NUMERICAL THREE-POINT SOLUTIONS

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ABSTRACT: It is sometimes necessary or desirable to solve the three-point problem numerically, instead of graphically. This paper gives two algebraic solutions, one for coplanar angles and the other for spherical angles.

INTRODUCTION

THE so-called three-point problem arises in determining the position of a point on the earth's surface from angles included between lines drawn from the observer to three other points whose positions are known. The problem may exist in several forms. The four points may be sufficiently close to one another, or the mapping may be on such a scale, that the lines connecting the four points may be regarded as coplanar. On the other hand, the inter-point distances or the scale of the map may be such that the lines connecting the four points must be regarded as curves on a spherical or a spheroidal surface.

The technique of the angular measurements gives rise to other variations in the problem. In plane table work, the three known points may be sufficiently close to the observer, or the scale of the map may be so small, that all four points can be plotted on a plane table sheet. Under such conditions. a graphic solution is usually made, utilizing any one of a number of well known methods. But if conditions are such that the four points can not be charted on a single sheet, or if for other reasons a graphical solution is not desired, the essential angles are measured instrumentally, or are read from the sheet by means of a vernier protractor, after which a numerical solution can be made.

The science of photogrammetry affords excellent illustrations of the need for such numerical solutions. A single oblique aerial photograph may include within its field three distant peaks, whose geodetic positions are known. A numerical solution will give the station-point of the airplane, at the instant when the exposure was made, with greater precision than a graphic solution. In Alaska, much aerial photogrammetry is being done by means of three wide-angle photographs, taken at the same instant, and spanning from horizon to nadir to horizon. In the two outer oblique views of such a set, this problem must necessarily arise, if control is taken from the photographs. Or again, an engineer may be landed by airplane in some unsurveyed country, with distant peaks in view whose positions are known, or will later be determined. With the transit he may be able to read, not three, but many angles to known points; and with such data he would be able to formulate normal equations, and to make a least square solution, thus determining his position with great precision.

This paper deals with two numerical solutions of the three-point problem, one for coplanar angles and the other for spherical angles. For ordinary work, a spheroidal solution is considered unnecessary.

COPLANAR ANGLES

Case I

In the solution of the three-point problem, as above outlined, three possible conditions occur, that change the form of the mathematical solution. These are shown in figures 1-a, 1-b, and 1-c. In figure 1-a, let the three stations on the earth's surface constitute a triangle of reference, with angles A, B, and C and sides a, b, and c. Let O be the observer's station, that is to be determined. Lines

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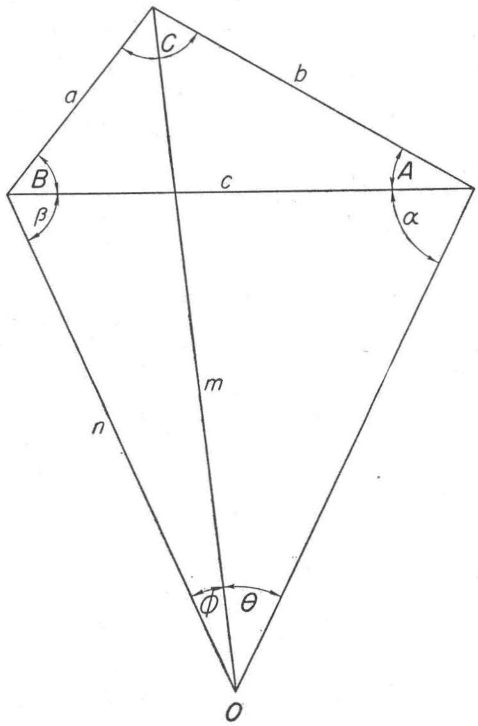

FIG. 1-a

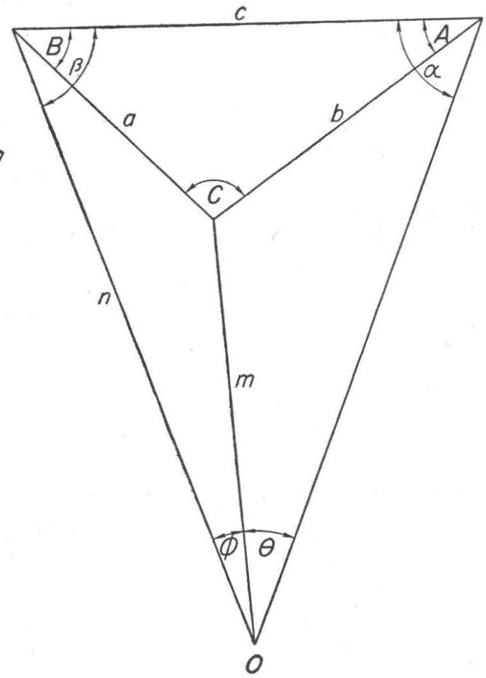

FIG. 1-b

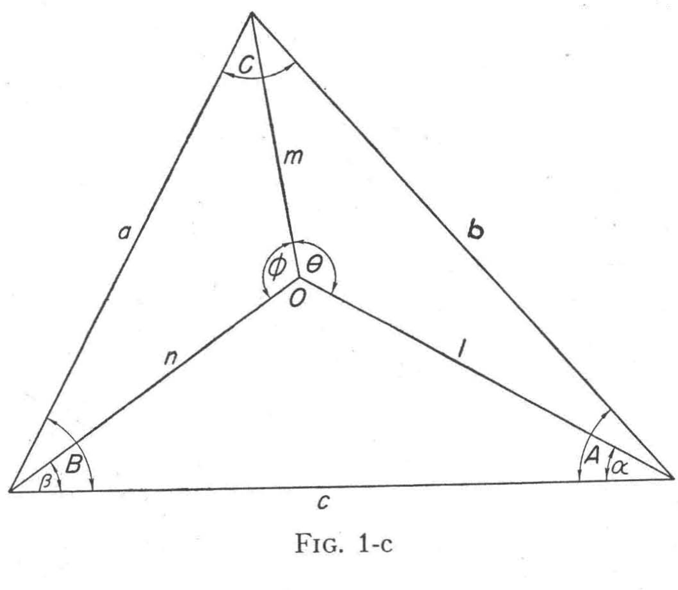

FIG. 1-c

from O to the three corners of the triangle of reference include the angles θ and ϕ, which therefore are known quantities. The angles α and β are not known, but their sum, designated as λ, is also a known quantity. It is desired to solve for α (or β), after which the distances l, m, and n are readily obtained. The solution is as follows:

$$m = \frac{b}{\sin\theta}\cdot\sin(\alpha + A) \quad \text{and} \quad m = \frac{a}{\sin\phi}\cdot\sin(\beta + B)$$

But
$$\alpha + \beta = \lambda \quad \text{whence} \quad \beta = \lambda - \alpha$$

$$\therefore \quad \frac{b}{\sin\theta}\cdot\sin(\alpha + A) = \frac{a}{\sin\phi}\cdot\sin[(\lambda + B) - \alpha]$$

$$\frac{b\sin\phi}{a\sin\theta} = K = \frac{\sin(\lambda + B)\cos\alpha - \cos(\lambda + B)\sin\alpha}{\sin A\cos\alpha + \cos A\sin\alpha}$$

$$K\sin A\cos\alpha + K\cos A\sin\alpha = \sin(\lambda + B)\cos\alpha - \cos(\lambda + B)\sin\alpha$$

$$[\cos(\lambda + B) + K\cos A]\sin\alpha = [\sin(\lambda + B) - K\sin A]\cos\alpha$$

$$\therefore \quad \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \frac{\sin(\lambda + B) - K\sin A}{\cos(\lambda + B) + K\cos A}$$

$$l = \frac{c\sin(\lambda - \alpha)}{\sin(\theta + \phi)} \qquad m = \frac{b\sin(\alpha + A)}{\sin\theta} \qquad n = \frac{c\sin\alpha}{\sin(\theta + \phi)}.$$

Cases II and III

The solutions for these conditions are so similar to the preceding case I, that it seems unnecessary to repeat the argument. Instead, there are stated below merely the solutions, as follows:

Case II.

$$\frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \frac{\sin(\lambda - B) + K\sin A}{\cos(\lambda - B) + K\cos A}$$

$$l = \frac{c\sin(\lambda - \alpha)}{\sin(\theta + \phi)} \qquad m = \frac{b\sin(\alpha - A)}{\sin\theta} \qquad n = \frac{c\sin\alpha}{\sin(\theta + \phi)}$$

Case III.

$$\frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \frac{K\sin A - \sin(B - \lambda)}{K\cos A + \cos(R - \lambda)}$$

$$l = \frac{c\sin(\lambda - \alpha)}{\sin(\theta + \phi)} \qquad m = \frac{b\sin(A - \alpha)}{\sin\theta} \qquad n = \frac{c\sin\alpha}{\sin(\theta + \phi)}.$$

Spherical Angles

For various reasons, it may be necessary or desirable to obtain a solution of this three-point problem in spherical angles. For this purpose, the angles and sides of the triangle of reference, if they are not already given as spherical angles, must be converted to such. This is readily done by applying the mean radius of curvature, for the particular latitude and azimuth involved, such radii being obtainable from any standard tabulation. The method given below assumes a

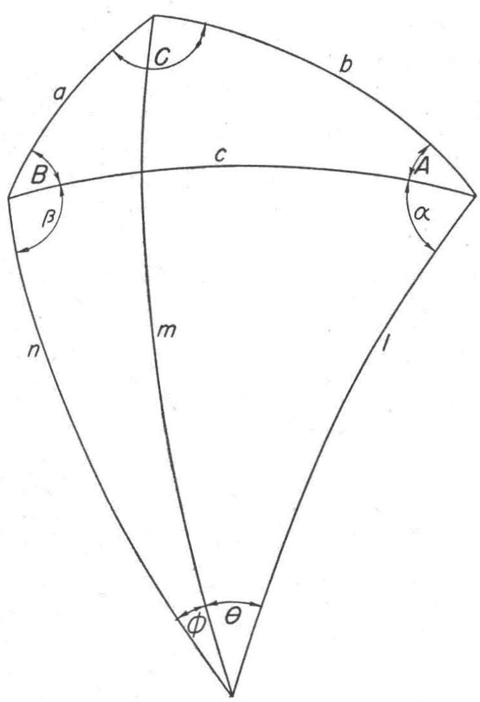

FIG. 2-a

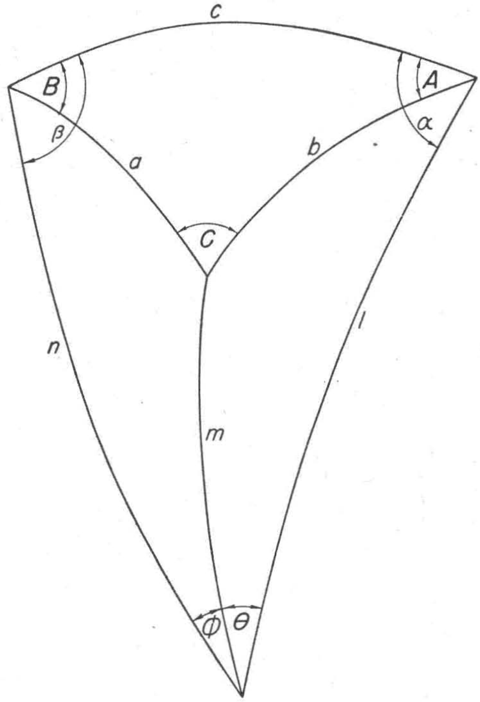

FIG. 2-b

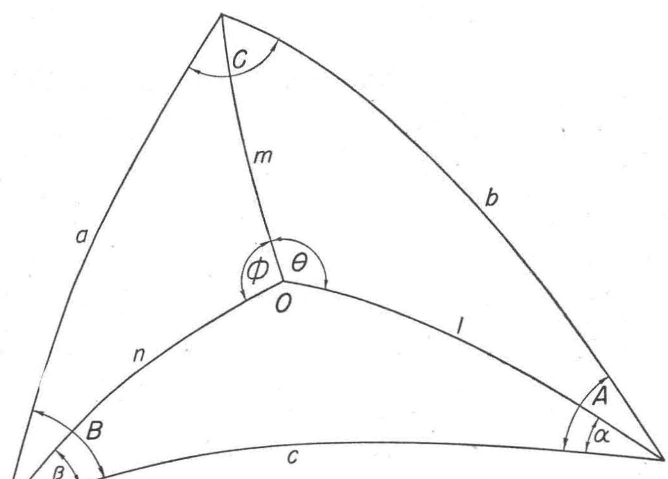

FIG. 2-c

preliminary computation of the angles α and β in plane angles, as a first approximation to the true values, to be obtained in spherical angles.

Case I

Three possible conditions occur in the spherical solution, as in the solution for coplanar angles. These are shown in figures 2-a, 2-b and 2-c. In figure 2-a, assume a triangle of reference, and an observer's station, exactly as in figure 1-a. The solution is as follows:

$$\sin m = \frac{\sin(\alpha + A)\sin b}{\sin\theta}$$

$$\sin m = \frac{\sin(\beta + B)\sin a}{\sin\phi}$$

$$F(\alpha, \beta) = \frac{\sin(\alpha + A)\sin b}{\sin\theta} - \frac{\sin(\beta + B)\sin a}{\sin\phi} = 0$$

$$G(\alpha, \beta) = \cos(\theta + \phi) + \cos\alpha\cos\beta - \sin\alpha\sin\beta\cos c = 0.$$

Now if α_0 and β_0 are the approximate roots of these two equations, and h and k are the desired corrections for a solution in spherical angles, we may write,

$$\alpha = \alpha_0 + h \qquad \beta = \beta_0 + k.$$

Then

$$F(\alpha_0 + h, \beta_0 + k) = 0 \quad \text{and} \quad G(\alpha_0 + h, \beta_0 + k) = 0.$$

Expanding by Taylor's theorem, we have,

$$F(\alpha_0 + h, \beta_0 + k) = F(\alpha_0, \beta_0) + h\left(\frac{\partial F}{\partial\alpha}\right)_0 + k\left(\frac{\partial F}{\partial\beta}\right)_0 + \frac{h^2}{2}\left(\frac{\partial^2 F}{\partial\alpha^2}\right)_0$$
$$+ hk\left(\frac{\partial^2 F}{\partial\alpha\partial\beta}\right)_0 + \frac{k^2}{2}\left(\frac{\partial^2 F}{\partial^2\beta}\right)_0 + \cdots + R.$$

Neglecting the squares, higher powers and products of h and k, we have,

$$F(\alpha_0 + h, \beta_0 + k) = F(\alpha_0, \beta_0) + h\left(\frac{\partial F}{\partial\alpha}\right)_0 + k\left(\frac{\partial F}{\partial\beta}\right)_0 = 0$$

Similarly,

$$G(\alpha_0 + h, \beta_0 + k) = G(\alpha_0, \beta_0) + h\left(\frac{\partial G}{\partial\alpha}\right)_0 + k\left(\frac{\partial G}{\partial\beta}\right)_0 = 0.$$

Solving for h and k, by Cramer's rule, we obtain,

$$h = \frac{\begin{vmatrix} -F(\alpha_0, \beta_0) & \left(\dfrac{\partial F}{\partial \beta}\right)_0 \\ -G(\alpha_0, \beta_0) & \left(\dfrac{\partial G}{\partial \beta}\right)_0 \end{vmatrix}}{\begin{vmatrix} \left(\dfrac{\partial F}{\partial \alpha}\right)_0 & \left(\dfrac{\partial F}{\partial \beta}\right)_0 \\ \left(\dfrac{\partial G}{\partial \alpha}\right)_0 & \left(\dfrac{\partial G}{\partial \beta}\right)_0 \end{vmatrix}}$$

$$k = \frac{\begin{vmatrix} \left(\dfrac{\partial F}{\partial \alpha}\right)_0 & -F(\alpha_0, \beta_0) \\ \left(\dfrac{\partial G}{\partial \alpha}\right)_0 & -G(\alpha_0, \beta_0) \end{vmatrix}}{\begin{vmatrix} \left(\dfrac{\partial F}{\partial \alpha}\right)_0 & \left(\dfrac{\partial F}{\partial \beta}\right)_0 \\ \left(\dfrac{\partial G}{\partial \alpha}\right)_0 & \left(\dfrac{\partial G}{\partial \beta}\right)_0 \end{vmatrix}}$$

where

$$\left(\frac{\partial F}{\partial \alpha}\right)_0 = \frac{\sin b}{\sin \theta} \cdot \cos(\alpha_0 + A)$$

$$\left(\frac{\partial F}{\partial \beta}\right)_0 = -\frac{\sin a}{\sin \phi} \cdot \cos(\beta_0 + B)$$

$$\left(\frac{\partial G}{\partial \alpha}\right)_0 = -\left[\sin \alpha_0 \cos \beta_0 + \cos \alpha_0 \sin \beta_0 \cos c\right]$$

$$\left(\frac{\partial G}{\partial \beta}\right)_0 = -\left[\cos \alpha_0 \sin \beta_0 + \sin \alpha_0 \cos \beta_0 \cos c\right].$$

Case II

The solutions for the other two conditions are similar to those for the preceding case I, so that it is unnecessary to repeat the argument in full. Instead, there are given merely the functions F and G, and their partial derivatives with regard to α and β.

$$F(\alpha, \beta) = \frac{\sin b \sin(\alpha - A)}{\sin \theta} - \frac{\sin a \sin(\beta - B)}{\sin \phi} = 0$$

$$G(\alpha, \beta) = \cos(\theta + \phi) + \cos \alpha \cos \beta - \sin \alpha \sin \beta \cos c = 0$$

$$\left(\frac{\partial F}{\partial \alpha}\right)_0 = \frac{\sin b}{\sin \theta} \cdot \cos(\alpha_0 - A)$$

$$\left(\frac{\partial F}{\partial \beta}\right)_0 = -\frac{\sin a}{\sin \phi} \cdot \cos(\beta_0 - B)$$

$$\left(\frac{\partial G}{\partial \alpha}\right)_0 = - \left[\sin \alpha_0 \cos \beta_0 + \cos \alpha_0 \sin \beta_0 \cos c\right]$$

$$\left(\frac{\partial G}{\partial \beta}\right)_0 = - \left[\cos \alpha_0 \sin \beta_0 + \sin \alpha_0 \cos \beta_0 \cos c\right].$$

Case III

The functions F and G, and their partial derivatives with regard to α and β, are as follows:

$$F(\alpha, \beta) = \frac{\sin (A - \alpha) \sin b}{\sin \theta} - \frac{\sin (B - \beta) \sin a}{\sin \phi} = 0$$

$$G(\alpha, \beta) = \cos (\theta + \phi) + \cos \alpha \cos \beta - \sin \alpha \sin \beta \cos c = 0$$

$$\left(\frac{\partial F}{\partial \alpha}\right)_0 = - \frac{\sin b}{\sin \theta} \cdot \cos (A - \alpha_0)$$

$$\left(\frac{\partial F}{\partial \beta}\right)_0 = \frac{\sin a}{\sin \phi} \cdot \cos (B - \beta_0)$$

$$\left(\frac{\partial G}{\partial \alpha}\right)_0 = - \left[\sin \alpha_0 \cos \beta_0 + \cos \alpha_0 \sin \beta_0 \cos c\right]$$

$$\left(\frac{\partial G}{\partial \beta}\right)_0 = - \left[\cos \alpha_0 \sin \beta_0 + \sin \alpha_0 \cos \beta_0 \cos c\right]$$

NEWS NOTES

The progress of photogrammetry was officially noted and rewarded on 20 Nov. 43 when the U. S. Government, represented by Michael W. Straus, First Asst. Sec. of the Department of the Interior, presented awards to two members of this society.

The awards, salary increases, were given to James G. Lewis and James L. Buckmaster of the Alaskan Branch, Geological Survey.

Mr. Buckmaster's award was based on the recognized value of two instruments, both of which he has developed from the "camera lucida" principle. These Sketchmasters, vertical and oblique, have materially aided the mapping and charting programs of the Army, Navy and Marine Corps. They have been important integral parts of the tri-metrogon mapping method.

Mr. Lewis' instrument, the Rectoblique Plotter, has furnished a cheap, simple instrument which permits the operator to determine true horizontal angles from oblique photographs. The plotter has also proved an important part of the tri-metrogon system since it released a much more costly instrument, the Wilson Photoalidade, for other uses.

It is noteworthy that two out of five awards made on this date were for advancements in the science this society represents.