SOME NOTES ON STEREOSCOPIC PARALLAX

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As POINTED out in an excellent paper by Albert L. Nowicki,[1] serious errors exist in the Master Parallax Table as used in stereocomparagraph plotting. Inasmuch as these tables are widely used, it would seem desirable to present alternate methods of preparing parallax data. Therefore it is the purpose of this paper to demonstrate mathematically means of compiling parallax information correctly fitting the plotting at hand.

According to Captain Nowicki's article, the errors which exist in the Master Parallax Table may be divided into two categories: (1) Elevation errors caused by form of equation used in preparing tables; (2) errors caused by fact that tables are based on an altitude of 25,000 feet and no provision is offered for adaptation to other altitudes.

The approximation causing the elevation error may be shown by the following reasoning. If we start with the basic parallax equation

$$p = B_m\left(\frac{h}{H - h}\right) \tag{1}$$

where p is the stereoscopic parallax
B_m is the stereo base distance at sea level
H is the altitude of air station above sea level
h is the elevation of the point above sea level

a relation for Δp, the increment of parallax may be obtained without any approximations. That relation is

$$\Delta p = \frac{B_m H \Delta h}{(H - h)(H - h - \Delta h)} \cdot \tag{2}$$

If Δh is considered to be small, the following form may be used with small error

$$\Delta p = \frac{B_m H \Delta h}{(H - h)^2} \cdot \tag{3}$$

By considering one of the $(H-h)$ factors in the denominator of (3) to be equivalent to H, the relation on which the Master Parallax Table is based will result, namely

$$\Delta p = \frac{B_m \Delta h}{(H - h)} \cdot \tag{4}$$

As pointed out in the forementioned paper, the above relation may be corrected by multiplying the tabular values by either $H/(H-h)$ or $H/(H-h-\Delta h)$. These factors serve to modify (4) into either (3) or (2).

If equation (1) is differentiated, an entirely correct relation in terms of differentials will result

[1] "The Stereocomparagraph Plotting Machine," PHOTOGRAMMETRIC ENGINEERING, April–May–June, 1943.

$$dp = B_m \frac{H\,dh}{(H-h)^2}. \tag{5}$$

By integration it is possible to determine the total parallax between H and $(H-h)$.

$$\begin{aligned} p = \Sigma\Delta p &= B_m H \int_{H-h}^{H} \frac{dh}{(H-h)^2} \\ &= -B_m H \left[\frac{1}{H-h} - \frac{1}{H}\right]. \end{aligned} \tag{6}$$

Equation (6)[2] is comparatively simple to handle and presents the obvious advantages of being adapted to the particular value of the flight altitude and of being mathematically accurate. The negative sign, of course, may be disregarded.

Table I has been prepared based on equation (6) to show a possible arrangement for the preparation of data to a given altitude and a B_m of 90 mm. Since tables of reciprocals are easily obtainable, preparation of this and similar tables may be reduced to routine mechanical methods.

TABLE I

Values of $\Sigma\Delta p$ and Δp Based on Relation $\Sigma\Delta p = B_m H\left[\frac{1}{H-h} - \frac{1}{H}\right]$.

$\left(\frac{1}{H} = 0.5\times10^{-4},\ B_m H = 180\times10^{4},\ H = 20{,}000 \text{ feet, and } B_m = 90 \text{ mm.}\right)$

(1)	(2)	(3)	(4)	(5)	(6)
$H-h$	h	$\frac{1}{H-h}(\times10^4)$	$\left[\frac{1}{H-h} - \frac{1}{H}\right]$ $(\times10^4)$	$(4)\times B_m H = \Sigma\Delta p$	Δp
18000	2000	0.5555556	0.0555556	10.0000	
					1.1236
17800	2200	0.5617978	0.0617978	11.1236	
					1.1491
17600	2400	0.5681818	0.0681818	12.2727	
					1.1756
17400	2600	0.5747126	0.0747126	13.4483	
					1.2029
17200	2800	0.5813953	0.0813953	14.6512	
					1.2312
17000	3000	0.5882353	0.0882353	15.8824	
					1.2605
16800	3200	0.5952381	0.0952381	17.1429	
					1.2908
16600	3400	0.6024096	0.1024096	18.4337	
					1.3224
16400	3600	0.6097561	0.1097561	19.7561	
					1.3550
16200	3800	0.6172840	0.1172840	21.1111	
					1.3889
16000	4000	0.6250000	0.1250000	22.5000	

[2] By gathering terms, it will be recognized that equation (6) is equivalent to equation (1).

Parallax tables similar mathematically to the Master Parallax Table wherein the flight altitude only is corrected may be prepared with similar ease. Equation (4) may be written in differential form as

$$dp = B_m \frac{dh}{(H-h)}. \tag{7}$$

This may also be integrated to give a p or $\Sigma\Delta p$ similar to the Master Parallax Table, but possessing the virtue that the data are adjusted to the correct altitude.

$$\begin{aligned} p = \Sigma\Delta p &= B_m \int_{H-h}^{H} \frac{dh}{(H-h)} \\ &= -B_m[\log_e H - \log_e (H-h)]. \end{aligned} \tag{8}$$

Equation (8) is quite as easy to apply and lends itself to convenient tabulation. Tables of natural logarithms are readily available.[3] Table II presents parallax data based on (8) for the same range and conditions as Table I. Here again, the negative sign may be disregarded.

TABLE II

Values of $\Sigma\Delta p$ and Δp Based on Relation $\Sigma\Delta p = B_m[\log_e H - \log_e (H-h)]$.
(H=20,000 feet, B_m=90 mm., and Log$_e$ 20,000=9.9034876.)

(1)	(2)	(3)	(4)	(5)	(6)
$H-h$	h	Log$_e(H-h)$	Log$_e H-(3)$	$\Sigma\Delta p=(4)\times 90$	Δp
18000	2000	9.7981270	0.1053606	9.4825	
					1.0056
17800	2200	9.7869537	0.1165339	10.4881	
					1.0169
17600	2400	9.7756542	0.1278334	11.5050	
					1.0286
17400	2600	9.7642255	0.1392621	12.5336	
					1.0405
17200	2800	9.7526647	0.1508229	13.5741	
					1.0526
17000	3000	9.7409686	0.1625190	14.6267	
					1.0651
16800	3200	9.7291342	0.1743534	15.6918	
					1.0779
16600	3400	9.7171580	0.1863296	16.7697	
					1.0909
16400	3600	9.7050366	0.1984510	17.8606	
					1.1043
16200	3800	9.6927665	0.2107211	18.9649	
					1.1180
16000	4000	9.6803440	0.2231436	20.0829	

Captain Nowicki also sets up another interesting set of data to show the fallacies of the Master Parallax Table. He computes, through the aid of a graph Δp for a thousand-foot difference of elevation between the values of $(H-h)$ of 8000 feet and 7000 feet at varying flight altitudes between 25,000 feet and 10,000 feet for a B_m of 100 mm. The Master Parallax Table gives a Δp of

[3] Works Progress Administration. *Tables of Natural Logarithms*, Volume I, Logarithms of Integers from 1–50,000. New York, 1941.

127.296−113.943=13.353 mm. regardless of the flight altitude. In the following tabulations relations (6) and (8) as presented in this paper are contracted for the above data.

H = 25,000 feet

$(H - h) = 7000$ feet

by (6) $$p = B_m H\left[\frac{1}{H-h} - \frac{1}{H}\right]$$

$$= 100(25{,}000)(0.000142857 - 0.000040000)$$
$$= 257.14 \text{ mm.}$$

$(H - h) = 8000$ feet

$$p = 100(25{,}000)(0.000125000 - 0.000040000)$$
$$= 212.50$$

$$\Delta p = 44.64 \text{ mm.}$$

$(H - h) = 7000$ feet

by (8) $$p = B_m[\log_e H - \log_e (H - h)]$$
$$= 100[10.1266311 - 8.8536654]$$
$$= 127.30$$

$(H - h) = 8000$ feet

$$p = 100[10.1266311 - 8.9871968]$$
$$= 113.94$$

$$\Delta p = 13.36 \text{ mm.}$$

H = 20,000 feet

$(H - h) = 7000$ feet

by (6) $$p = B_m H\left[\frac{1}{H-h} - \frac{1}{H}\right]$$

$$= 100(20{,}000)(0.000142857 - 0.000050000)$$
$$= 185.71$$

$(H - h) = 8000$ feet

$$p = 100(20{,}000)(0.000125000 - 0.000050000)$$
$$= 150.00$$

$$\Delta p = 35.71 \text{ mm.}$$

$(H - h) = 7000$ feet

by (8) $$p = B_m[\log_e H - \log_e (H - h)]$$
$$= 100(9.9034876 - 8.8536654)$$
$$= 104.98$$

$(H - h) = 8000$ feet

$$p = 100(9.9034876 - 8.9871968)$$
$$= 91.63$$

$$\Delta p = 13.35 \text{ mm.}$$

H = 15,000 feet

$(H - h) = 7000$ feet

by (6) $$p = B_m H\left[\frac{1}{H-h} - \frac{1}{H}\right]$$

$$= 100(15{,}000)(0.000142857 - 0.000066667)$$
$$= 114.28$$

$(H - h) = 8000$ feet

$$p = 100(15{,}000)(0.000125000 - 0.000066667)$$
$$= 87.50$$

$$\Delta p = 26.78 \text{ mm.}$$

$(H - h) = 7000$ feet

by (8) $p = B_m(\log_e H - \log_e (H - h))$
$= 100(9.6158055 - 8.8536654)$
$= 76.22$

$(H - h) = 8000$ feet

$p = 100(9.6158055 - 8.981968$
$= 62.86$

$\Delta p = 13.36$ mm.

$H = $ *10,000 feet*

$(H - h) = 7000$ feet

by (6)
$$p = B_m H\left(\frac{1}{H - h} - \frac{1}{H}\right)$$
$= 100(10,000)(0.000142857 - 0.000100000)$
$= 42.86$

$(H - h) = 8000$ feet

$p = 100(10,000)(0.000125000 - 0.000100000)$
$= 25.00$

$p = 17.86$ mm.

$(H - h) = 7000$ feet

by (8) $p = B_m(\log_e H - \log_e (H - h))$
$= 100(9.2103404 - 8.8536654)$
$= 35.67$

$(H - h) = 8000$ feet

$p = 100(9.2103404 - 8.9871968)$
$= 22.31$

$p = 13.36$ mm.

TABLE III

Comparison of Δp Between $(H-h)$ of 8000 and 7000 Feet as Obtained From Various Sources for $B_m = 100$ mm.

H	Nowicki's Graph	Formlua (6)	Master Parallax Table	Formula (8)
25,000 feet	44.5 mm.	44.64 mm.	13.353 mm.	13.36 mm.
20,000	35.8	35.71	13.353	13.35
15,000	26.7	26.78	13.353	13.36
10,000	17.8	17.86	13.353	13.36

From the mathematical discussions in this paper, various conclusions may be drawn. First and foremost seems to be the inadequacy of the Master Parallax Table. Any attempt to adapt the basic form of the equation behind the tables to flight altitudes other than that for which they have been prepared seems fruitless. This may be expressed mathematically by the fact that since a table of natural logarithms represents a parallax table of this type, data are extracted from large values where the common difference is constant for tremendous ranges.

The most satisfactory solution seems to be the use of the reciprocal relation $\Sigma\Delta p = B_m H(1/H-h) - (1/H)$ to prepare individual sets of data for the compilation being done. The use of this relation as shown in Table I would seem to indicate that the establishment of a satisfactory table may be done routinely and with comparative ease.