EQUATION OF A LENS DISTORTION CURVE

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IT IS commonly recognized by users and designers of lenses that it is impossible to eliminate entirely, or to compensate for, all types of lens aberrations. For many purposes perfection in this respect can be so nearly attained that the effects of the residual errors are of no significance. In the case of wide angle camera lenses employed in securing vertical photographs for use in preparing topographic maps with multiplex instruments, small residual distortion errors cannot be ignored as they introduce inaccuracies in the map in excess of those permitted by commonly accepted standards of map accuracy. Therefore some means of compensating for the distortion of the aerial camera lens must be provided if the resulting maps are to pass the customary tests of map accuracy.

The necessity of compensating for distortion errors was called to attention when multiplex operators of the Geological Survey demonstrated that the surface of stereoscopic models was systematically warped by the uncompensated effects of distortion inherent in the camera lens. Although the effects of distortion had been recognized prior to this time, it was not generally recognized what a serious effect on map accuracy would result from neglecting these relatively small residual errors. It thus became necessary to reduce the effects of camera lens distortion, to values that would be commensurate with errors permitted by the map accuracy specifications.

The first method of accomplishing this end was by use of a distortion correction chart computed, for, and plotted, to the scale of the stereoscopic model. This chart was used as a guide for the introduction of corrections to measurement of elevations in various parts of the model of such amounts as to compensate for the errors caused by the distortion of the aerial camera lens.[1] At best this was a tedious and not altogether satisfactory method. A much more complete compensation has since been devised, and is now in general use, based on the introduction of distortion of an equal amount but opposite sign by employing in the multiplex reduction printer a lens specially designed for that purpose. This special lens yields diapositive plates essentially free of the effect of camera lens distortion, thus making it possible to attain a satisfactory degree of map accuracy.

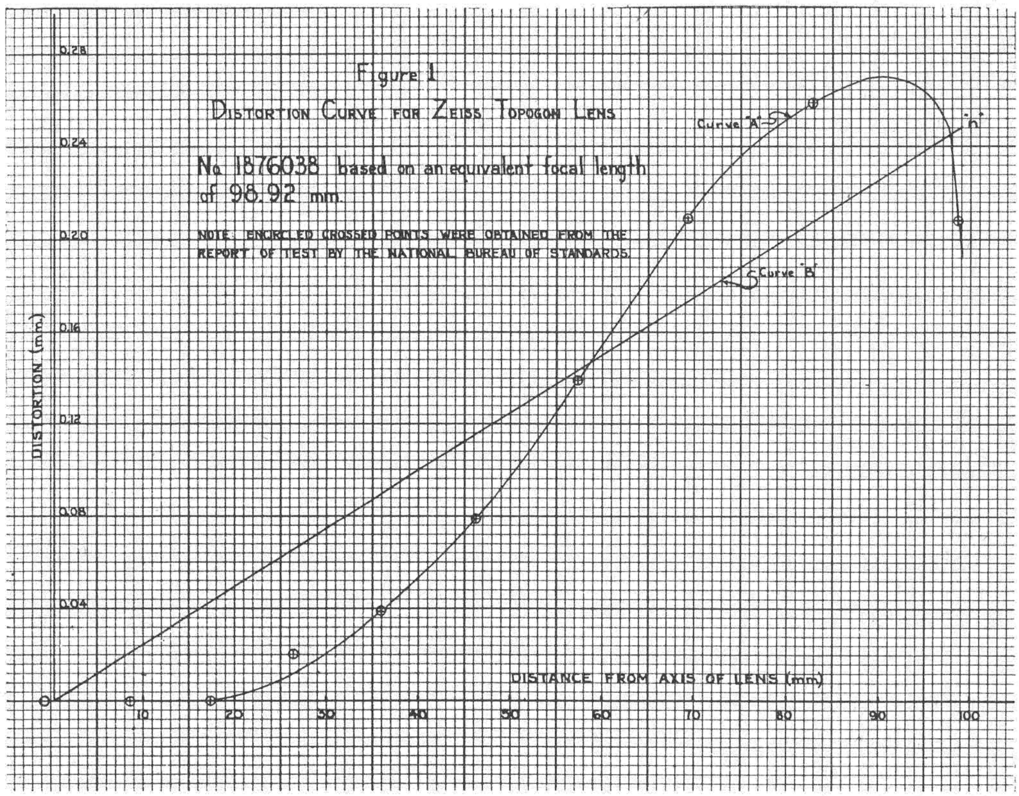
The first step necessary to the design of this special printer lens was a test of the aerial camera lens by the National Bureau of Standards. The lens selected for the experiment was the Zeiss Topogon lens which is installed in a Geological Survey camera. This lens had an equivalent focal length of 98.92 millimeters and distortions in the amounts indicated in Table I. Lines 1 and 2 of this table indicate the distances from the lens axis, in degrees and millimeters respectively, of the points at which distortion measurements were made. Line 3 gives the amount of distortion measured at these chosen points.

TABLE I. DISTORTION OF TOPOGON LENS NO. 1876038
Equivalent Focal Length 98.92 mm

Angular distance from axis	5°	10°	15°	20°	25°	30°	35°	40°	45°
Linear distance from axis (mm)	8.6	17.4	26.5	36.0	46.1	57.1	69.2	82.9	98.9
Distortion in millimeters	0.00	0.00	+0.02	+0.04	+0.08	+0.14	+0.21	+0.26	+0.21

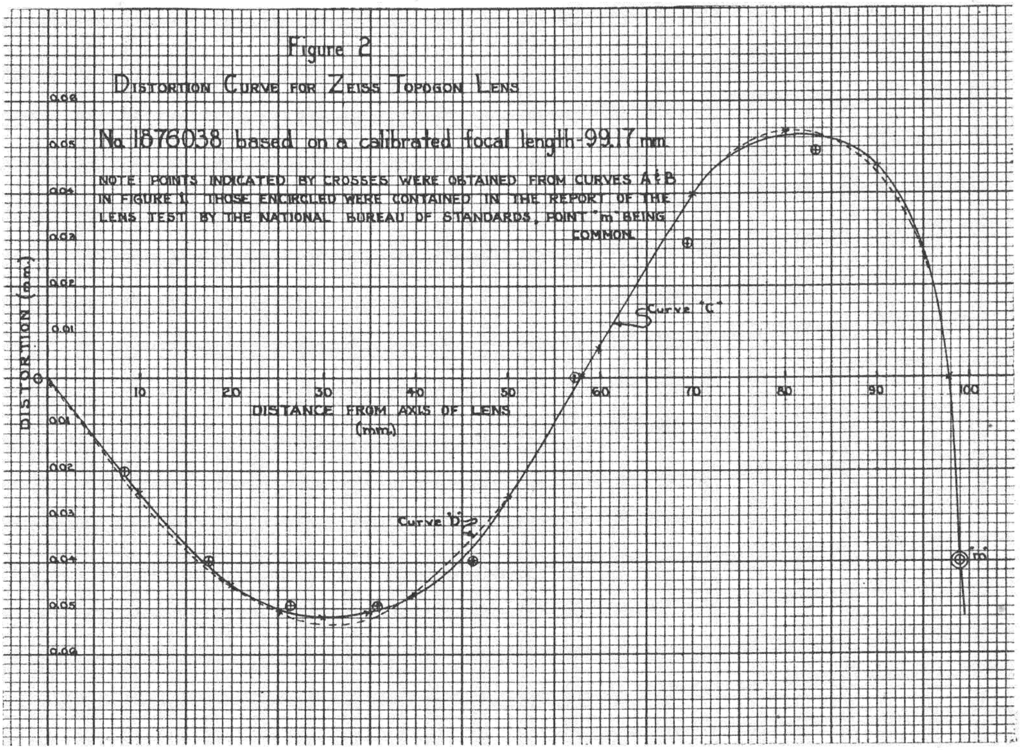
[1] Bean, R. K. Errors Affecting Multiplex Mapping. PHOTOGRAMMETRIC ENGINEERING, Vol. VI, No. 2.

If the values on line 2 are plotted as abscissae and those on line 3 as ordinates a distortion curve is delineated as shown by Curve "*A*" in Figure 1. This curve indicates that the distortion is always positive within the angular limits of the lens and is generally increasing as one proceeds outward, attaining a maximum value near the outer limits of the field of view. It was desirable in this case to adopt a calibrated focal length of such magnitude as to make the positive and



negative values of distortion equal, thus reducing greatly the maximum ordinate values. The method of converting values based on the equivalent focal length to values based on a calibrated focal length is demonstrated diagrammatically in Figure 3.

Points *a* and *d* indicate the intersection of rays *La* and *Lb* with the negative plane if no distortion were present. These rays, however, actually intersect the negative plane at points *c* and *f* and the distances $\overline{ac}$ and $\overline{df}$ are the amounts of distortion at points *a* and *d* respectively. If the equivalent focal length is considered to be increased by the amount Δf, the true positions of *a* and *d* would move to *b* and *e* and the value of the distortion at these points would then be *bc* and *ef* respectively, which by inspection is observed to be less than before. It now follows that by regulation of Δf the distortion of any single point can be reduced to zero. Now, if the change in the abscissa, as demonstrated by Figure 3, is plotted as shown in Figure 1, the result will be the definition of Curve "*B*." It is readily observed that the ordinate value of point *n* (Fig. 1) could be assigned any value, equal to Δf, that would not cause the calibrated focal length to exceed the depth of focus of that particular lens. In this particular case the calibrated focal length becomes equal to the equivalent focal length plus Δf or 99.17 mm. Obviously to obtain this value graphically with only Curve "A" known, Curve "B" must be drawn in such position as to nearly equalize algebraically the differences between the two curves, thus determining the value of Δf. If Curve "A" is subtracted from Curve "B" and the differences plotted, Curve "C" as shown in Figure 2 is obtained.



This compares quite accurately with the data determined by the National Bureau of Standards and shown in Table II.

TABLE II. DISTORTION OF TOPOGON LENS NO. 1876038

Calibrated Focal Length 99.17 mm

Angular distance from lens axis	5°	10°	15°	20°	25°	30°	35°	40°	45°
Linear distance from lens axis (mm)	8.6	17.5	26.6	36.1	46.2	57.3	69.5	83.2	99.2
Distortion in millimeters	−0.02	−0.04	−0.05	−0.05	−0.04	0.00	+0.03	+0.05	+0.04

Curve "C" (Fig. 2), being indicative of the refined values of distortions actually appearing in the stereoscopic model, was used in the determination of corrections embodied graphically in distortion correction charts of the same scale as the conventional stereoscopic model. It is the purpose of this paper to demonstrate a simple method of determining the rectangular equation of this curve.

It is assumed that the curve most nearly approaching Curve "C" is of the fifth power, whose roots are 0, ± 58, ± 98. Its general equation is:

$$Y = x(x^2 - \overline{58}^2)(x^2 - \overline{98}^2)C \text{ or}$$

$$= x(x^2 - 3364)(x^2 - 9604)C \qquad \text{Equation 1}$$

From Curve "C" substitution of "X" and "Y" values are made in Equation 1 and the equation solved for C to determine its value for each point at which

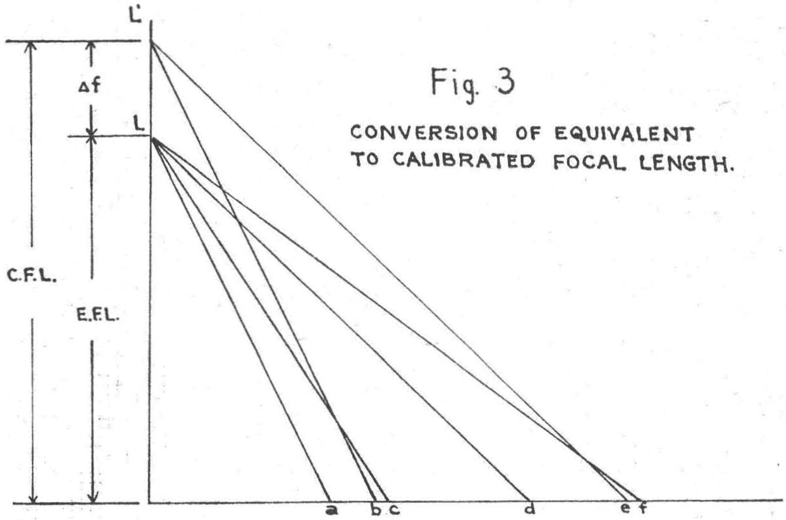


Fig. 3
CONVERSION OF EQUIVALENT TO CALIBRATED FOCAL LENGTH.

the substitution was made. A record of the values of C determined in this way, and the average adopted is recorded in Table III.

TABLE III. VALUES OF C

Expressed as Reciprocals with the Numerator, Unity Omitted

C_{10}	−12,400,000,000
C_{20}	−12,110,000,000
C_{30}	−12,120,000,000
C_{40}	−11,780,000,000
C_{50}	−11,790,000,000
C_{60}	−14,170,000,000
C_{70}	−12,940,000,000
C_{80}	−14,660,000,000
C_{90}	−13,900,000,000
C_{av}	−12,870,000,000

Table IV is a comparison of corresponding values taken from Curves "A" and "B" and that determined by Equation 1 utilizing the average value of C. The maximum values of divergence as shown in this table occur in the vicinity of the maximum and minimum points of Curve "D." It was found that the equation would fit the plotted curve more closely when the absolute values of Y in Equation 1 were reduced 10%. After application of this correction the final equation may be expressed as either:

$$Y = 0.9x\,(x^2 - \overline{58^2})\,(x^2 - \overline{98^2})C$$

$$\text{where } C = \frac{1}{1287 \times 10^7} \qquad \text{Equation 2}$$

or

$$Y = x(x^2 - \overline{58})^2\,(x^2 - \overline{98^2})C$$

$$\text{where } C = \frac{1}{143 \times 10^8} \qquad \text{Equation 3}$$

It is obvious that the discrepancy between the two curves is too small to be of consequence.

TABLE IV. COMPARISON OF CURVES "*C*" AND "*D*"

x	10	20	30	40	50	60	70	80	90
	mm	mm	mm	mm	mm	mm	mm	mm	mm
Y from Curves "*A*" and "*B*"	−0.025	−0.045	−0.053	−0.048	−0.026	+0.0060	+0.039	+0.053	+0.046
Y from Equation 1	−0.024	−0.042	−0.050	−0.043	−0.024	+0.0066	+0.039	+0.060	+0.050
Difference	0.001	0.003	0.003	0.005	0.002	0.0006	0	0.007	0.004
Y from Equation 3	−0.026	−0.046	−0.055	−0.047	−0.026	+0.006	+0.035	+0.054	+0.045

It is well understood that due to imperfections in grinding, no lens is likely to follow any predetermined mathematical law. Nor will a given lens possess the same distortion characteristics at all points equidistant from its axis. While the procedure described is highly empirical it is of sufficient accuracy for those cases where graphical methods are employed. It would be more practical to first establish the equation of Curve "*A*" (Fig. 2). If this equation were diffierentiated and its maxima, minima and flex points determined, Curve "*B*" and Δf could be established analytically. The equation of the curve also lends itself to an exact relationship not afforded by the plotted curve.