THE EFFECT OF ATMOSPHERIC REFRACTION IN MULTIPLEX MAPPING

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THAT errors due to atmospheric refraction are present in aerial photography is a fact known to photogrammetric engineers. Equally apparent is the lack of provision for the correction of this error. The justification for this depends on the magnitude of the refraction and its attendent displacement of image points in the multiplex model. The magnitude of this error is not commonly known among photogrammetrists, due perhaps to the lack of its discussion in the readily available sources of information. For this reason I believe this paper is in order.

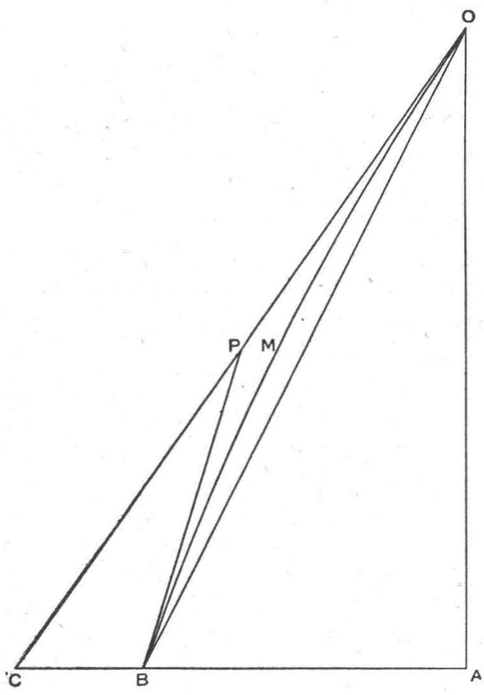

FIG 1. Displacement in a vertical photograph due to refraction.

In the following we shall be concerned with areial photography to be used in multiplex compilation on a map scale of 1:10,000. The optimum flight altitude for this scale is 11,810 feet. As shown in the diagram below, we shall suppose the camera station to be at O. The distance OA represents the flight altitude, A being the ground nadir point. Let us consider the light ray from a point on the ground designated B. Due to atmospheric refraction, the path of the light ray from B to O is not given by the straight line BO, but by the curve BMO which at every point is curving away from the normal dropped toward the earth. Thus the ray from point B appears to be coming from a point C, farther away from A, along the line CO which is tangent to the curve at O. The distance BC is the error due to the refraction.

We shall consider the wide angle lens of six inch focal length. The group of light rays which is used in photography for multiplex work forms a cone well under 100° in angular extent. The greatest refractive error is present in those rays which make the largest angle with the plumb line. Thus the refraction error increases as the object point considered is taken farther from the nadir point A. Let us examine the most unfavorable case, namely the ray which makes an angle of 50° with the plumb line. Referring to Figure 1, the angle OCA is therefore equal to 40°.

To calculate the atmospheric refraction is a difficult problem in any event and quite unnecessary here. Let us assume the refraction to be given by the mean astronomical refraction, a value which is much larger obviously than is actually experienced. Designate by α the angle COA which the ray from point B makes with OA and let the distance AC be called x. Let H be the height of the camera above the ground; it is represented by the distance OA in the diagram. From the right triangle OAC we obtain the equation:

$$x = H \tan \alpha \tag{1}$$

from the differential calculus we obtain directly:

$$dx = H \sec^2 \alpha d\alpha \tag{2}$$

($d\alpha$ expressed in radians).

We denote by Δx the increment in x due to an increment in angle α. Evidently $BC=\Delta x$ and angle $BOC=\Delta\alpha$. Directly from (2) we get:

$$\Delta x = H \sec^2 \alpha \Delta\alpha \tag{3}$$

where $\Delta\alpha$ is expressed in radians.

The atmospheric refraction is the angle which the line CO makes with the tangent to the curve BMO at B. This angle is obviously larger than the angle $\Delta\alpha$. Substituting in equation (3) H=11,810 and $\alpha=50°$ and using for the value of $\Delta\alpha$ the mean astronomical refraction for an apparent altitude of 40° (temperature 50° F, barometer 29.6 inches) which is equal to 1′09″ of arc, we obtain:

$$\Delta x = 9.56 \text{ feet.}$$

Thus we see that the maximum possible error in position for any point in the vertical photograph is under ten feet. If the photograph contains tilt, the position error is increased still further for those points which are furthest from the nadir point. However, in multiplex photography, the tilt is held within three degrees, so that this additional source of error has been accounted for by assuming a larger cone of rays than actually used in practice. This error of less than ten feet may be safely ignored since it represents an error of less than .02 inches on a 1:10,000 scale map. Since the flight altitude is inversely proportional to the scale of the multiplex manuscript map, the position error due to atmospheric refraction is less than .02 inches on any map scale. This becomes evident by inspecting the formulas obtained above, which are general.

The Geological Survey has done a considerable amount of multiplex mapping using high oblique photographs. These obliques contain on the average a tilt of 60 degrees. The resulting errors are of a magnitude sufficient to make it necessary to use a better approximation for $\Delta\alpha$. This can be done by considering the curve BMO. The angle which the chord BO makes with OC is $\Delta\alpha$. The angle which the tangent BP makes with CO is the atmospheric refraction. Thus it is the value of angle CPB (further approximated by the mean astronomic refraction) which we have used as an approximation to $\Delta\alpha$ in order to compute the position error in the vertical photograph. We shall now assume the curve BMO to be the arc of a circle. If this assumption arrives at a position error slightly less than that which the actual curve would produce, it is compensated for by the fact that we are using the value of the astronomic refraction rather than the atmospheric refraction. Since BMO is the arc of a circle angle $BPC=2\Delta\alpha$.

Let us suppose now that BMO is the ray to a point on the line parallel to the horizon which intersects the principal point of the photograph. The greatest value of α satisfying this condition is 65 degrees. The mean astronomic refraction for an apparent altitude of 25 degrees is 2′04″. Therefore $\Delta\alpha=1'02''$. Substituting in (3) this value of $\Delta\alpha$ (expressed in radians), $H=11{,}810$ and $\alpha=65°$ we obtain $\Delta x=19.88$ feet. On the map scale of 1:10,000 this represents an error of .024 inches. If α is increased by five degrees Δx is doubled, representing a error of .048 inches.

Thus far in this paper we have considered the position error produced in a single photograph. Let us carry our investigation further and consider the multiplex model which is made up of points which are the intersections of conjugate rays of two overlapping photographs. In the diagram below let O and O' be camera stations for photographs A and B which overlap 60%. Let N and N' represent the corresponding nadir points for the photographs and let the object point P be a point in the area of overlap. Due to refraction, the ray from point P registers on photograph A as though it came from point P_A. Likewise the ray from point P registers on photograph B as though it came from point P_B. As indicated in the diagram, the displacements PP_A and PP_B are radial from the nadir points of the photographs. The plane determined by the lines ON and NP is a vertical plane since it contains the plumb line. For the same reason the plane determined by $O'N'$ and $N'P'$ is a vertical plane. Any two vertical planes must intersect in a plumb line. Since OP_A and $O'P_B$ lie in the two aforementioned planes, their intersection P' must lie on the intersection of the two planes. Thus it is readily seen that there is no position error due to refraction in the multiplex model. However the displacement PP' represents a vertical error, call it Δh, which we proceed to calculate. Triangle $PP'P_A$ is similar to triangle ONP_A. Defining α as the angle NOP_A, we get $PP_A = PP' \tan \alpha$. Using the same notation as for figure 1, $PP_A = \Delta x$ since PP_A represents the position error in photograph A, we get $\Delta x = \Delta h \tan \alpha$. Solving this for Δh we obtain the equation:

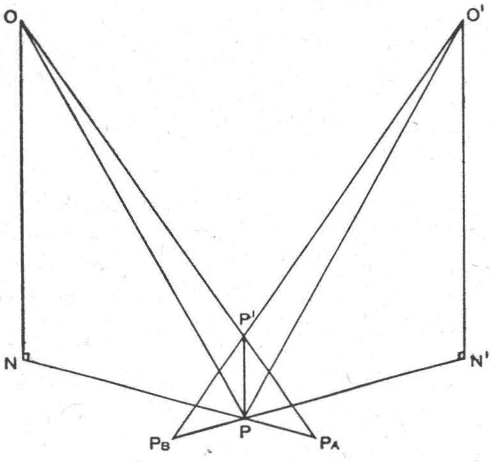

FIG. 2. Displacement in the multiplex model due to refraction.

$$\Delta h = \Delta x \cot \alpha. \tag{4}$$

But

$$\Delta x = H \sec^2 \alpha \Delta \alpha.$$

Substituting this in (4) we get $\Delta h = H \sec^2 \alpha \Delta \alpha \cot \alpha$ which can be simplified further to give:

$$\Delta h = \frac{2H\Delta\alpha}{\sin 2\alpha} \cdot \tag{5}$$

If the value of Δh were constant for every point in the model, the total effect would be merely to raise the entire model by an amount equal to Δh. However the value of Δh is not constant throughout the model; the value of Δh for a point P increases as the distance of P from the air-base increases. The value of Δh is least for those points which lie in the vertical plane passing through the air-base, and takes on its minimum value for the point which is equidistant from both camera stations. Let us determine this minimum value for Δh. Assuming 60 per cent overlap, α is easily calculated to be 16°42′. The value of $\Delta\alpha$ for this angle is 18″ (.00008726 radians). As before $H=11{,}810$. Substituting these values in equation (5) we get $\Delta h=3.75$ feet.

Let us now determine the maximum value of Δh. This value occurs for the point in the corner of the model. Angle α is calculated and found to be 38°40′. The corresponding $\Delta\alpha$ is 47″ (.0002279 radians). Substituting these values in equation (5) we get $\Delta h=5.52$ feet.

The difference between the maximum and minimum values of Δh is 1.77 feet and gives the total amount of distortion in the model. On a 1:10,000 scale this error amounts to .05 mm.

A more careful study of figure 2 reveals that in general the rays OP and $O'P$ do not intersect unless they make equal angles with their respective plumb lines. This non-intersection causes Y-parallax in the model which is of theoretical interest only, because the amount of Y-parallax is so small as to remain undetected.

Our investigation leads us to conclude that no error or distortion of any kind which is caused by atmospheric refraction can be detected in the multiplex model.