uralter Unsinn! (Aschenbrenner 1926!)

PIVOT POINTS FOR RADIAL LINE PLOTTING

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BECAUSE it is readily obtainable from registered fiducial marks, the principal point of an aerial photograph is commonly used as the pivot in making radial line plots and in setting slotted templates. Often, a well defined image point of an object, lying near the center, is used instead, because of the ease with which that point can be spotted on overlapping photographs. Since neither of these points is the best point to use as pivot, it is proposed to analyze herein the errors arising in direction lines drawn, on slightly tilted vertical aerial photographs, from the principal point, from the center of distortion (isocenter), and from the vertical or plumb point.

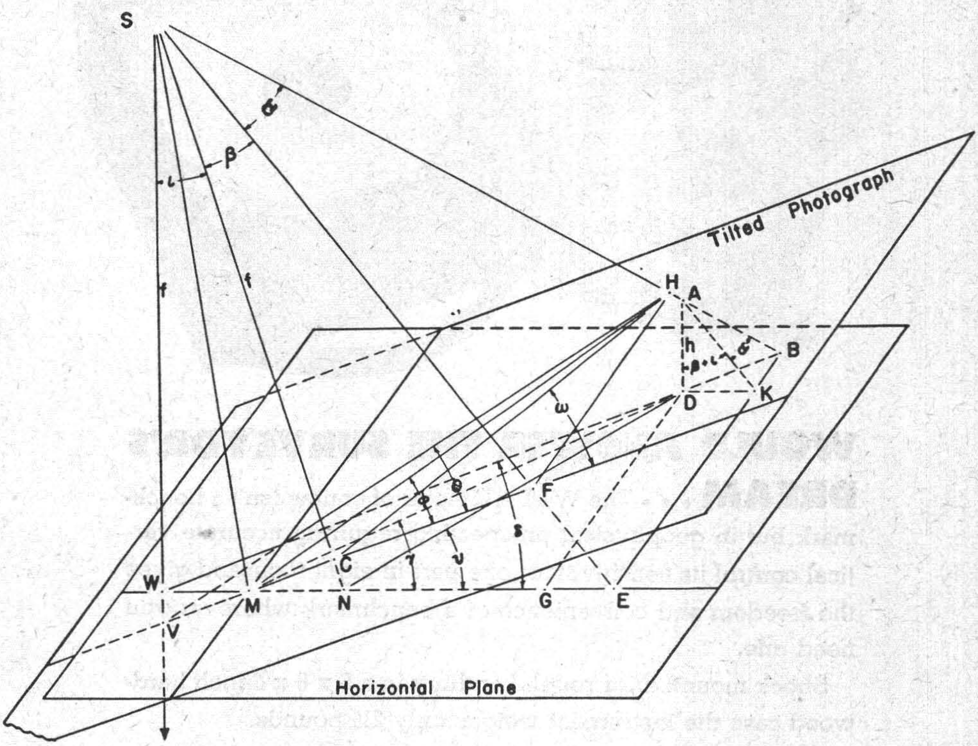

FIG. 1

Figure 1 is the setup that will be employed to deduce fundamental equations. In it S is the perspective center, C the principal point, M the center of distortion, and V the vertical point of a tilted aerial photograph whose focal distance SC is denoted by f and whose angle of tilt is denoted by i. Intersecting the tilted photograph at the point M is a horizontal plane which is to be used as the reference plane. Angles measured in this plane will equal corresponding horizontal angles measured on the ground, hence it will afford the required comparisons. Any object of the terrain, not in the datum plane, that is represented in the aerial photograph and in the intersecting horizontal plane, provided its image is not in the principal line of the photograph, will serve for the analysis. Let A

represent such an object. It stands at a height h above the horizontal plane and its image is caught in the photograph at H. The ray SA prolonged strikes the horizontal plane at B.

The line containing the points V, M, and C is the principal line of the photograph, and the principal plane cuts the line WE from the horizontal plane. Lines perpendicular to WE are drawn from D and B, and the line HF is drawn perpendicular to the principal line. The other auxiliary lines of the figure, which connect significant points, are drawn to facilitate the analysis.

The principal line is used as origin of angles measured on the photograph and the line WE as origin of corresponding angles measured in the horizontal plane. The equations desired are ones which will give:

(1) The value of angle HVF with respect to angle DWE. For this purpose let θ denote angle HVF and δ denote angle DWE.

(2) The value of angle HMF with respect to angle DME. For this purpose let ϕ denote angle HMF and γ denote angle DME.

(3) The value of angle HCF with respect to angle DNE. For this purpose let ω denote angle HCF and υ denote angle DNE.

The line WB, which is cut from the horizontal plane by the vertical plane SWB, necessarily passes through the point D. The figure is completed by letting α and β denote the two auxiliary angles as shown, and by drawing DK perpendicular to BE and AK parallel to SE. Then angle $DAK=\beta+i$, and angle $BAK=\alpha$.

For the Vertical Point V, and its corresponding position W in the horizontal plane

$$\tan\theta = \frac{HF}{VF} = \frac{\tan\alpha}{\sin\beta + \cos\beta\tan i}$$

$$\tan\delta = \frac{DG}{WG} = \frac{\tan\alpha}{\sin(\beta + i)}\cdot$$

$$\tan\theta = \frac{\tan\delta\sin(\beta + i)}{\sin\beta + \cos\beta\tan i} = \tan\delta\cos i \qquad (1)$$

$$\text{or } \tan\delta = \frac{\tan\theta}{\cos i}\cdot \qquad (2)$$

It will be noted that equation (1) is the same as that obtained for the vertical point when there is no relief. This means, of course, that relief has no effect on angles measured about the vertical point.

At the Center of Distortion, M

$$\tan\phi = \frac{HF}{MF} = \frac{\dfrac{f\tan\alpha}{\cos\beta}}{f\left(\tan\beta + \tan\dfrac{i}{2}\right)} = \frac{\tan\alpha}{\sin\beta + \cos\beta\tan\dfrac{i}{2}}$$

$$\tan \gamma = \frac{DG}{MG} = \frac{\dfrac{(f-h)\tan\alpha}{\cos(\beta+i)}}{(f-h)\tan(\beta+i) - f\tan\dfrac{i}{2}}$$

$$= \frac{(f-h)\tan\alpha}{(f-h)\sin(\beta+i) - f\cos(\beta+i)\tan\dfrac{i}{2}}$$

$$= \frac{\tan\alpha}{\sin(\beta+i) - \dfrac{f}{f-h}\cos(\beta+i)\tan\dfrac{i}{2}} \tag{3}$$

and

$$\tan\phi = \frac{\tan\gamma\left[\sin(\beta+i) - \dfrac{f}{f-h}\cos(\beta+i)\tan\dfrac{i}{2}\right]}{\sin\beta + \cos\beta\tan\dfrac{i}{2}} \tag{3a}$$

or

$$\tan\gamma = \frac{\tan\phi\left(\sin\beta + \cos\beta\tan\dfrac{i}{2}\right)}{\sin(\beta+i) - \dfrac{f}{f-h}\cos(\beta+i)\tan\dfrac{i}{2}}. \tag{3b}$$

For the Principal Point, *C*, and its corresponding position, *N*, in the horizontal plane

$$\tan\omega \frac{HF}{CF} \quad \frac{f\tan\alpha}{f\cos\beta\tan\beta} = \frac{\tan\alpha}{\sin\beta}$$

$$\tan\upsilon = \frac{DG}{NG} = \frac{BE - BK}{WE - WN - GE} = \frac{\dfrac{(f-h)\tan\alpha}{\cos(\beta+i)}}{f\tan(\beta+i) - f\tan i - h\tan(\beta+i)}$$

$$= \frac{\tan\alpha}{\sin(\beta+i) - \dfrac{f}{f-h}\tan i\cos(\beta+i)} \tag{4}$$

and

$$\tan\omega = \frac{\tan\upsilon\left[\sin(\beta+i) - \dfrac{f}{f-h}\tan i\cos(\beta+i)\right]}{\sin\beta} \tag{4a}$$

or

$$\tan\upsilon = \frac{\tan\omega\sin\beta}{\sin(\beta+i) - \dfrac{f}{f-h}\tan i\cos(\beta+i)}. \tag{4b}$$

Equations (1) and (2) hold good for all values of θ and δ. The other equations are in proper form for the upper side of the photograph, the angles being turned from the upper branch of the principal line, but when β is less than i the term $\beta+i$ changes to $i-\beta$. For the lower side of the photograph, the terms commencing with $f/f-h$ change from minus to plus and in general $\beta+i$ changes to $\beta-i$.

When $h=0$, equation (3b) reduces to $\tan \gamma = \tan \phi$, and equation (4b) to $\tan v = \tan \omega \cos i$. The latter then takes the same form as equation (1), showing that the effect of tilt is the same for the vertical point as for the principal point and that when there is no relief the maximum error occurs for a direction at 45° to the principal line.

Computations made with equations (3) to (4b) inclusive are laborious. It would be desirable to have tables made up from the results of enough computations to show readily the amount of relief and tilt that can be tolerated in plotting at the scales commonly used, but instead, only some general ideas of limits can be given here. Before proceeding with this, however, the great advantage to be got from using the vertical point as pivot should be emphasized. Since relief has no effect on radial lines drawn from the vertical point, the importance of having a quick method of finding the position of this point on each photograph is evident. The maximum error in a direction line drawn from the vertical point, that can be caused by tilt, is, for $i=2°$, but one minute, and for $i=3°$, two minutes and twenty-one seconds. In terms of displacement the corresponding errors are for $i=2°$, one part in 3500, for $i=3°$, one part in 1460, the latter being equivalent to one hundredth of an inch in 14.6 inches, which is accurate enough for standard mapping at the larger scales commonly used.

The limits of 2° and 3° for tilt are used here because tilt determinations made at the Institute of Geographical Exploration of Harvard University indicate that about ten per cent of the photographs tested had tilt exceeding 2°, while none were tilted more than $2\frac{1}{2}$°. About 60% of the photographs were tilted less than 1°. The fact that the maximum effect of tilt occurs along only four radial lines of each photograph and there are but four sectors, totaling a third or a half of each photograph, that are appreciably affected by moderate tilt, deserves some consideration. The question which arises is this: Is it safe to assume that of all points of a photograph selected for location none will fall near enough to any one of the four lines of maximum tilt effect to cause that full effect to come into play? It seems that "no" must be the answer, for this reason: In laying down templets for a radial line plot, or in making a radial line plot on a transparent sheet of paper or acetate, one affected direction line spreads its effect through and will materially reduce the accuracy of the entire plot. This is having in view the making of standard maps. Reconnaissance, exploratory, or other maps of small scale have to be made under conditions which seldom warrant the care, time, and expense necessary to maintain a high degree of accuracy. Under these conditions, sub-standard practices are necessary and allowable to permit a job to be finished at reasonable cost and in an allotted period of time. Since tolerances then must in general be widened, there is no reason why the limit for tilt, or the effects of tilt, should not be stretched.

Next the isocenter will be considered as pivot point. It may be dismissed with brief comment. The only condition under which it becomes available for use is that which also provides the position of the vertical point, namely, when the principal line and degree of tilt are known. If there is relief (the general case) the vertical point is always a better point to use as pivot than the isocenter. Only when relief is lacking and the tilt excessive does the isocenter give any advantage over the vertical point. This occurs so rarely that the isocenter may be excluded for general use as the pivot point in radial line plotting.

The chief merit of the principal point is that it is provided by the camera. It will suffice to indicate briefly the magnitude of errors caused by tilt and relief when this point is used as pivot in radial line plotting. Inasmuch as focal lengths have little influence on the effects of tilt, average approximations will serve to indicate the errors liable to occur. From these quantities allowable limits for commonly-used scales may be estimated.

The angular error (for the locus of maximum values) is greatest near the principal point. Going outward from that point, it diminishes rapidly at first but as the distance increases the rate of change in the angular error grows less. On the other hand, the rate of change of the lateral displacement is very slow. Lateral displacement is here used to denote the amount the image point is shifted by tilt and relief away from a radial line drawn from the principal point that is true in direction with reference to the principal line of the photograph. Its value is nearly constant for distances spanned by verticals and transformed obliques. Therefore in the following table, average lateral displacements are given instead of angular errors, for thereby a short table will serve in lieu of a lengthy one.

Tilt	Scale of Photo	Relief in feet	Approx. maximum lateral displacement of image point in inches
1°	1:24,000	600	0.006
2°	1:24,000	600	0.013
3°	1:24,000	600	0.022

For the scale 1:12,000 and 600 feet of relief, the maximum lateral displacements are about double those given in the table; for the scale 1:48,000 about half the amounts shown. If the allowable amount of lateral displacement is .01 inch and tilt up to 3° is to be accepted, it is seen that for plotting at the scale 1:24,000 the relief should not exceed 300 feet. For plotting at the 1:48,000 scale under the same conditions 600 feet of relief can be satisfactorily dealt with.

Actually the lateral displacement gradually diminishes as the distance between the principal point and the image increases, but the effect of distance is so slight it may be neglected. However, increase in distance is important in the use of high obliques for small-scale plotting to obtain radial line to distant objects, for the greater the distance the smaller the angular error, and the length of the radial line will usually not be more than a small fraction of its length for standard mapping. This is simply another way of stating that the smaller the scale of the plotting the greater may be the allowable relief and tilt.

One more item, relating to the principal point as pivot, deserves consideration. This is the effect of relief and tilt in displacing an image point which coincides with the principal point. It does not affect the location of the pivot point itself, for that point is in effect located by projecting the optical axis to the datum plane of the map. However, it is customary (and usually necessary) to spot a pivot point on overlapping photographs by image details, and when so spotted its position on any one of those photographs will be in error if the object producing the image details is higher or lower than the datum plane. The amount of error is the elevation difference involved multiplied by the tangent of the angle of tilt. For an elevation difference of 600 feet and a tilt of 2° this would be about one-hundredth of an inch at the 1:24,000 scale. In any particular case the entire amount of the error might come into play or its effect might be some amount between the whole and zero.

No similar error can affect the vertical point because the plumb line passes through the top and base of any object at the corresponding position of the terrain. The advantage to be gained from using the vertical point as pivot in radial line plotting warrants the expenditure of every effort and money necessary to obtain a means of quickly locating it on every photograph to be used. The horizon image seems to offer the surest and easiest solution. Through the manufacture of faster and more sensitive film, the prospects of overcoming atmospheric difficulties have been greatly improved during the last few years. It is suggested that attention should be given to the development of auxiliary horizon cameras to go with the principal camera. Such cameras can not be small; they should have a focal length long enough to assure determination of tilt with error no greater than a few minutes.