THE *AMS* METHOD OF TILT ANALYSIS*

Robert E. Altenhofen, Capt. C. E.

THE WRITER wishes to acknowledge the contribution of Mr. Andrew Bendixen of the Army Map Service to the development of the AMS method of tilt analysis. His original idea of rotating the resection circles was expanded by joint investigation into the practical procedure described in this article.

A NEW and unique method of tilt analysis has been developed at the Army Map Service. Photogrammetric research was undertaken because of the need for a rapid semi-graphical procedure to determine the tilt of an aerial photograph. Prerequisites of the method were that the resulting data be suitable for the computation of rectifier settings; that the procedure be simple; the geometric construction, self-checking; and that the data be accurate and applicable to various types of control. The AMS method possesses all of these characteristics.

The specific project which prompted the search for a simple analysis technique was the laying of a series of 1:25,000 photomaps from wide angle, high altitude photography. Specifications required agreement with existing line maps of the same scale. Since these maps were found to be consistent with trig lists, they were considered an adequate source of control points. It was realized that map points selected at random would not possess the accuracy of geodetic positions; however, the AMS method can be modified to yield an average plumb point position. Cartographers having topographic surveying experience are familiar with the three-point method of plane table orientation and know how a fourth ray might influence a location. A similar problem is encountered in a tilt analysis by the AMS method when the number of control points used exceeds the required minimum. This analogy between tilt analysis and plane table orientation is drawn to demonstrate the versatility of a method which can achieve accurate results by the sole use of map control. Fortunately the photomapping project undertaken by the Army Map Service was accomplished with a minimum of average plumb point determination. This favorable circumstance was attributed to the accuracy of the 1:25,000 line maps.

THEORY

The AMS method of tilt analysis is a graphical resection in space applying hitherto unexploited properties of the circle. Like the analytical procedure for space resection, this technique uses the angles formed at the perspective center by image points on the photograph, to effect a group of semi-graphic three-point resections. Descriptive geometric procedures are useful in the constructions.

Consider a simple geometric theorem concerning the circle. In Figure 1, the locus of the apex k of the angle ϕ subtending the chord ae is a circle whose radius O_1a can be determined from the formula:

$$O_1a = \frac{ae}{2} \csc \phi. \tag{1}$$

In Figure 2, the circle O_2 is the locus of the apex of angle ψ subtending the chord eb. The circles O_1 and O_2 intersect at k. This is the semi-graphical method of solving the three-point problem.

* Reprinted from A.M.S. Bulletin No. 12, October 1944. Release granted by the War Department for republication in this Journal.

A rotation of the circles O_1 and O_2 about their respective chords *ae* and *ed* through 180° places them at O_1' and O_2' with their intersection at *m* (Figure 3). If the circles are rotated simultaneously so as to intersect continually, that

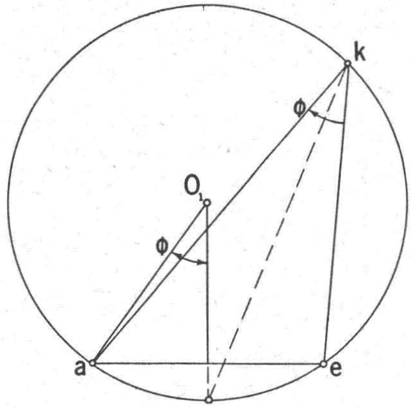

FIGURE 1

intersection in moving from *k* to *m* describes a curve in space, the properties of which are the foundation of the AMS method of tilt analysis.

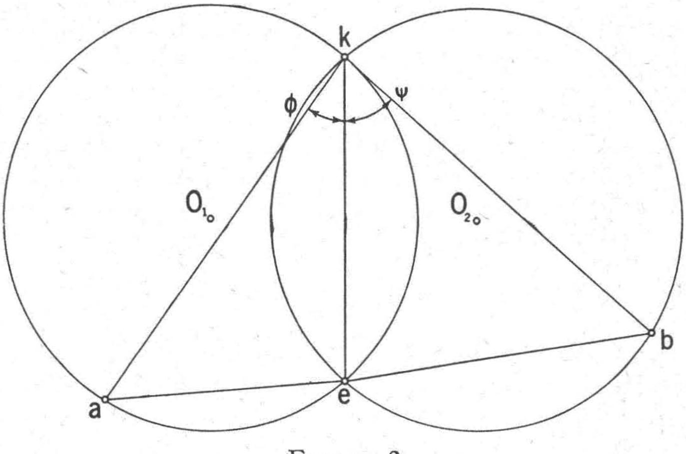

FIGURE 2

Fundamental Theorem: The locus of the intersection of the circles O_1 and O_2 as they are rotated about their respective chords ae and eb is a space curve (circle under certain conditions) passing through the perspective center and the points k and m which define a plane perpendicular to the plane determined by a, e, and b.

This curve is defined also as the intersection of the torii generated by the circles as they are rotated about their chords. The characteristics of this curve may be established by a geometric construction which will be described later under the discussion of flight height determination. The intersection of the planes *Lkm* and *aeb* is the line *km* which passes through the principal point *p*.

Figure 4 represents the positive position of a tilted photograph with its exposure station at *L* and its plumb line and plate perpendicular, defined by *LnN* and *LpP*, respectively. Angle *pLn* is the tilt *t*; *Lp* is the principal distance *f*. Other elements of the diagram are appropriately labelled. *A*, *B*, *C*, *D*, and *E* are map points which are imaged on the photograph at *a*, *b*, *c*, *d*, and *e*. For purposes of this discussion these tilt analysis points are assumed to be at a constant elevation. The peripheral points *a*, *b*, *c*, and *d* are selected to form a strong quadrilateral containing the principal point *p*. The fifth or central analysis point *e* is selected within 50 millimeters of the principal point.

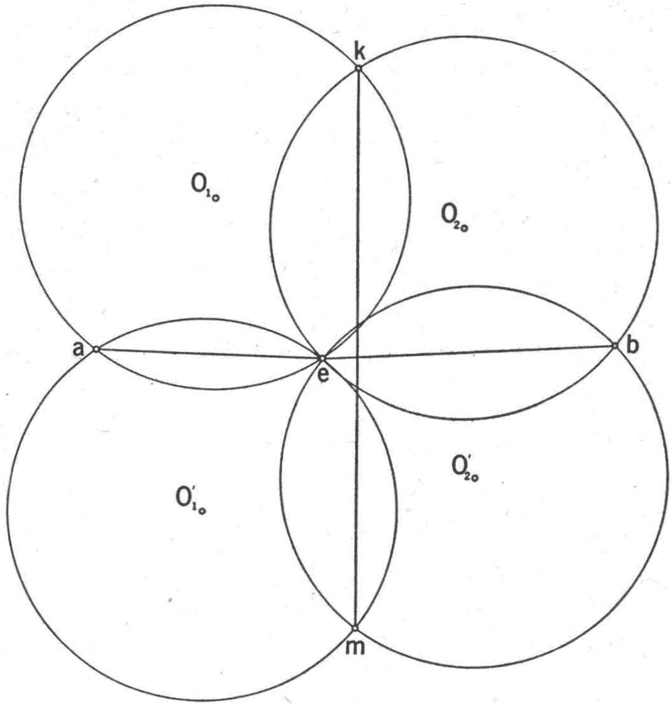

FIGURE 3

It is emphasized that this description of the AMS method of tilt analysis is based on the use of maps as the source of control points. It is convenient to so describe the technique since it was devised to accomplish the job of preparing photomaps consistent with existing line maps. However, the method is equally workable with control derived from a slotted templet assembly.

Again, considering Figure 4, the application of the Fundamental Theorem involves a computation of the radii of two circles, the first passing through points *L*, *a*, and *e*, and the second, through points *L*, *e*, and *b*. When rotated about their chords *ae* and *eb* into the plane of the photograph, these circles assume the positions O_1' and O_2' of Figure 5. They intersect at *m*. The perspective center *L* takes the positions L_1 and L_2.

Computation of the radii of circles O_1' and O_2' is performed semi-graphically as follows: In Figure 5, perpendiculars are dropped from the principal point *p* to the chords *ae* and *eb* intersecting them at *r* and *s*, respectively. These perpendiculars appear also in Figure 4, to form the right triangles *Lpr* and *Lps*. *Lra*, *Lsb*, and *Lse* are right angles. The apex angles *aLe* and *eLb* must be computed to determine the radii of circles O_1' and O_2'. Since *Lr* and *Ls* are hypotenuses of right triangles, their lengths may be determined by constructing right triangles prL_1' and psL_2' where $rL_1' = sL_2' = f$. The lengths sought are $L_1'p$ and

$L_2'p$ (Figure 5). These are plotted along the extended perpendiculars pr and ps from the points r and s to locate the points L_1 and L_2.

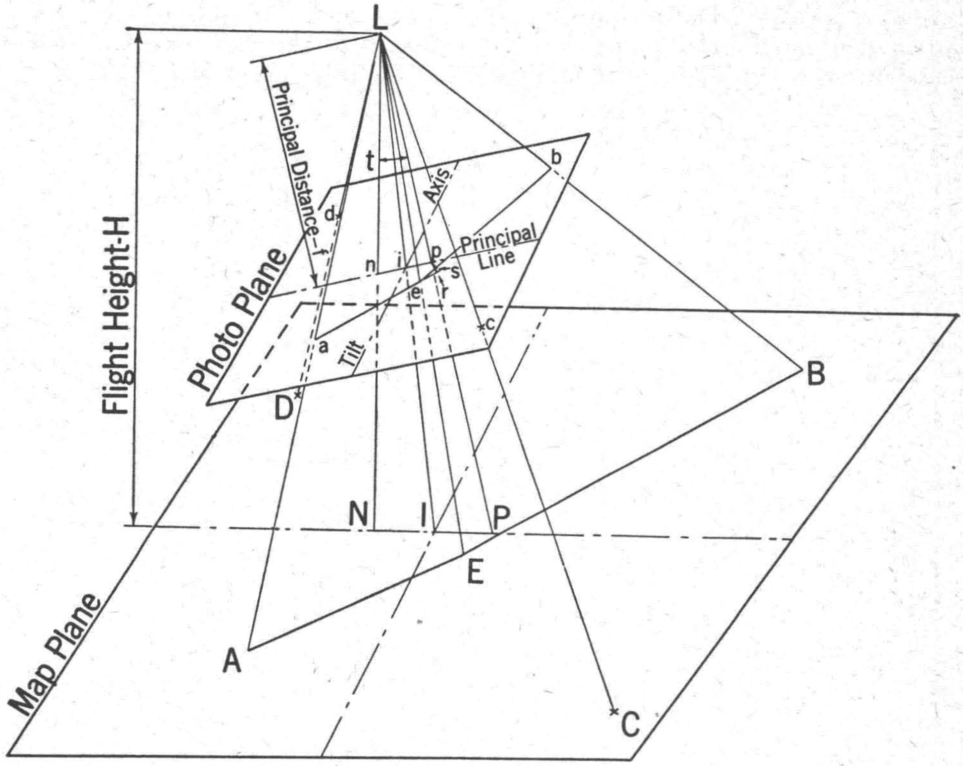

FIGURE 4

Now, the apex angles have been constructed in a common plane and their values determined by scaling appropriate distance. The following equations are apparent:

$$\angle aL_1e = \angle aL_1r - \angle rL_1e$$

$$\angle eL_2b = \angle sL_2e + \angle sL_2b$$

$$\tan aL_1r = \frac{ar}{rL_1}; \quad \tan rL_1e = \frac{re}{rL_1}$$

$$\tan sL_2e = \frac{se}{sL_2}; \quad \tan sL_2b = \frac{sb}{sL_2}.$$

The apex angles aL_1e and eL_2b computed from these equations are substituted in (1) to determine the required radii. Thus:

$$\text{Radius of } O_1' = \frac{ae}{2} \csc aL_1e$$

$$\text{Radius of } O_2' = \frac{eb}{2} \csc eL_2b.$$

These circles intersect at m. When rotated through 180° about their chords these circles assume the positions O_1 and O_2 with the intersection at k. Obviously the intersection of the rotating circles traces a curve which passes through the perspective center L. From the Fundamental Theorem it follows that km must pass through the principal point.

If the tilt analysis points c, e, and d of Figure 4 were plotted in Figure 5, a construction similar to the foregoing would determine a second line inter-

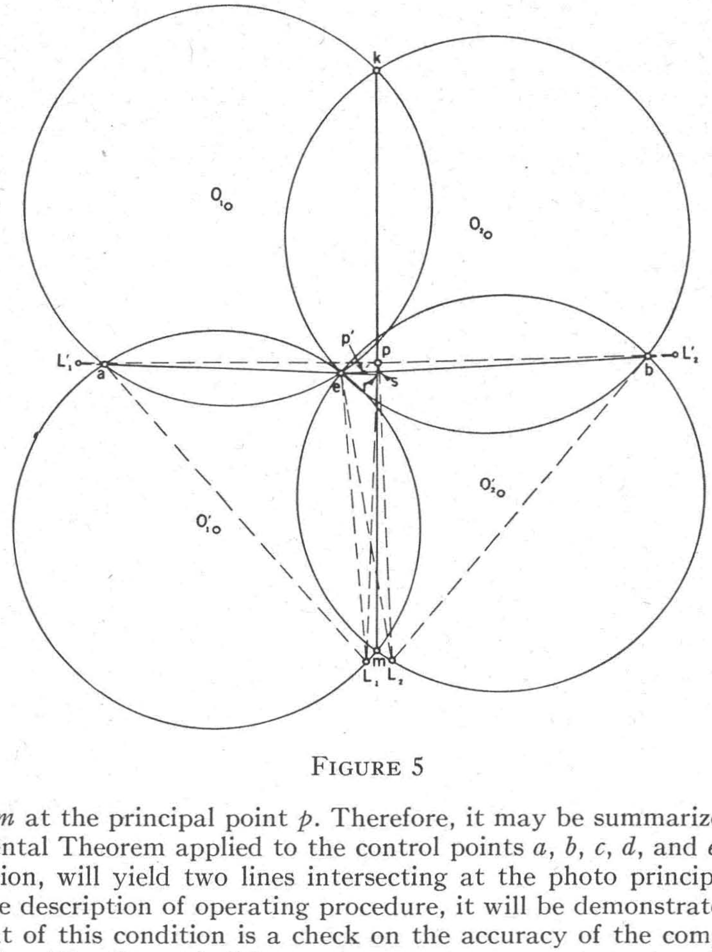

FIGURE 5

secting km at the principal point p. Therefore, it may be summarized that the Fundamental Theorem applied to the control points a, b, c, d, and e, in proper combination, will yield two lines intersecting at the photo principal point p. Under the description of operating procedure, it will be demonstrated that the fulfillment of this condition is a check on the accuracy of the computed apex angles aLe, eLb, cLe, and eLd.

This geometric construction establishes a corollary to the Fundamental Theorem—*The space curves intersect at the perspective center L and pierce the plane abcd at points which define lines intersecting at the foot of the perpendicular dropped from L to the base plane of the pyramid Labcd.* This corollary validates the application of the construction to the *horizontal* plane, cutting the extended edges of the perspective pyramid at the ground positions of the analysis points A, B, C, and D. E represents the map plot of the central control point. It follows, therefore, that the construction applied to the pyramid $LABCD$ will determine the map position of the foot the perpendicular dropped from L; that is, the nadir point N.

Briefly, the construction consists of determining the apex angles from the aerial photograph as demonstrated in Figure 5. These angles *aLe*, *bLe*, *cLe*, and *dLe* are substituted with their appropriate chord values *AE*, *BE*, *CE*, and *DE*, in equation (1) to calculate the radii of the resection circles. The complete construction is diagrammed in Figure 6. Nadir locus lines *KM* and *UV* intersect at the point sought, *N*. This location of the nadir point constitutes the first phase of the AMS method of tilt analysis, namely, a determination of the horizontal position of the exposure station.

The second phase embraces a computation of the flight height and tilt. These elements complete the analysis and establish the space orientation of the aerial photograph. Flight height, at map scale, is given by the length of the perpendicular from the exposure station *L* to the nadir point *N*. Since the space curve passes through *L*, the determination of altitude consists of finding the vertical distance from the nadir point to the curve. Analytical investigation of the torii intersection curve is beyond the scope of this article and is unnecessary

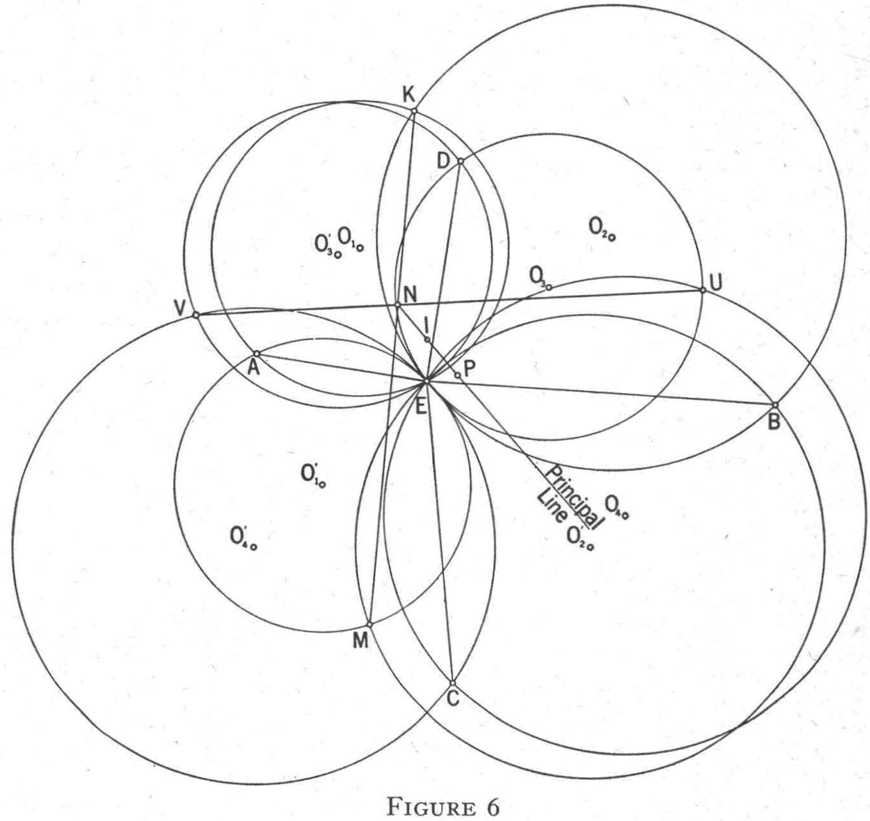

FIGURE 6

to determine flight height. The graphical method is demonstrated in Figure 7. An explanation of the construction follows.

Perpendiculars through the nadir point *N* are drawn to the chords *AE* and *DE*, and are extended to intersect circles O_1 and O_3' at points *F* and *G*, respectively. The line *FN* is the projection on the horizontal plane of the circle described by the point *F* as the circle O_1 is rotated about its chord *AE*. At some point in its path, *F* will be vertically above *N* at a distance equal to the flight height *H*. Now, whatever the position of *F*, its distance from *E* on the axis of rotation is constant. Therefore, when *F* is vertically above *N*, a right triangle is defined with a hypotenuse equal to FE, and sides equal to NE and the flight height H. Expressed in equation form:

$$H = \sqrt{(FE)^2 - (NE)^2}.$$

This equation is solved geometrically by constructing a circle on FE as a diameter. With E as center and NE as a radius, strike an arc intersecting this circle at J. The distance FJ is equal to the flight height H as plotted at the map scale.

A similar construction based on the perpendicular GN will produce another value, GW, for the flight height. Equality of the two determinations verifies the accuracy of the plotting. It is unnecessary to carry the second construction beyond the fixing of G, since GE must equal FE to assure identical values of H.

Computation of the tilt t will complete the orientation of the aerial photograph. The map position P of the principal point p (Figure 4) is plotted by

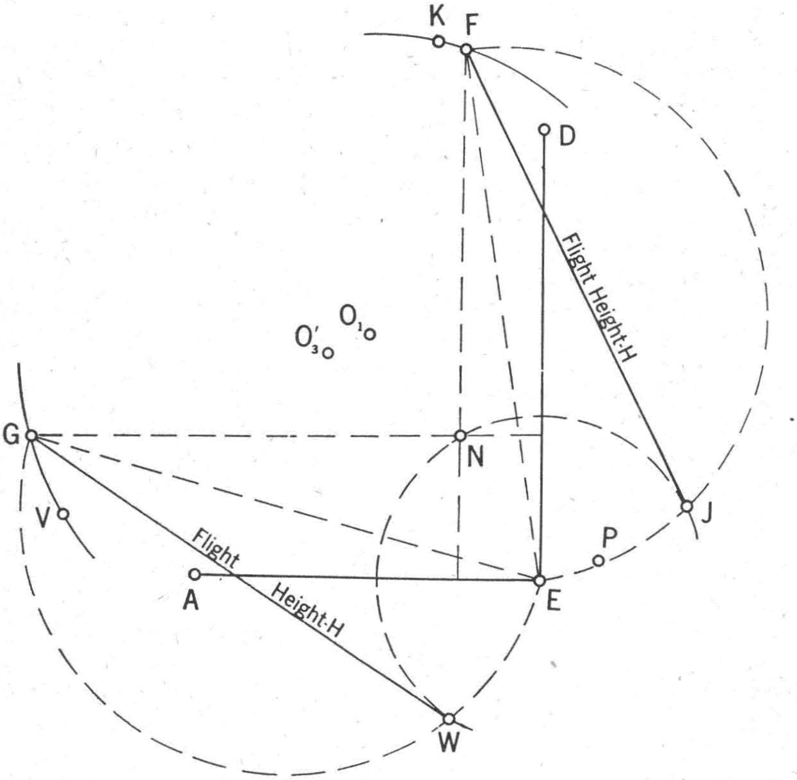

FIGURE 7

means of the four-point or paper strip method.[1] Location of P fixes the principal line NP in Figure 6. The tilt t is computed from the equation:

$$\tan t = \frac{NP}{H}.$$

A transfer of the nadir point N from the map templet to the photograph will establish the principal line upon the photograph. This is accomplished by first plotting the isocenter I on the line NP (Figure 6) at a distance given by the formula:

[1] Described in FM 21–26, *Advanced Map and Aerial Photograph Reading*, page 169, paragraph 112.

$$NI = h \tan \tfrac{1}{2}t.$$

A transparent overlay is then placed on the map templet and, with I as center, rays are drawn to at least three of the points A, B, C, D, E, and P. A "tracing cloth" resection is made on the photograph by passing the overlay rays through corresponding points and pricking the position of the isocenter I. This point fixes the principal line pi. The nadir point N is plotted along pi extended at the distance computed from the formula:

$$pn = f \tan t.$$

The derivation of the ground position of the exposure station, flight height, tilt, and principal line establishes the orientation of the aerial photograph. These data are required to compute the Ziess Automatic Rectifier settings. Simultaneous enlargement and rectification of the 6-inch photography is considered an appropriate subject for another discussion. However, at this point, it should be stated that the rectified enlargements were accurate enough to permit the assembly of 1:25,000 photomaps matching the line maps well within the specified tolerance. Data pertaining to specific exposures possessing extreme tilts will be presented in a concluding tabulation to demonstrate the utility of the AMS method of tilt analysis.

Operational Procedures

Further evidence of utility is the ease with which the analysis procedure can be adapted to the use of mechanical and graphical devices to increase production. The system of geometric construction has been considered under *THEORY*. Although academic, it is the only method available to those whose tools are the straight edge, scale, and beam compass. However, if warranted by the volume of work, geometry should be applied using special devices such as the metal three-arm protractor designed to plot three-point sextant fixes in hydrographic surveying. Operational procedures developed at the Army Map Service will be considered in sequence from the initial task of selecting analysis points on the maps to the completion of the map templet for the rectifying operation.

1. Tilt analysis points are selected by an examination of the photograph and map, emphasis being placed on the certainty of identification and the strength of configuration. The four peripheral points, A, B, C, D (Figure 4), must form a strong quadrilateral about the principal point P, and the central point E should fall within 50 millimeters of P.

2. Analysis points are transferred to the map templet by the use of grid squares. The average 6-inch exposure embraces approximately the area of one map sheet and its random orientation requires the selection of control points from as many as four maps. To minimize error due to paper distortion and possible mismatches between sheets, the transfer of analysis points is accomplished by an initial assembly on a master grid overlay. This procedure permits a location of each point square by square. The final templet is made by transferring the control from the grid plot to an acetate overlay.

3. The configuration of the tilt analysis points as they appear on the photograph is enlarged to the approximate scale of the map templet. This operation is performed by means of the AMS Ratiograph described in Photogrammetric Engineering, Vol. X, No. 2. A suitable ratio of enlargement, R, may be computed from the equation:

$$R = \frac{AB}{ab} \text{ (Figure 4).} \tag{2}$$

This inverse use of the Ratiograph provides a rapid graphical method of enlarging the photo templet to permit a more precise measure of apex angles. The photograph is punched at the principal point. Photograph, Ratiograph, and blank acetate are placed on a stud in that order. Radial lines are drawn through the analysis points and extended far enough to accommodate plotting of the enlarged radial distances. Figure 8 diagrams the ratioing procedure.

4. The principal point is transferred from the enlarged photo templet to the map templet by the paper strip method.

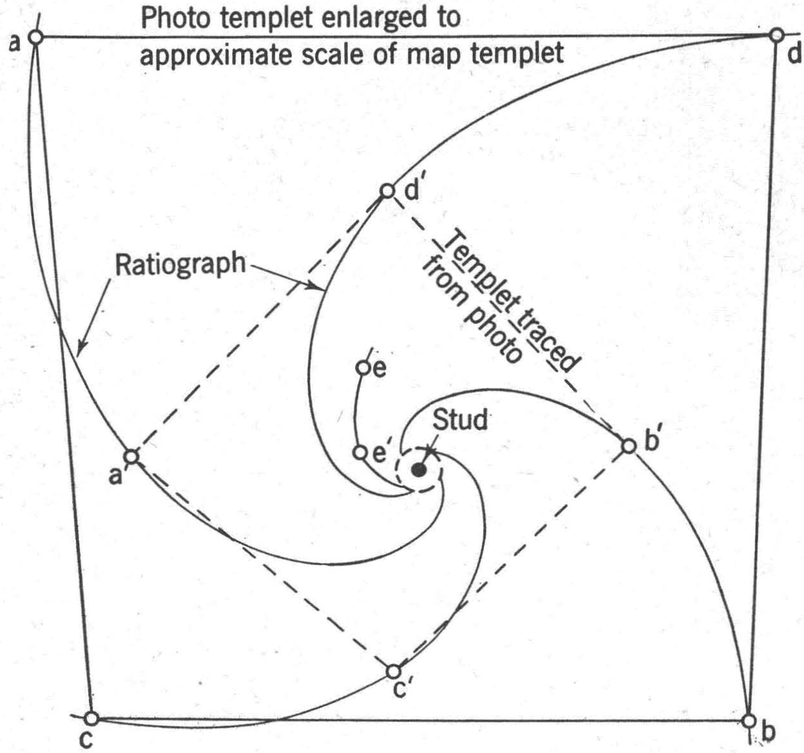

FIGURE 8

5. Apex angles are determined by use of the three-arm protractor on the enlarged photo templet. The semi-graphical method as demonstrated in Figure 5 is replaced by an instrumental procedure which places the protractor center successively at L_1 and L_2. With the center at L_1 and the middle arm of the protractor laid along the perpendicular prL_1, the left arm is rotated to pass through a. With the left arm in this position, the center arm is rotated to pass through e. The left motion is clamped, fixing angle aL_1e on the left sector of the protractor. A similar procedure with the instrument center at L_2 records the angle eL_2b. If the protractor is positioned with the three arms passing through a, e, and b, the point m is fixed. A corresponding point, k, is plotted if the angles are reversed on the protractor and the instrument oriented in the opposing position. The line km will pass through the principal point p if the plotting is accurate. In practice, master perpendiculars on a sheet of vinylite eliminate the need to draw the lines ae, eb, ce, and ed on the enlarged photo templet. The templet is placed over the guide perpendiculars with one axis passing through a and e and the other through p. Location of L_1 is facilitated by graduating the center

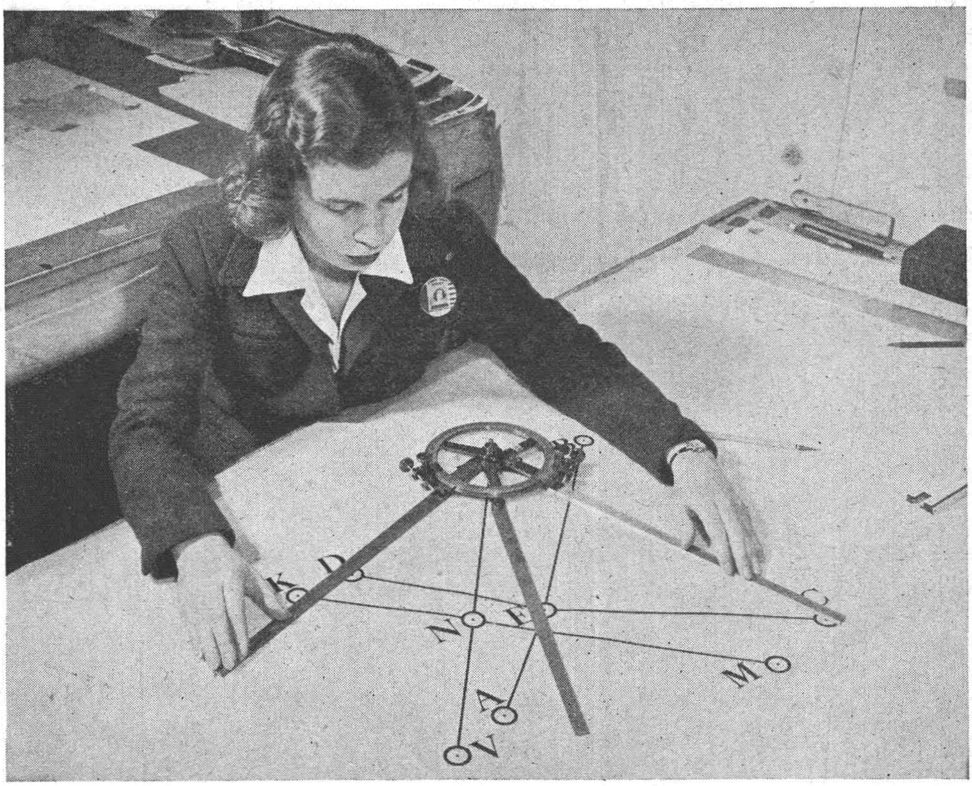

FIGURE 9

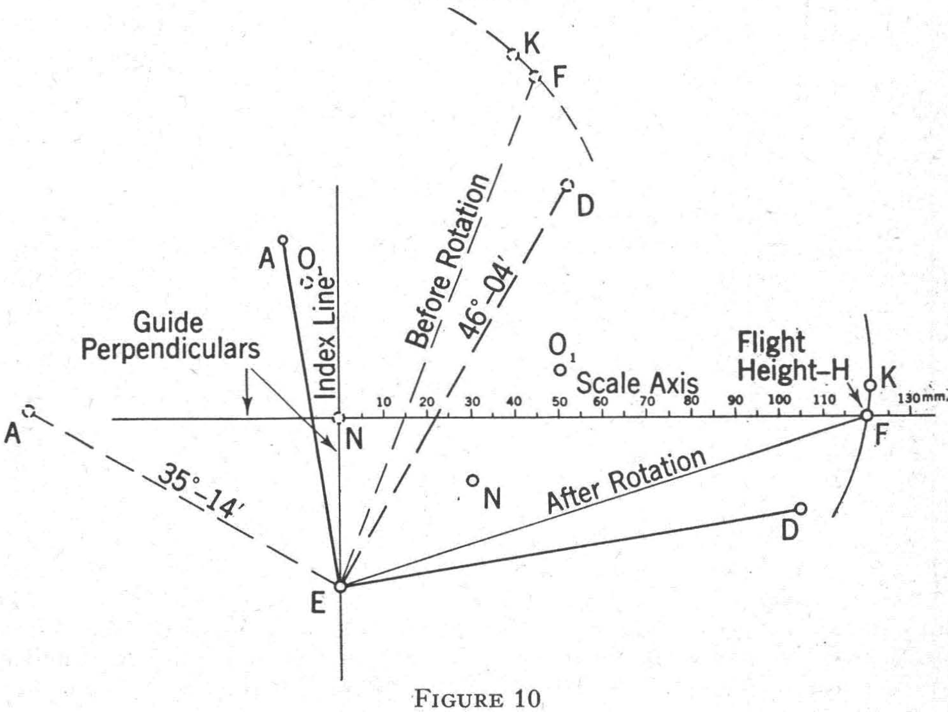

FIGURE 10

protractor arm in millimeters with zero at the center. With r as center and pr as radius, an arc is struck intersecting ae at p'. The center arm of the protractor is laid along the perpendicular pr with the scale at r, reading the value Rf where R is the enlargement ratio derived from equation (2) and f is the aerial photo principal distance. While the protractor center is held fixed, the center arm is rotated to p' and the required hypotenuse read off the scale. Thus, by means of the protractor, a right triangle is solved which is equivalent to the triangle $L_1'pr$ solved by the geometric method discussed under *THEORY*. The center arm is repositioned along the perpendicular with the hypotenuse length registering at r. The instrument center is now at L_1.

6. With apex angles set on the protractor, it is positioned over the map templet to fix the points K, M, U, and V which determine lines KM and UV intersecting at the required nadir point N. One position of the protractor is in Figure 9.

7. Flight height H is determined by a modification of the geometric method previously described. Apex angles are recorded along their respective chords (Figure 10). Radii of the circles O_1 and O_1' are computed from equation (1):

$$\text{Radius of } O_1 = \frac{AE}{2} \csc 35°14'$$

$$\text{Radius of } O_3' = \frac{DE}{2} \csc 46°04'.$$

Segments of circles O_1 and O_3' are drawn through the points K and V, respectively. As in the geometric construction, perpendiculars must be drawn through N to chords AE and DE and extended to intersect the circles O_1 and O_3' at F and G, respectively. Instead of drawing these lines, guide perpendiculars are placed under the templet with one axis through A and E and the other through N. The point F on circle O_1 may be marked without drawing the line FN. Measurement of flight height H is simplified by plotting a millimeter scale along one axis of the guide perpendiculars. The other axis serves as an index line along which the point E is fixed as a pivot to rotate the templet. In Figure 10, the point N is placed at the zero point of the scale and E is positioned on the index line. The point E is pierced by a needle and the templet is rotated about this pivot until the point F falls on the scale axis at the value of the flight height H in millimeters at the scale of the map templet. Continued rotation of the templet will swing the point G to the scale at a flight height value which should equal the first. This procedure is simply an expedient substitute for the geometric construction to determine one side of a right triangle (flight height H) when the other side, NE, and the hypotenuse EF are given.

After the nadir point and the flight height have been determined, the tilt and position of the principal line are derived by the procedure previously described. These elements complete the orientation of the aerial photograph. The computation of the rectifier settings using these data will be discussed in another article. The results of such rectifications from excessively tilted negatives demonstrate the utility and accuracy of the AMS method of tilt analysis. The following is a tabulation of data concerning several exposures.

Results

The following tabulation lists the characteristics of two excessively tilted exposures. These were processed in the Zeiss Automatic Rectifier set in accordance with computed data. The resulting prints were used in the preparation of a 1:25,000 photomap.

Exposure No.	1035	1036
Tilt	20°45′	13°19′
Flight height (meters)	8288	8558
Magnification along rectifier lens axis	2.47	2.30
Maximum position error of identifiable image points (millimeters)	2.5	2.0
Average position error	1.5	0.9
Variation in terrain elevation (meters)	230–380	230–380

Neither of these rectified prints showed discrepancies between photo and map positions in excess of the specified maximum of 2.5 millimeters. The average error is remarkably low when it is considered that the check points were random map images and that the analysis disregarded relief. In the case of exposure 1035, the rectification of 25°45′ and simultaneous enlargement of 2.47 diameters demonstrates the latitude of the automatic rectifier. These results cannot be duplicated by the outmoded method of tilt analysis using a multiplicity of radial ratios coupled with inadequate rectifying equipment.

Summary

The Army Map Service is aware of the qualifying aspects of the description of this new method of tilt analysis. The procedure was streamlined to fit a photo mapping project covering moderately rolling terrain; hence, the disregard for elevation differences. Also, the control for the analyses was taken from printed maps and consequently was subject to inaccuracies. Niceties of construction therefore were avoided. Neither of these circumstances, however, detract from the basic soundness of the method. When accuracy requires the procedure may be modified to allow for relief.

A review of this new method of analyzing aerial photographs for tilt reveals that it is a definite contribution to photogrammetry because of the following characteristics:

1. It is correct theoretically and involves no assumptions which limit its use to small tilts.
2. It involves a minimum of analytical computations. Due to its graphical nature, it can be taught to inexperienced operators quickly.
3. It possesses several self-checking operations.
4. It permits the determination of mean nadir point locations by the use of more than four peripheral analysis points. This feature is useful when using maps for the control of mosaics since any one point, seriously in error, will not adversely affect the location based on a series of points.
5. It produces a map templet carrying all the data necessary to compute the automatic rectifier settings.