \frac{t_c}{\tan \tau_c} \right]$$

If the point chosen is the corner of the model,

$$\tan \tau_c \cos \sigma_c = \frac{.5S}{h}$$

$$\tan^2 \tau \cos^2 \sigma = \left(\frac{.5S}{h}\right)^2 [(1 - \rho) + \rho A]^2$$

when the left-hand projector is tipped.

$$dZ = \frac{h^2}{.5S^2} \frac{1}{(1 - \rho)} \left[\frac{t_c'}{\tan \tau_c'} - \frac{t_c}{\tan \tau_c}\right] \left\{1 + \left(\frac{.5S}{h}\right)^2 [(1 - \rho) + \rho A]^2\right\}$$

Let

$$K_c = \frac{t_c'}{\tan \tau_c'} - \frac{t_c}{\tan \tau_c}$$

$$dZ = K_c \frac{.5}{(1 - \rho)} \left\{\frac{h^2}{.25S^2} + [(1 - \rho) + \rho A]^2\right\}$$

If the parallax is removed from the other corner by tipping the right hand projector, the total effect is

$$dZ_t = K_c \frac{.5}{(1 - \rho)} \left\{\frac{2h^2}{.25S^2} + [(1 - \rho) + \rho A]^2 + [(1 - \rho) - \rho A]^2\right\}$$

$$= K_c \frac{1}{(1 - \rho)} \left\{\frac{h^2}{.25S^2} +' (1 - \rho)^2 + \rho^2 A^2\right\}$$

This shows a shift of the whole model up or down plus a change of shape. If the center of coordinates is shifted accordingly, the change of shape becomes

$$dZ_t = K_c \frac{\rho^2}{(1 - \rho)} A^2$$

To evaluate K_c, let s' be the length of the model in the y direction. Then

$$\tan \tau_c = \frac{.5}{h} \sqrt{s^2 + s'^2}$$

$$\tan \tau_c' = \frac{.5}{h} \sqrt{s^2 (2\rho - 1)^2 + s'^2}$$

A commonly used method of removing y parallax consists of a vertical adjustment of one projector to remove the parallax on its side of the model, and a tip of the same projector to remove it on the other side. In this case the tip of the projector must be twice that required by the previous method. Thus the tip required produces a change in the model of

$$dZ = K_c \frac{1}{(1 - \rho)} \left\{\frac{h^2}{.25S^2} + [(1 - \rho) + \rho A]^2\right\}$$

The vertical adjustment required is

$$dh = \frac{h}{y_c} dy_c$$

In this case

$$dy_c = -K_c$$

and

$$y_c = h \tan \tau_c \sin \sigma_c$$

$$dh = -\frac{K_c}{\tan \tau_c \sin \sigma_c}$$

$$= -\frac{K_c}{\tan \tau_c \cos \sigma_c \tan \sigma_c}$$

$$\tan \sigma_c = \frac{S'}{S(.5 - \rho)}$$

$$dh = -\frac{K_c}{\dfrac{.5S}{h} \dfrac{S'}{s(.5 - \rho)}} = -\frac{K_c h (.5 - \rho)}{.5S'}$$

$$dZ = -\frac{K_c h (.5 - \rho)}{S'(1 - \rho)} (1 - \rho) + \rho A$$

$$dZ_t = \frac{K_c}{(1 - \rho)} \left\{ \frac{h^2}{.25S^2} - \frac{h(.5 - \rho)(1 - \rho)}{S'} + [(1 - \rho) + \rho A]^2 - \frac{h(.5 - \rho)}{S'} \rho A \right\}$$

$$dZ_t = \frac{K_c}{(1 - \rho)} \left\{ \frac{h^2}{.25S^2} - \frac{h(.5 - \rho)(1 - \rho)}{S'} + (1 - \rho)^2 + 2\rho(1 - \rho)A + \rho^2 A^2 - \frac{h(.5 - \rho)}{S'} \rho A \right\}$$

Neglecting the constant terms,

$$dZ_t = \frac{K_c \rho}{(1 - \rho)} A \left[\rho A + 2(1 - \rho) - \frac{h(.5 - \rho)}{S'} \right]$$

When this is horizontalized

$$dZ_t = K_c \frac{\rho^2}{(1 - \rho)} A^2$$

The two methods are equivalent in so far as change of shape of the model is concerned. They differ by the amount of the vertical shift of the model.

A third method of eliminating y parallax consists of a vertical adjustment of one projector and a tip of the other projector. This method again produces the same change of shape of the model when a horizontalizing operation is added, but the vertical shift of the whole model is different from either of the other two models.

It seems likely that further study of these methods of removing y parallax might be useful in connection with the problem of extending beyond control. If the second method is used, adjusting only the last added projector, the horizontalizing operation is not possible. The result is that each successive model is tipped with respect to its predecessor and the extended terrain assumes a cylindrical shape instead of the proper plane.

To use equation 3 in conjunction with the curves of Fig. 3, let B be the fractional part the length of the model that is the y coordinate of the point P. Then

$$y = .5B\, s'$$

$$\tan \tau = \frac{.5}{h} \sqrt{B^2S'^2 + S^2[(1-\rho) + \rho A]^2}$$

$$\tan \tau' = \frac{.5}{h} \sqrt{B^2S'^2 + S^2[(1-\rho) - \rho A]^2}$$

Fig. 4 shows a series of cross sections of the model for each of the three types of distortion correction. These are cross sections taken parallel to the X

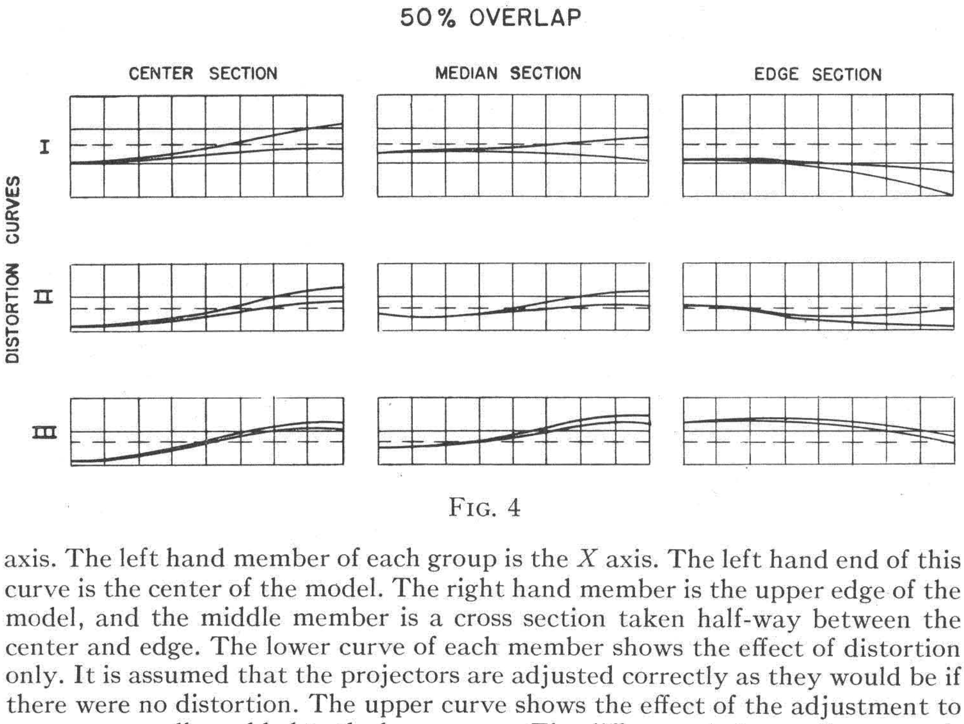

FIG. 4

axis. The left hand member of each group is the X axis. The left hand end of this curve is the center of the model. The right hand member is the upper edge of the model, and the middle member is a cross section taken half-way between the center and edge. The lower curve of each member shows the effect of distortion only. It is assumed that the projectors are adjusted correctly as they would be if there were no distortion. The upper curve shows the effect of the adjustment to remove y parallax added to the lower curve. The difference between the curves is the change of cross section due to adjustment.

It is noteworthy that there is little significant difference in shape of corresponding curves for the different distortion corrections when the effect of adjustment is added. There is, however, a difference in the relative placement of the members of a group. In changing from type I to type III distortion correction the upper edge of the model rises to a better average position. The change in this respect is less between II and III than between I and II. These curves are drawn for a 50% overlap.

Fig. 5 shows a similar set of curves for a 75% overlap. The warp of the model is seen to be increased by a considerable amount. Otherwise same remarks apply for this set as the previous set.

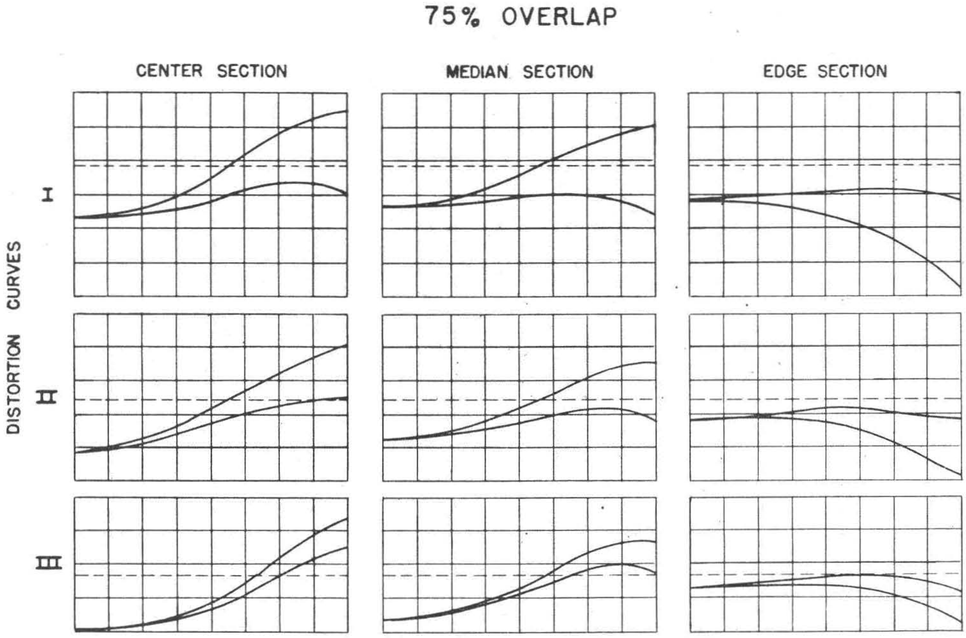

FIG. 5

A study of these curves indicates some advantage in using a type of distortion correction similar to II or III. It also indicates that there is a fairly large permissible variation in the distortion corrections of the optical systems used for Multiplex mapping.

The large increase in warp of the model when the overlap is increased is perhaps surprising and warrants investigation. Equation 3 can be written

$$Z = .5\left[\frac{t}{\tan\tau} + \frac{t'}{\tan\tau'}\right] + \frac{.5\rho A}{(1-\rho)}\left[\frac{t}{\tan\tau} - \frac{t'}{\tan\tau'}\right]$$

When $\rho = 1$, the first term is $t/\tan\,\tau$ and the second term is indeterminate.

$$\underset{\rho \doteq 1}{L}\ Z = \frac{t}{\tan\tau} + \frac{.5A\left[\frac{t}{\tan\tau} - \frac{t'}{\tan\tau'}\right] + .5\rho A\, d\left[\frac{t}{\tan\tau} - \frac{t'}{\tan\tau'}\right]/d\rho}{-1}$$

$$= \frac{t}{\tan\tau} - .5\mathrm{A}\, d\left[\frac{t}{\tan\tau} - \frac{t'}{\tan\tau'}\right]/d\rho$$

$$\tan^2\tau = \frac{.25}{h^2}\left\{B^2S'^2 + S^2[(1-\rho) + \rho A]^2\right\}$$

$$2\tan\tau\, d(\tan\tau) = \frac{.5S^2}{h^2}\,[(1-\rho) + \rho A]\,(A-1)dp$$

$$dp = \frac{4h^2}{S^2}\,\frac{\tan\tau}{[(1-\rho)+\rho A](A-1)}\,d(\tan\tau)$$

$$= 4\frac{h^2}{S^2}\,\frac{\tan\tau}{A(A-1)}\,d(\tan\tau)$$

when $\rho=1$
Similarly

$$dp = 4\frac{h^2}{S^2}\frac{\tan\tau'}{A(A+1)}\,d\,(\tan\tau')$$

So

$$\underset{P\doteq 1}{Z} = \frac{t}{\tan\tau} + \frac{.25A^2}{\tan\tau}\frac{S^2}{h^2}\frac{d\left(\dfrac{t}{\tan\tau}\right)}{d\,(\tan\tau)}$$

This shows that the warp of the model does not become zero when the overlap is 100%.

To investigate the effect on the shape of the model of a misadjustment of tip or tilt, the constant terms of the equations can be disregarded. The equations for tip can be put in the form

$$d\omega = \frac{S(1-\rho)}{h^2}\frac{1}{\tan^2\tau\cos^2\sigma}\,dZ$$

$$dy_\omega = \frac{S(1-\rho)}{h}\tan\sigma\,dZ$$

Similar equations for tilt are

$$d\nu = \frac{S(1-\rho)}{h^2}\frac{1}{\tan^2\tau\sin\sigma\cos\sigma}\,dZ$$

$$dy_\nu = \frac{S(1-\rho)}{h}\frac{dZ}{\tan\sigma}$$

For a square, 45° model, 360 mm. projection distance, and 50% overlap these become, at the corner of the model

$$d\omega = .00393\,dZ$$
$$dy_\omega = .707\,dZ$$

and

$$d\nu = .00393\,dZ$$
$$dy_\nu = .707\,dZ$$

In these equations, the model has not been horizontalized after tipping the projector. If this is done, the error for each side of the model with respect to the center becomes .05 mm. or $\pm.025$ mm. In the tilting operation no horizontalizing operation is possible, since one corner goes up and the other directly above or below it in the plane of the model goes down, while the two corners on the side of the model of the projector that is tilted are unchanged. This means that the error is $\pm.1$ mm. Thus an angular error, or y parallax error of tilt causes four times the warp of the model as an equal error of tip. If the model is to be held to $\pm.05$ mm., the y parallax must be reduced to .035 mm. This is less than .002″. It seems entirely possible that accidental errors of adjusting the projectors are this large or even greater.