TILT DETERMINATION BY COMPARISON OF LINE DIRECTIONS*

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THE equipment needed for this method includes a standard drafting machine, or other protractor-equipped device capable of measurements to at least ten minutes of arc, and the easily improvised "tilt parallelogram" described herein. The procedure of determining amount and direction of tilt consists of four steps:

1. Mark the principal point on the photograph and select three well separated control points on the photograph and in the control plane. These points should be of equal elevation for greatest accuracy. Connect the points to form the photo-triangle and the control triangle. On the photograph draw perpendiculars from the principal point to each line, and number these intersections or "base points" 1, 2 and 3. A template facilitates this operation.

2. Fasten photograph and map to a drawing board, orienting the photograph by eye. With the arm and protractor of the drafting machine, measure the angle between each of the three legs of the triangle on the map and the arbitrary zero of the machine. Also measure the angles between each of the three legs of the triangle on the photograph and the arbitrary zero. Subtract the respective angles to give the difference between map-angle and photo-angle. From these three differences subtract the approximate mean of the three. Example:

Line	*Map Angle*	*Photo Angle*	*Difference*	*D*
1	59°30′	48°00′	+11°30′	−0°30′
2	107°30′	97°40′	+ 9°50′	−1°10′
3	190°30′	178°20′	+12°10′	+1°10′

3. On the tilt parallelogram set the three movable cross-members at their respective positions along the side scales. In the example, since the greatest values are −1°10′ and +1°10′, scale *C* is used.

4. Next place the tilt parallelogram over the photograph orienting it and collapsing it until the cross-members pass through their respective base points 1, 2 and 3. Read the angle of tilt from scale *C* at the corner of the parallelogram. Draw a line from the principal point parallel to the cross-members in the direction indicated by the arrow on the parallelogram, indicating the direction of tilt.

This method is entirely independent of camera altitude and map scale. The scale or magnification factor required for mosaic construction is easily computed after tilt is determined by using control points along the isoline. The accuracy of this method is highest when tilts are small. It has a workable range from one-half degree to ten degrees tilt, which is sufficiently wide for normal conditions. Other factors which limit its application and scope are discussed in the development of the equations.

Development of Equations

In Figure 1 the negative plane, tilted θ degrees, and the horizon plane through the lens intersect in line *st*. Line *pt* is the principal line, and line *rs* is

* *Editor's Note:* Determination of tilt in military topographic organizations requires a simple method, easily taught and adaptable to rapid production. During the closing weeks of the European campaign, Technical Sergeant George Hagedorn, assigned to a special research section of the 941st Engineer Aviation Topographic Battalion, devised a novel method of determining approximate tilt. Although not tested outside the research section, nor applied in any extensive photomapping projects, the method seems to warrant its publication because of its unique approach and the simplicity of its application.

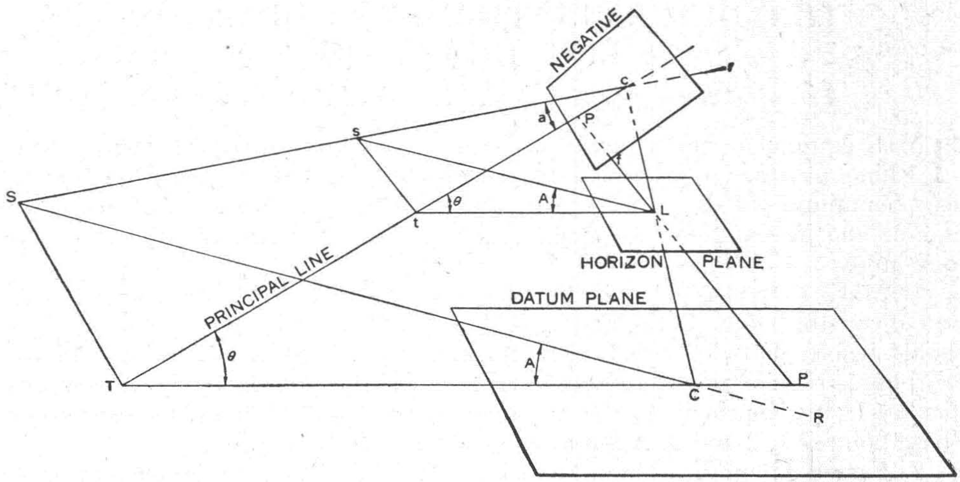

FIGURE 1

any line on the photograph intersecting the principal line. Its trace in the datum plane is RS. The following relationships are evident:

$$pt = \frac{f}{\tan\theta} \qquad Lt = \frac{f}{\sin\theta}.$$

Since the datum plane and the horizon plane are parallel

$$\angle A = \angle sLt = \angle SCT.$$

Revolving the tilted negative plane about the line st as an axis until $\theta = 0$ the figure $Lcts$ becomes Figure 2.

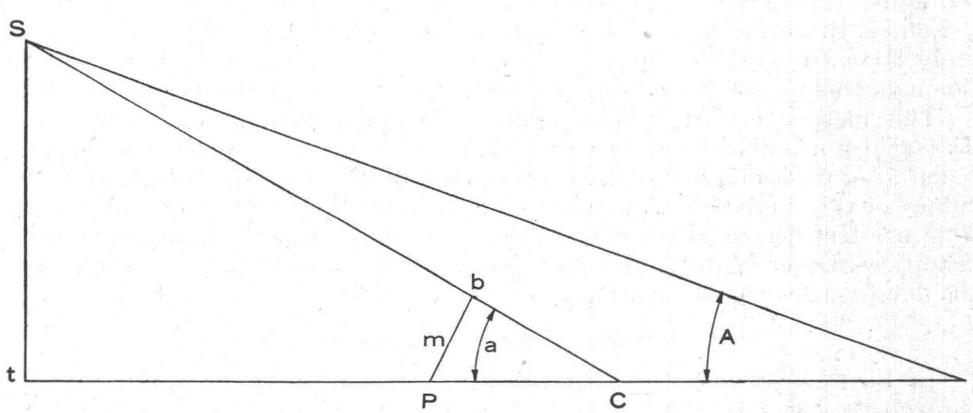

FIGURE 2

Construct line pb perpendicular to line sc and call its length m. Then:

$$pc = \frac{m}{\sin a}$$

And

$$ct = cp + pt = \frac{m}{\sin a} + \frac{f}{\tan \theta}$$

$$Lt = \frac{f}{\sin \theta}$$

$$st = \frac{f \tan A}{\sin \theta} = \left(\frac{m}{\sin a} + \frac{f}{\tan \theta}\right) \tan a$$

or

$$\tan A = \sec a \sin \theta \left(\frac{m}{f}\right) + \tan a \cos \theta. \qquad (1)$$

If A and a were known, equation (1) could be used to determine θ, but since A and a must be measured from the principal line, which is unknown, the angle measurements must be taken from some arbitrary base line. If in Figure 3 we let B and b equal the measured angles from the arbitrary base line to the map and photo lines, and let S and s equal the angles between the arbitrary base line and the principal line, then:

$$A = B - S$$

$$a = b - s$$

Equation (1) becomes:

$$\tan (B - S) = \cos \theta \tan (b - s) + \frac{m}{f} \sin \theta \sec (b - s). \qquad (2)$$

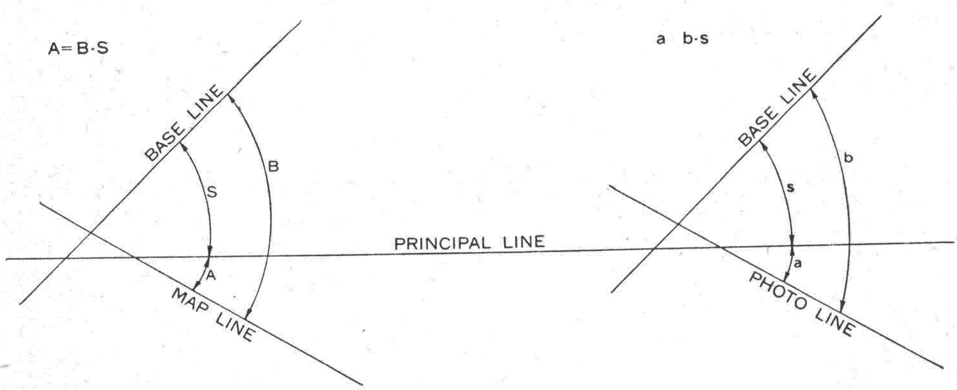

FIGURE 3

In this equation B, b and m may be measured. S, s and θ are unknown, requiring that values for three lines must be measured before the problem becomes determinate. With three identified control points per photograph, the three lines connecting them will provide sufficient knowns to determine S and θ. Such a solution, though theoretically possible, is too lengthy and complex for use under operating conditions in military organizations.

To bring the method within practical application, equation (1) may be simplified by writing it:

$$A = a + \frac{m}{f}(\cos a)\theta. \tag{3}$$

This simplification is accomplished by applying "MacLaurin's Series"

$$A = A_{(\theta=0)} + \theta \frac{\delta A}{\delta\theta_{(\theta=0)}} + \cdots .$$

The first term of the expansion, the value of A when $\theta=0$, is of course a. This may be verified by substituting 0 for θ in equation (1).

To evaluate the derivative we return to (1) and differentiate both sides with respect to θ

$$\sec^2 A \frac{\delta A}{\delta\theta} = \sec a \cos\theta \frac{m}{f} - \tan a \sin\theta.$$

When $\theta - 0$;

$$\sec^2 A = \sec^2 a, \quad \cos\theta = 1, \quad \text{and} \quad \sin\theta = 0.$$

Substituting these values and simplifying:

$$\frac{\delta A}{\delta\theta_{(\theta=0)}} = \frac{m}{f}\cos a.$$

By substituting the values derived for the coefficients of the first two terms in the series, and ignoring subsequent terms

$$A = a + \frac{m}{f}(\cos a)\theta. \tag{3}$$

The following table compares results of using equation (3) instead of (1) in computing values of A.

θ	m/f	a	(A) equation (1)		(A) equation (3)		Error
5 degrees	.50	0 deg.	2 deg.	29 min.	2 deg.	30 min.	1 min.
		45	46	37	46	47	10
		90	90	0	90	0	0
10	.50	0	4	59	5	0	1
		45	47	56	48	32	36
		90	90	0	90	0	0
5	.10	0	0	27	0	30	3
		45	45	14	45	21	7
		90	90	0	90	0	0
10	.10	0	0	58	1	0	2
		45	45	34	45	42	8
		90	90	0	90	0	0

It is evident that the maximum error, which occurs in lines which cross the principal line at 45 degrees, is caused principally by dropping the expression $(\cos\theta-1)\frac{1}{2}\sin 2a$ in the simplification of the equation.

Equation (3) may also be written:

$$\frac{A-a}{\theta}=\frac{m\cos a}{f}. \tag{4}$$

The left side of equation (4) expresses the difference between corresponding angles measured on the map and on the photograph divided by the angle of tilt. The expression ($m\cos a$) is interpreted graphically in Figure 4, by dropping a perpendicular from b, the "base point" in line rs to the principal line pt. The length of this perpendicular equals ($m\cos a$). Therefore the right side of equation (4) is this perpendicular distance expressed as a fraction of the focal length.

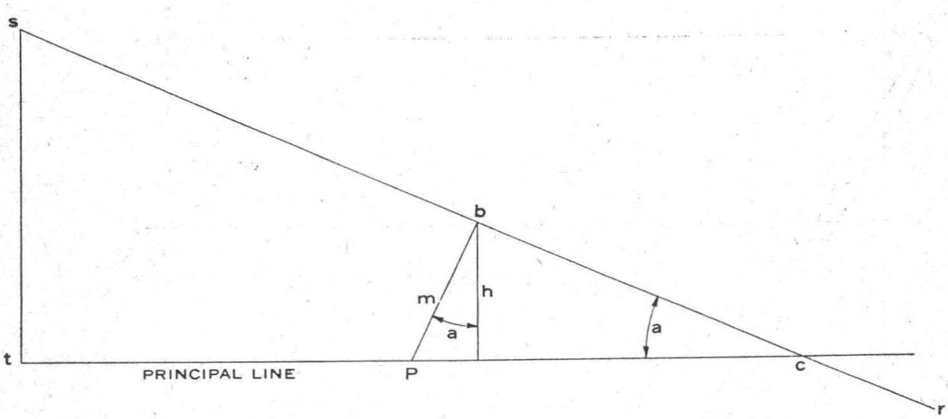

FIGURE 4

Using the same substitutions as in equation (2), equation (4) may be written:

$$\frac{(B-b)+(s-S)}{\theta}=\frac{m\cos(b-s)}{f}. \tag{5}$$

Let $(B-b)=D$; $(s-S)=j$ and $m\cos(b-s)=h$. Then:

$$\frac{D+j}{\theta}=\frac{h}{f}. \tag{6}$$

Assume that θ will be greater than 1 degree. In Figure 5, line pt is the principal line, and (1), (2) and (3) are the base points of the three selected lines. Their respective measured differences are D_1, D_2 and D_3. Through each base point a line parallel to the principal line is drawn. Intersecting these parallels are two additional parallel lines forming angle α. Let α be the angle whose cosecant is equal to θ in degrees. Along one of these lines between its intersection with the principal line and each of the other three parallels, the distances are designated respectively g_1, g_2 and g_3. It is evident that:

$$h_1=g_1\sin\alpha.$$

Dividing through by f:

$$\frac{g_1}{f} \sin \alpha = \frac{h_1}{f}.$$

And since $csc\alpha = \theta$ in degrees, $\sin \alpha = 1/\theta$. Then:

$$\frac{\frac{g_1}{f}}{\theta} = \frac{h_1}{f}. \tag{7}$$

The similarity between equations (6) and (7) is evident. If, instead of measuring the distance g_1 from the intersection with the principal line, it is measured

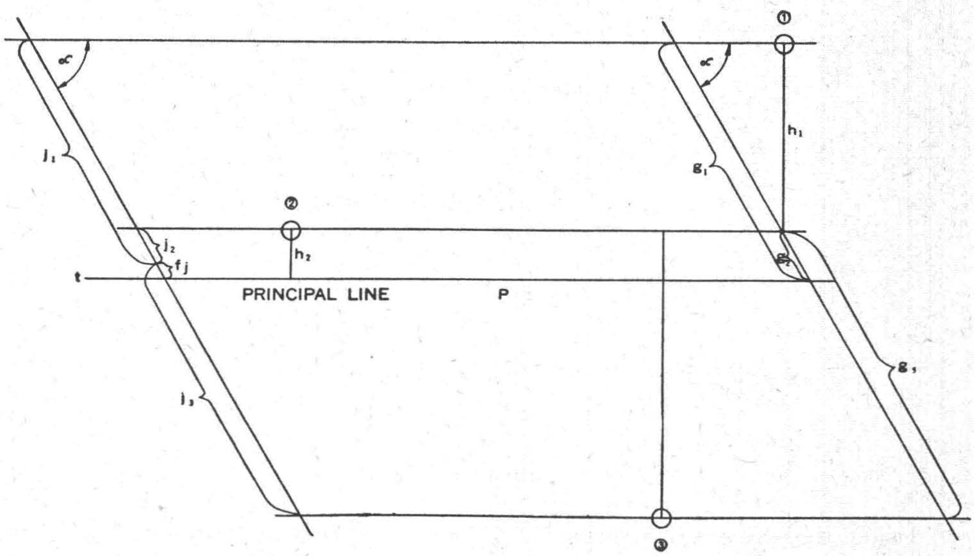

FIGURE 5

from a point at a distance f_j from that intersection, and this distance is called j_1, then:

$$j_1 = g_1 - f_j \quad \text{and} \quad \frac{j_1}{f} + j = \frac{g_1}{f}$$

Substituting in (7) we have:

$$\frac{\frac{j_1}{f} + j}{\theta} = \frac{h_1}{f}. \tag{8}$$

By comparing (8) with (6)

$$\frac{j_1}{f} = D_1 \qquad \frac{j_2}{f} = D_2 \quad \text{and} \quad \frac{j_3}{f} = D_3.$$

Whence:

$$j_1 = fD_1 \qquad j_2 = fD_2 \quad \text{and} \quad j_3 = fD_3.$$

Since f is known, and D_1, D_2 and D_3 may be measured, j_1, j_2 and j_3 may be computed. If the parallelogram formed by the two sets of parallel lines in Figure 5 is straightened into a rectangle, Figure 6 will result. With D_1, D_2 and D_3 measured the three parallel lines may be constructed at their respective distances. With the base points marked on the three lines in the photograph, the

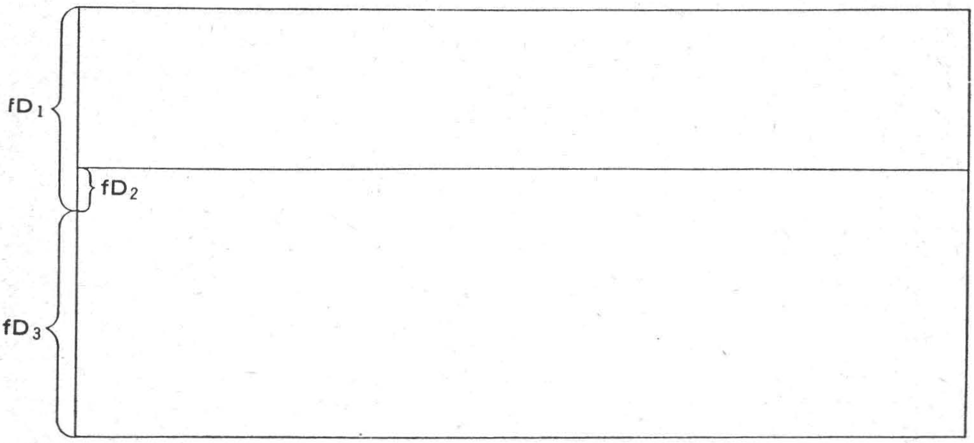

FIGURE 6

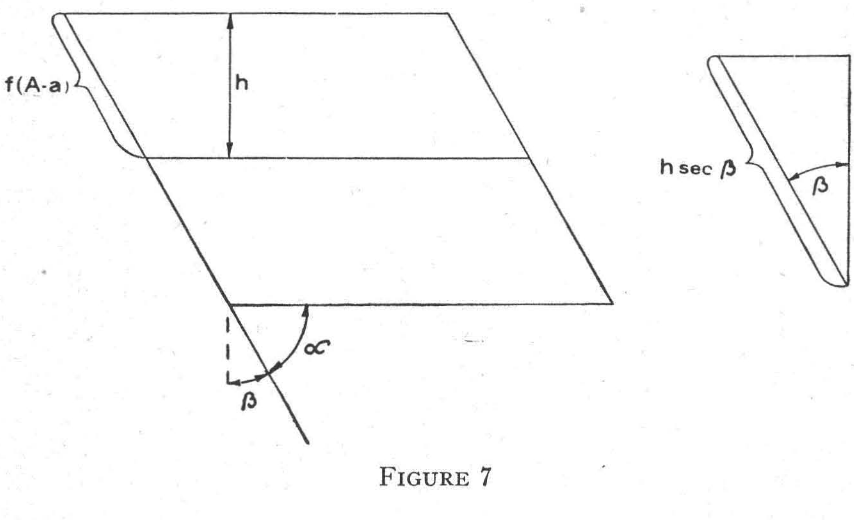

FIGURE 7

rectangle may be placed over the photograph and distorted until each line passes through its respective base point. The principal line is parallel to the three parallels, and since θ in degrees is equal to the angle (α) which the parallelogram has been distorted, the tilt of the photograph can be determined.

To facilitate the graphical construction of the parallelogram, an easily improvised device has been designed, consisting of four rigid members bolted together at their extremities to form a collapsible rectangular frame (Fig. 8).

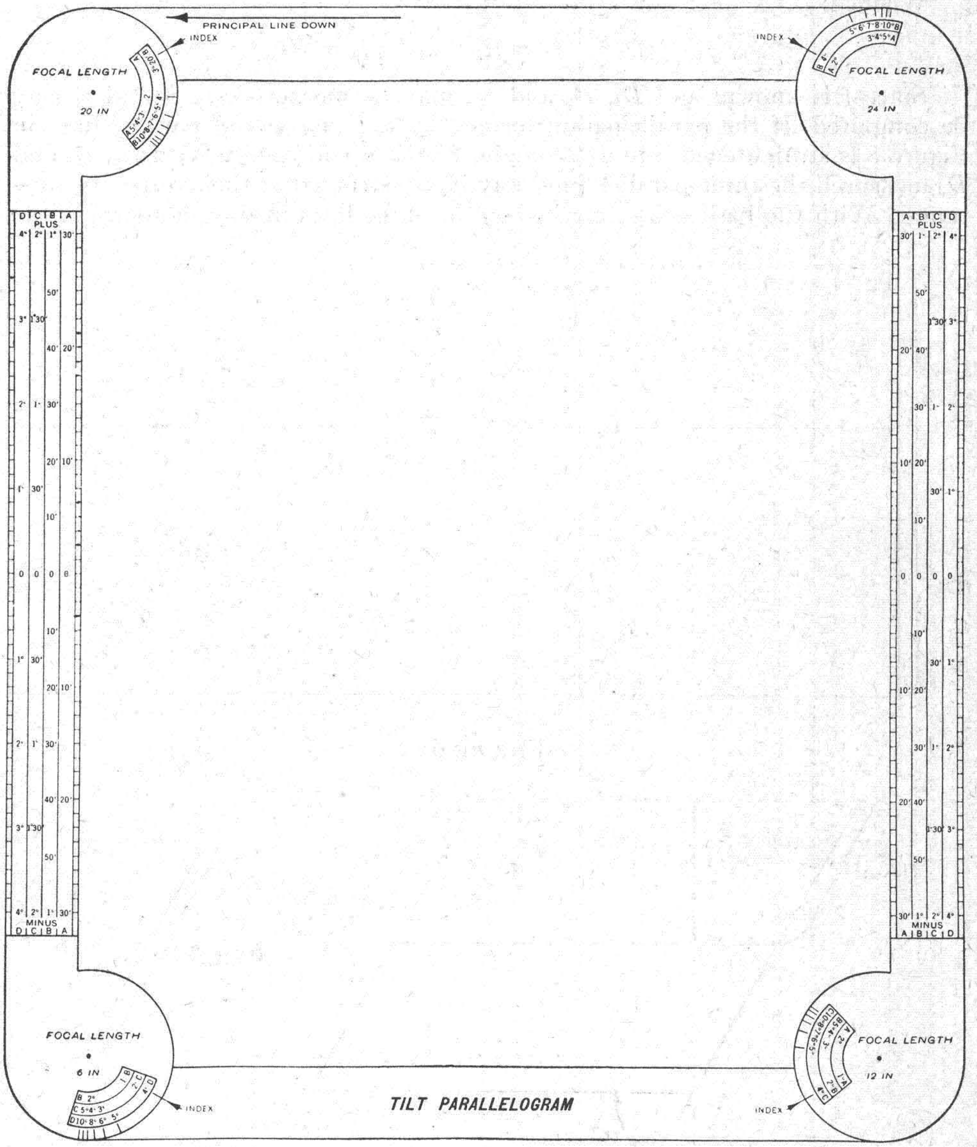

FIGURE 8

Three movable cross-members are provided which can be set at their proper distances over scales on two opposite sides of the frame. At each corner of the frame are included scales for reading tilt in degrees. The construction of these scales is complicated by the necessity of keeping the device compact.

In constructing the side scales equation (3) is used as the basis. Since h, the perpendicular distance from base point to the principal line, is equal to $m \cos a$, equation (3) may be written:

$$f(A - a) = h\theta. \tag{9}$$

If the side scales are calibrated so that the distance from 0 to +1 degree is equal to the focal length, the actual distance from zero to a movable cross-member set on its $A-a$ value will equal $f(A-a)$, which is the left side of equation (9). Also when the three cross-members are set over their respective base points without distorting the parallelogram, that is, if the angle in Figure 7 is zero, then: $f(A-a)=h$ and from equation (9) it is evident that $\theta=1$ degree.

In developing equation (7) we let α be the angle whose cosecant is equal to θ in degrees in Figure 7 and since $\beta=90-\alpha$, then:

$$\sec\beta = \theta \text{ in degrees.} \tag{10}$$

Under the conditions outlined above, that is with the side scales calibrated for $f=6$ inches, the $(A-a)$ scale from -1 to $+1$ degree will be 12 inches long, and the tilt scale at the corner of the parallelogram may be calibrated according to (10). However, it is desirable to give the parallelogram greater range, since the three values of $(A-a)$ will not always lie between -1 and $+1$ degrees. Either the frame must be longer, or the scales must be expanded. Figure 8 is a drawing of the scales for a parallelogram which willo accmmodate photographs

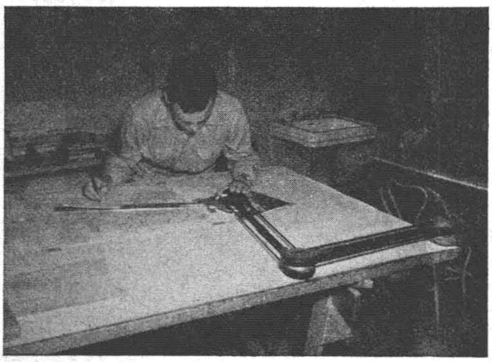

FIG. 9. Measuring a map angle with drafting machine.

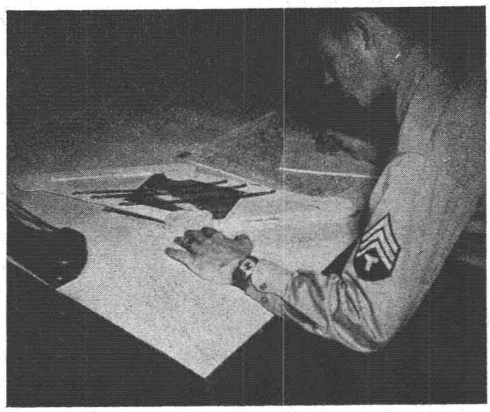

FIG. 10. The tile parallelogram in operation.

of 6, 12, 20 and 24 inch focal lengths. It is designed to fit a parallelogram having side scales 12 inches long. The scales were developed as follows:

Equation (9) may be written:

$$\frac{f(A-a)}{\theta_n} = h\frac{\theta}{\theta_n} \tag{11}$$

where θn is any value greater than unity. If the side scales are calibrated to make the distance from 0 degrees to 1 degree in inches $=f/\theta_n$, it is evident that the tilt scale equation (10) must be altered to:

$$\sec\beta = \frac{\theta}{\theta_n} \quad \text{or} \quad \cos\beta = \frac{\theta_n}{\theta}\cdot \tag{12}$$

From which it is evident that no tilts of less than θ_n degrees can be measured. The difficulties of arriving at a workable scale length may be explained in terms of θ_n. By making θ_n large, that is, by compressing the side scales in order to contain all probable values of $(A-a)$, it becomes impossible to measure small tilts. If θ_n is reduced, small tilts can be measured, but the side scales will either become too long or not contain all the probable values of $(A-a)$. The solution lies in providing multiple scales for both the $(A-a)$ or side scales and the tilt (β) scales, as shown in Figure 8. The limitations of the four scales are tabulated below:

	Range of (A−a) in 12″ scale	Minimum value of θ measurable for $f=6''$	$f=12''$	$f=20''$	$F=24''$
Scale A	−30′ to +30′	30′	1°	1°40′	2°
B	− 1° to + 1°	1°	2°	3°20′	4°
C	− 2° to + 2°	2°	4°	6°40′	8°
D	− 4° to + 4°	4°	8°	12°00′	16°

In practice, scale *A* is not used for focal lengths of 6″ since scale *B* will measure tilts down to 1 degree. Scale *D* is not used for 12″ and scale *C* and *D* are not used for 24″ since the minimum tilts they will measure are too great.

As an example of the manner in which the circular scales are constructed, the computations for scale *D* with $f=6''$ are tabulated below. Letting $\theta=4$ in equation (12) we have:

$$\cos\beta = \frac{\text{Scale Reading}}{4}$$

Scale reading	cos β	Angle β
4°	1.00000	0°00′
5°	.80000	36°52′
6°	.66667	48°10′
7°	.57143	55°10′
8°	.50000	60°00′
9°	.44444	63°39′
10°	.40000	66°25′

ANNOUNCEMENT

THE Secretary has received a request from the Swiss Association of Documentation through Mr. Robert Zurlinden, who is a member of this Society, for one copy of the April-May-June, 1942 issue of PHOTOGRAMMETRIC ENGINEERING. That issue is out of print and the Society has no copies left. Inasmuch as that issue is desired for the benefit of Swiss readers in general, it would be greatly appreciated if any member who has an extra copy and would be willing to relinqish it would communicate with the Secretary. Payment received for the issue would of course be turned over to the person furnishing it.