A GEOMETRICAL NOTE ON THE ISOCENTER

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AUTHOR'S NOTE: In this article I have noted the existence of a point which is of some importance in the geometry of the tilted photograph, but which, so far as I know, seems to have escaped previous consideration. I have called this new point the *negative isocenter*, and have, for contrast, called the ordinary isocenter the *positive isocenter*. The positive and negative isocenters, employed jointly, yield a theoretically simple graphical solution to the problem of rectifying a tilted photograph. Perhaps the reader will be able to find further uses for the negative isocenter.

AN IMPORTANT point in the geometry of aerial photogrammetry is the so-called isocenter. This is a point on a tilted photograph of flat terrain at which angles are equal to the corresponding angles on the ground. Are there other points on the tilted photograph possessing this isocenter property? This, and kindred questions, are natural and interesting questions to raise when considering the isocenter and its possible use in mapping processes. The following note is devoted to a brief examination of such questions. For the sake of completeness the subject is generalized beyond the situations that actually occur in photogrammetry.

We shall consider a central projection of plane p on plane p', L being the center of projection. We adopt the convention that angular directions on p (or p') are positive if they are counterclockwise when p (or p') is viewed from point L.

Definition 1. We define any point on p to be a *positive isocenter* on p if all angles on p having the point for vertex are invariant under the projection.

Theorem 1. If P is a positive isocenter on p, then its image, P', is a positive isocenter on p'.

Theorem 2. If p and p' are parallel, L not lying between p and p', then every point of p is a positive isocenter.

For p' is a perfect map of p.

Theorem 3. If p and p' are not parallel, then p cannot have two positive isocenters.

For suppose P and Q are two positive isocenters on p. Draw any circle s on p passing through P and Q, and let R be any other point on s. Then, since $\measuredangle QPR = \measuredangle Q'P'R'$ and $\measuredangle PQR = \measuredangle P'Q'R'$, it follows that $\measuredangle PRQ = \measuredangle P'R'Q'$, whence s projects into a circle s'. Thus all circles on p through P and Q project into circles on p' through P' and Q'. But this is impossible. For consider the circle on p, passing through P and Q and touching the vanishing line on p. This circle must project into a parabola. Thus the original assumption of the existence of two positive isocenters is incorrect.

Theorem 4. Let p and p' be non-parallel and let O be the foot of the perpendicular from L onto p, and let V be the image of V', the foot of the perpendicular from L onto p'. Then any line on p perpendicular to VO is parallel to any line on p' perpendicular to $V'O'$.

For (see Figure 1) draw VQ on p perpendicular to VO. Then

$$LQ^2 = LO^2 + OQ^2 = LO^2 + VO^2 + VQ^2 = LV^2 + VQ^2.$$

Therefore VQ is perpendicular to LV, and therefore to plane VLO. Hence any line on p perpendicular to VO is perpendicular to plane VLO. Similarly, any line on p' perpendicular to $O'V'$ is perpendicular to plane VLO. This proves the theorem.

Theorem 5. If p and p' are not parallel, then p has exactly one positive iso-

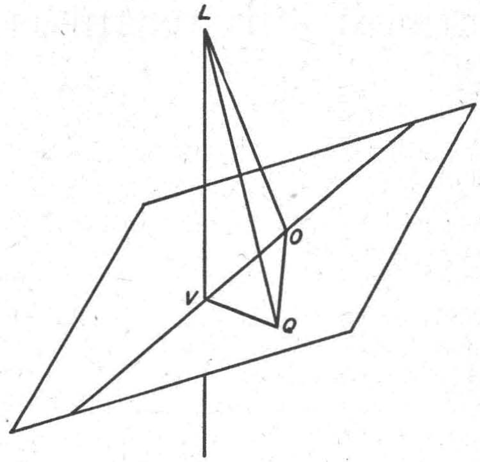

FIGURE 1

center, and it is the intersection with p of the bisector of the angle formed by the perpendiculars dropped from L onto p and p'.

We adopt the notation of theorem 4. Let P be any point on p and let W be the foot of the perpendicular from P on VO. Then, by theorem 4, W' is the foot of the perpendicular from P' on $V'O'$. Let I be the intersection with VO of the bisector of angle VLO. Draw (see Figure 2) WK perpendicular to LV to cut LV in K and LI in J. Then

$$IW/I'W' = JW/I'W' = LW/LW' = WP/W'P'.$$

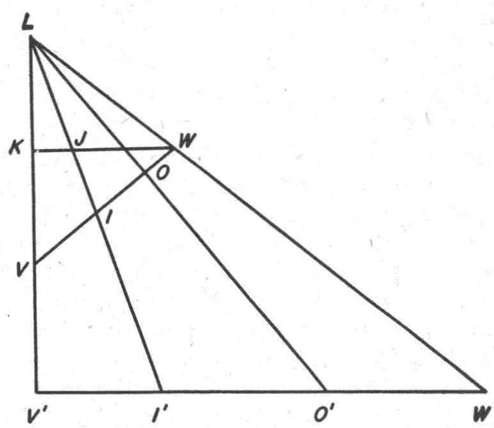

FIGURE 2

Hence $\measuredangle WIP = \measuredangle W'I'P'$. Similarly, if Q is any other point on p, $\measuredangle QIW = \measuredangle Q'I'W'$. Hence $\measuredangle QIP = \measuredangle Q'I'P'$, and I is a positive isocenter on p. By theorem 3, I is the only positive isocenter on p.

(For an analytical demonstration of the fact that I is a positive isocenter on p see page 24 of Church's *Elements of Aerial Photogrammetry*.)

Definition 2. We define any point on p to be a *negative isocenter* if all angles on p having the point for vertex project into equal but oppositely directed angles on p'.

Theorem 6. If P is a negative isocenter on p, then its image, P', is a negative isocenter on p'.

Theorem 7. If p and p' are parallel, L lying between p and p', then every point of p is a negative isocenter.

Theorem 8. If p and p' are not parallel, then p has exactly one negative isocenter, and it is the intersection with p of the external bisector of the angle formed by the perpendiculars dropped from L onto p and p'.

We may give a proof of this theorem analogous to that given for theorem 5.

An application to tilted photographs. The two isocenters may be jointly employed to graphically produce a rectified picture from a tilted photograph—after the tilt and swing of the photograph have been determined. We only briefly sketch the process here merely to show the theoretical possibility of the construction and do not consider any of the practical drafting difficulties and limitations that will most certainly arise, especially in the important case of small tilts.

Let p be the photograph and let V, I, O, and N be the nadir point, positive isocenter, principal point, and negative isocenter respectively. Let P be any point on p. On a piece of paper draw a line $O'I'V'N'$ similar and oppositely directed to $VION$. Draw $\sphericalangle O'I'P' = \sphericalangle OIP$ and $\sphericalangle V'N'P' = -\sphericalangle VNP$. Then P' is the map of P on a rectified picture which has V' for principal point. In this manner the photograph p may be graphically corrected for tilt without using any tilt displacement computations. (The above argument becomes apparent after an examination of Figure 3.)

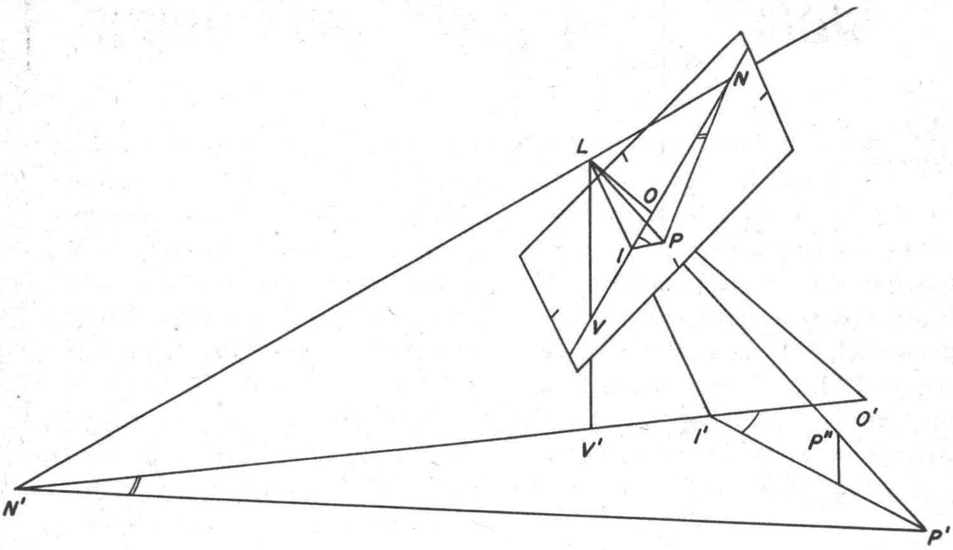

FIGURE 3

Each photograph of a flight strip may be rectified for tilt in the above way, and drawn to a common scale by taking $V'I' = kH \tan \frac{1}{2}t$, where k is a convenient constant of proportionality, H is the flight altitude, and t is the tilt of the photograph being rectified. We may then go on and eliminate relief displacement by the usual method of radial plotting, thus finally securing a true map of the ground.