MATHEMATICS OF THE RATIOGRAPH

Robert E. Altenhofen, Capt., Corps of Engineers

EDITOR'S NOTE: In Volume X, No. 2, PHOTOGRAMMETRIC ENGINEERING, the construction and use of the Ratiograph device was discussed. The use of this device eliminates laborious scaling of photo and map distances and the subsequent computations. A knowledge of the mathematics from which the device is derived is not essential for its use. However, the editor is of the opinion that the mathematical basis of the device is of sufficient interest to be published in our Journal.

THE Ratiograph device can be developed as an application of a classic curve . . . the logarithmic or equi-angular spiral. The analysis presented here is based on this assumption. The equation of this curve will be derived in its polar form. It will be demonstrated that the equation follows from the definition that the ratiograph curve is the locus of the end point of a radius vector which when swept through a unit angle increases its length by a constant ratio.

In Figure 1 consider the initial vector $OA=\rho_0$ and the subsequent vectors OB, OC, and OD resulting from the rotation of OA through the unit angle, θ, and multiples thereof.

From the definition of the curve, assuming the constant ratio to be R, it follows that $OB=R\rho_0$; $OC=R^2\rho_0$; $OD=R^3\rho_0$ since the vector has swept through the angles θ, 2θ, and 3θ respectively. It is seen that the exponent of R equals the coefficient of θ.

Therefore, for any angle, $n\theta$, the vector, $\rho=R^n\rho_0$.

As pointed out in the description of the Ratiograph, the ratio of a vector to the initial vector, ρ_0, is recorded opposite the intersection of its prolongation with a circle of convenient radius. This circular scale is represented in Figure 1 by $A'B'C'D'$ labeled respectively R^0, R, R^2 and R^3.

Properties of the curve $ABCD$ follow from a consideration of the angles ABO, BCO and CDO between chords and their respective radii vectors. If θ is a small angle the following equations are acceptable approximations:

$$\tan ABO = \frac{\rho_0\theta}{R\rho_0-\rho_0} = \frac{\theta}{R-1}$$

$$\tan BCO = \frac{R\rho_0\theta}{R^2\rho_0-R\rho_0} = \frac{\theta}{R-1}$$

$$\tan CDO = \frac{R^2\rho_0\theta}{R^3\rho_0-R^2\rho_0} = \frac{\theta}{R-1}$$

$$\therefore \quad \angle ABO = \angle BCO = \angle CDO.$$

Hence it follows that the angle between a tangent to the curve and the radius vector to the point of tangency is a constant. In passing from the chord to the tangent it is assumed that the angle θ approaches a limiting value of zero.

This assumption leads to a consideration of the differential analysis in deriving the equation of the Ratiograph curve.

From Figure 2,

$$\tan\phi = \frac{\rho d\theta}{d\rho} = K \text{ a constant.}$$

Rearranging terms

$$\frac{d\rho}{\rho} = \frac{1}{K}\,d\theta.$$

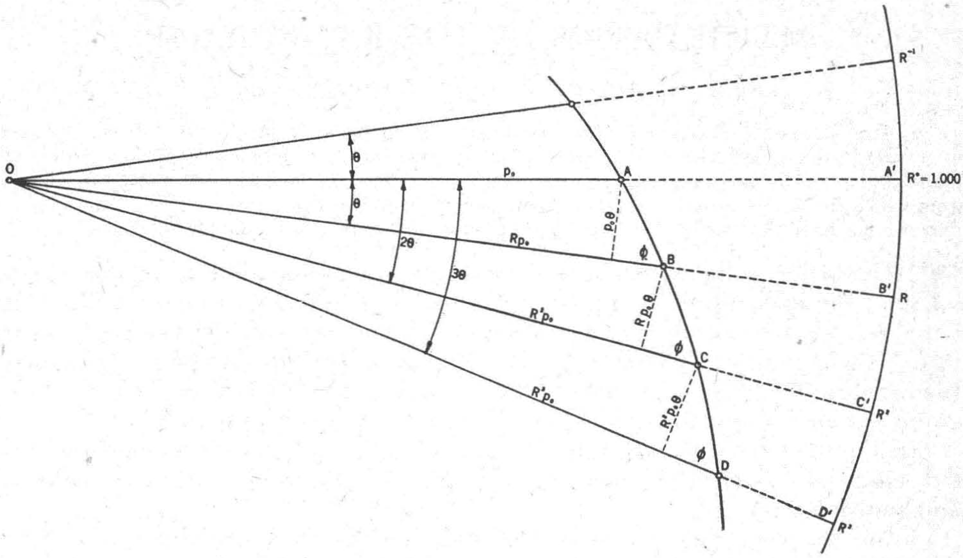

FIGURE 1

Integrating

$$\log \rho = \frac{\theta}{K} + \log c$$

$$\log \frac{\rho}{c} = \frac{\theta}{K}$$

$$\frac{\rho}{c} = e^{\theta/K}$$

or

$$\rho = ce^{\theta/K}. \tag{1}$$

Equation (1) defines a logarithmic spiral.

Constants c amd k may be determined by assuming both analytical and empirical conditions.

Analytical—When $\theta=0$, $\rho=\rho_0$

ρ_0 would be the average radial distance likely to be encountered on aerial photographs in common usage.

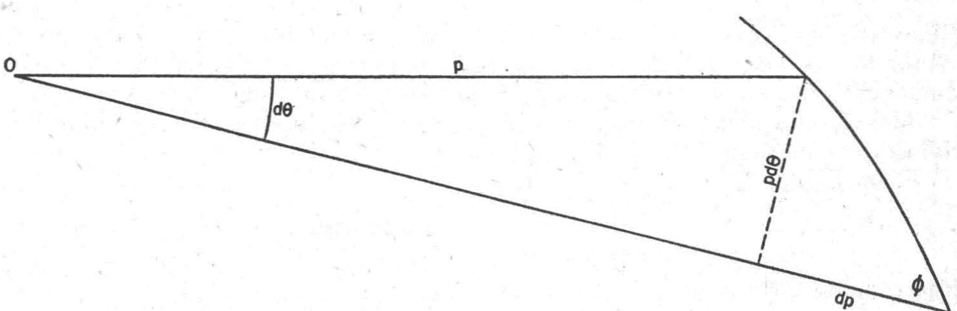

FIGURE 2

Substituting:

$$c = \rho_0$$

$$\rho = \rho_0 e^{\theta/K}.$$

Empirical—It was determined by trial that a useable spiral was fixed by assuming that the length of the radius vector increased 1% when rotated through 1°.

$$\therefore \quad \text{When } \theta = 1°, \qquad \rho = 1.01\rho_0.$$

Substituting

$$1.01\rho_0 = \rho_0 e^{1/K}$$

$$\frac{1}{K} = \log 1.01$$

$$\therefore \quad \rho = \rho_0 e^{\theta \log 1.01}$$

But the ratio,

$$R = \frac{\rho}{\rho_0} = e^{\theta \log 1.01}$$

$$\therefore \quad \log R = \theta \log 1.01$$

$$\theta = \frac{\log R}{\log 1.01} \tag{2}$$

and

$$\rho = R\rho_0. \tag{3}$$

The Ratiograph curve is computed from equations (2) and (3).
The calculation of θ for a few values of R is indicated as follows:

R	$\log R$	$\dfrac{\log R}{\log 1.01}$	θ	diff.
0.75	−0.1249387	−28.911625°	−28°54′42″	+1°19′52″
0.76	−0.1191864	−27.580506°	−27°34′50″	+1°18′50″
0.77	−0.1135093	−26.266788°	−26°16′00″	
0.97	−0.0132283	−3.061114°	−3°03′40″	+1°01′51″
0.98	−0.0087739	−2.030337°	−2°01′49″	+1°01′13″
0.99	−0.0043648	−1.010043°	−1°00′36″	+1°00′36″
1.00	0.0000000	0.000000°	0°00′00″	+1°00′00″
1.01	0.0043214	1.000000°	1°00′00″	+0°59′25″
1.02	0.0086002	1.990142°	1°59′25″	+0°58′49″
1.03	0.0128372	2.970611°	2°58′14″	
1.48	0.1702617	39.399662°	39°23′59″	+0°40′36″
1.49	0.1731863	40.076433°	40°04′35″	+0°40′20″
1.50	0.1760913	40.748669°	40°44′45″	