ANOTHER PROOF FOR THE ANDERSON TILT EQUATION

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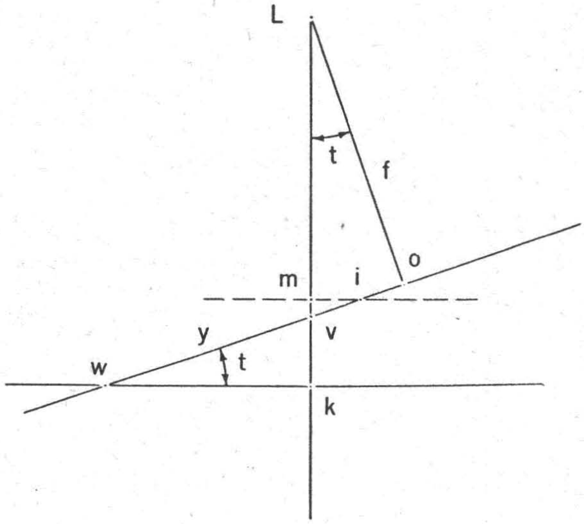
THE figure represents the principal plane of a tilted photographic print. The principal point is at o, nadir point at v, isocenter at i, and the point of perspective at L. The tilt of the photograph is the angle $t = oLv$, the principal line is vo, and the focal length is $f = Lo$. Let a (not necessarily on the principal line) be the photographic image of any object whose elevation is h above sea level. Let H be the height of L above sea level.



Let the line kw be the trace in the principal plane of a horizontal plane which contains the image a. Then w may be said to be the projection of a upon the principal line, and w is the foot of a perpendicular to the principal line from a. Let y represent the distance vw, positive in the direction ov.

Then the image a may be considered as lying in a truly vertical photograph whose focal length is Lk. The scale at a may be expressed in the customary manner

$$S = \frac{Lk}{H - h}. \tag{1}$$

In the right triangle Lov

$$Lv = f \sec t. \tag{2}$$

In the right triangle wkv,

$$\text{angle } kwv = \text{angle } t$$

and

$$kv = vw \sin t = y \sin t.$$

Hence

$$Lk = f \sec t + y \sin t. \tag{3}$$

Substituting in (1)[1]

$$S = \frac{f \sec t + y \sin t}{H - h} \cdot \qquad (4)$$

Or

$$S = \frac{f \sec t}{H - h} + \frac{\sin t}{H - h} y.$$

By calculus, the differential of the expression regarding S and y as variables, is

$$dS = O + \frac{\sin t}{H - h} dy$$

and the form of the derivative regarding y as the independent variable is

$$\frac{dS}{dy} = \frac{\sin t}{H - h} \cdot \qquad (5)$$

The resulting form is the rate of change of scale for images in the photograph with relation to their change in positions on, or parallel to, the principal line. This rate of change is considered as obtained by the method of dropped perpendiculars.[2]

Solving (5) for sin t,

$$\sin t = \frac{dS}{dy}(H - h). \qquad (6)$$

The scale at the isocenter i may also be customarily expressed as

$$S_i = \frac{Lm}{H - h} \cdot \qquad (7)$$

But obviously

$$Lm = Lo = f$$

and

$$S_i = \frac{f}{H - h}$$

whence

$$H - h = \frac{f}{S_i} \cdot \qquad (8)$$

Substituting in (6),

$$\sin t = \frac{f\left(\frac{dS}{dy}\right)}{S_i} \cdot \qquad (9)$$

[1] Church, Earl, *Elements of Aerial Photogrammetry*, page 22, Syracuse University Press, Syracuse, N. Y., 1943.

[2] Anderson, R. O., *Applied Photogrammetry*, 1939, Chapter 3, Edwards Brothers, Ann Arbor, Michigan.

Equation (9) is essentially the same as that of Mr. Anderson.[3] It is exact since no approximations were involved in the derivation. Equation (6) might be easier to employ in connection with tilt determination.

Equation (4) is significant in that it incorporates in a simple manner all the factors of tilt, relief, and flying height, into a single expression for the scale at any point on a photograph.

Equation (5) demonstrates that the rate of change in scale due to tilt is a constant along a straight line parallel to the principal line. It can be shown by further analysis that the statement is also true for any straight line across a photograph.

[3] Ibid., page 98, equation (E).