MATHEMATICAL PROOF OF THE VALIDITY OF THE GRAPHIC METHOD TO DETERMINE THE GREAT CIRCLE DISTANCE BETWEEN TWO GIVEN POINTS ON OR NEAR THE SURFACE OF THE EARTH

Henry H. Silverstein

AS a preliminary step in our proof, the following simple theorems must be proven:

Theorem I—When two great circles are cut by two parallels perpendicular to the great circles, the great circle arcs joining the opposite vertices of the spherical quadrilateral formed are equal.

That is from Figure 1: Given two great circles *O-NADS* and *O-NBCS*

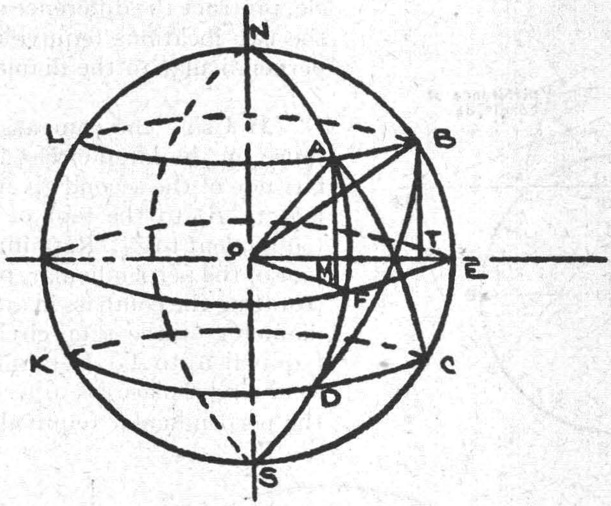

FIGURE 1

and two parallels *LAB* and *KDC* perpendicular to the great circles, to prove that arc *AC* equals arc *BD* of great circles passing through these points.

From points *A* and *B* drop perpendiculars to the great circle parallel to the given parallels intersecting at *M* and *T* respectively. Since perpendiculars between parallel planes are equal, $\overline{AM}$ equals $\overline{BT}$. $\overline{OA}$ and $\overline{OB}$ being radii of the sphere are equal. $\overline{AM}$ is perpendicular to $\overline{OM}$ and $\overline{BT}$ is perpendicular to $\overline{OT}$ since a line perpendicular to a plane is perpendicular to every line in the plane passing through its foot. Hence angles *AMO* and *BTO* are right angles and are also equal. Therefore triangles *AOM* and *BOT* are congruent since the hypotenuse and an arm of one are equal respectively to the hypotenuse and an arm of the other. Hence angle *AOM* equals angle *BOT* being corresponding parts of congruent triangles. Since all great circles of a sphere are equal, equal central angles subtend equal arcs and arc *AF* equals arc *BE*. Similarly arc *FD* equals arc *EC* and by adding equals to equal arc *AD* equals arc *BC*.

A spherical triangle is formed by the intersection of three great circles on the surface of a sphere. After passing great circles through points *A* and *C* and through points *B* and *D* let us examine spherical triangles *ACD* and *BCD*.

Spherical angle *ADC* is measured by the dihedral angle between the planes of parallel *KDC* and great circle *O-NAD*. Since these circles are given perpendicular we have that spherical angle *ADC* is a right angle. Similarly spherical angle *BCD* is measured by the dihedral angle between the planes of parallel *KDC* and great circle *O-NBC* and is also a right angle.

By the first portion of our proof arc *AD* equals arc *BC*. Angle *ADC* equals angle *BCD* and arc *DC* equals arc *DC* by identity. Therefore spherical triangle *ADC* is congruent to spherical triangle *BCD*. (Two spherical triangles are congruent if two arcs and the included angle of one triangle are equal respectively to two arcs and the included angle of the other triangle.) Therefore arc *AC* equals arc *BD* being corresponding parts of congruent spherical triangles—Q.E.D.

Theorem II—Since all great circles are equal and equal arcs of equal circles have equal chords, chord $\overline{AC}$ equals chord $\overline{BD}$.

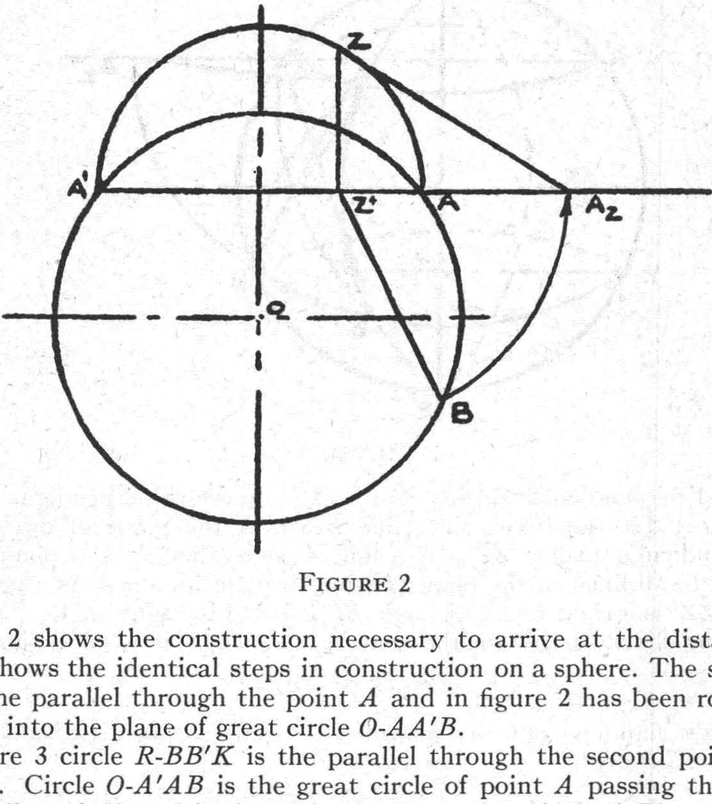

FIGURE 2

Figure 2 shows the construction necessary to arrive at the distance $\overline{A_2Z}$. Figure 3 shows the identical steps in construction on a sphere. The semi-circle *AZA'* is the parallel through the point *A* and in figure 2 has been rotated 90° to bring it into the plane of great circle *O-AA'B*.

In figure 3 circle *R-BB'K* is the parallel through the second point on the sphere, *B'*. Circle *O-A'AB* is the great circle of point *A* passing through the pole (meridian of *A*) and is shown in the same plane in both Figures 2 and 3. Circle *O-NB'S* is the great circle of point *B'* passing through the pole (meridian of *B'*). Point *B* is the point on great circle *O-A'AB* with the same latitude as point *B'*. Similar points in Figures 2 and 3 have been lettered identically to show the cross correspondence.

In Figure 2 arc *AZ* has been designated as the difference in longitude between points *A* and *B'*. This is shown more clearly by noting arc *AZ* of Figure 3. The perpendicular *ZZ'* of Figure 2 is similarly shown in Figure 3. Line *Z'B* is identical in both Figures. The point A_2 is also constructed in Figure 3 as in Figure 2.

The graphic method of computation states that the distance $\overline{A_2Z}$ when used as a chord of circle *O-ABA'* gives an arc equal to the arc of the great circle through points *A* and *B'*.

Therefore, we must prove that the distance $\overline{A_2Z}$ equals the chord $\overline{AB'}$ of great circle $O\text{-}AB'$.

From Figure 2, applying the Pythagorean theorem on right triangle A_2ZZ', we find that

$$\overline{A_2Z}^2 = \overline{ZZ'}^2 + \overline{Z'A_2}^2.$$

Since $\overline{BZ'}$ equals $\overline{Z'A_2}$ we may substitute in the above equation and find that

(1) $$\overline{A_2Z}^2 = \overline{ZZ'}^2 + \overline{BZ'}^2.$$

Now in Figure 3 we examine triangle BZZ'. The planes of circles $V\text{-}A'ZA$ and $O\text{-}A'AB$ are perpendicular. Line $\overline{ZZ'}$ in plane of circle $V\text{-}A'ZA$ has been

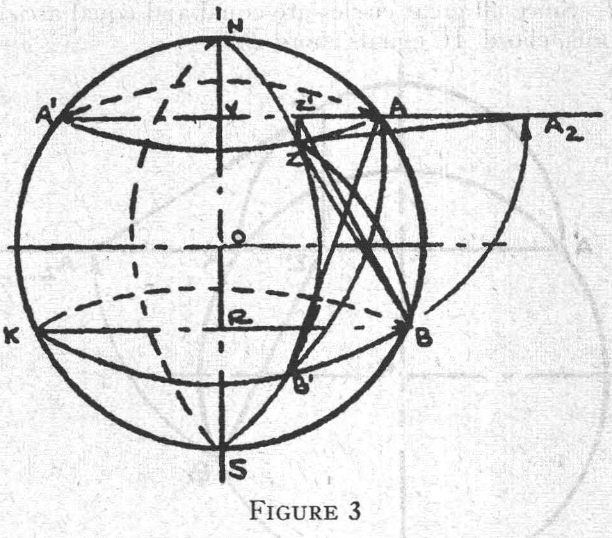

FIGURE 3

constructed perpendicular to line $A'A$ and is therefore perpendicular to plane of circle $O\text{-}A'AB$. Therefore since line $\overline{BZ'}$ is in the plane of circle $O\text{-}A'AB$ it is perpendicular to line $\overline{ZZ'}$. (If a line is perpendicular to a plane it is perpendicular to all lines on the plane passing through its foot.) As a consequence triangle BZZ' is a right triangle; angle $BZ'Z$ being its right angle.

Applying the Pythagorean theorem to triangle BZZ' we find that

(2) $$\overline{BZ}^2 = \overline{ZZ'}^2 + \overline{BZ'}^2.$$

The right hand portion of equations (1) and (2) are identical. Therefore' $\overline{BZ}^2$ equals $\overline{A_2Z}^2$ or

$$BZ \text{ equals } A_2Z.$$

Points A, Z, B and B' of Figure 3 are identical to points B, A, C and D of Figure 1. Since we have already proven by Theorems I and II that chord $\overline{BD}$ equals chord $\overline{AC}$ of Figure 1 we have now that chord $\overline{AB'}$ equals chord $\overline{BZ}$. Since $\overline{BZ}$ equals A_2Z and BZ also equals $\overline{AB'}$ we have finally that

$$\text{Chord } \overline{A_2Z} \text{ equals chord } \overline{AB'}.$$

As has been mentioned on Theorems I and II, all great circles of a sphere are equal and that equal chords subtend equal arcs in equal circles, we find that the arc subtended by chord $\overline{A_2Z}$ on great circle $O\text{-}A'AB$ equals the arc subtended by chord $\overline{AB'}$ on great circle $O\text{-}AB'$ which is the angular distance between the designated points.—Q.E.D.