ANALYSIS OF THE MULTIPLEX MODEL*

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THE purpose of this paper is to determine the exact effects in the multiplex model of any change in the elements of relative orientation. It has been found convenient to solve the problem first for the general case. The specific cases follow directly from the general solution.

Intersecting rays from projectors I and II form a model. The Cartesian axes are so chosen that the coordinates of the perspective centers are (0, 0, 0), and (a, b, c). A point in the model, (x_1, y_1, z_1), is determined by the intersection of ray I and ray II. In vector notation, (Fig. 1),

$$\boldsymbol{B} + \boldsymbol{r}_2 = \boldsymbol{r}_1. \tag{1}$$

The model can be altered only by a translation and/or rotation of one projector relative to the other. The construction of the multiplex requires that the translation be achieved by the three axial movements Δx, Δy, Δz. The rotation must be achieved by the three rotations α, β, γ about the x-axis, y-axis, z-axis, respectively.

If projector II is translated through

$$\boldsymbol{s} = (\Delta x)\boldsymbol{i} + (\Delta y)\boldsymbol{j} + (\Delta z)\boldsymbol{k},$$

and rotated an arbitrary amount, (1) becomes (fig. 2),

$$[a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}] + [(\Delta x)\boldsymbol{i} + (\Delta y)\boldsymbol{j} + (\Delta z)\boldsymbol{k}] - (1+\lambda)[x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}]$$
$$- (1+\mu)[(a - x_1)\mathbf{i} + (b - y_1)\mathbf{j} + (c - z_1)\mathbf{k}] = \boldsymbol{o}, \tag{2}$$

provided that ray I and ray II still intersect.

The axes **i**, **j**, **k** are defined as

$$\mathbf{i} = \alpha_{11}\boldsymbol{i} + \alpha_{12}\boldsymbol{j} + \alpha_{13}\boldsymbol{k}$$
$$\mathbf{j} = \alpha_{21}\boldsymbol{i} + \alpha_{22}\boldsymbol{j} + \alpha_{23}\boldsymbol{k}$$
$$\mathbf{k} = \alpha_{31}\boldsymbol{i} + \alpha_{32}\boldsymbol{j} + \alpha_{33}\boldsymbol{k},$$

where

$$|\alpha_{\rho\sigma}| = 1.$$

This orthogonal transformation can be interpreted geometrically. Projector II first is described with respect to the **i**, **j**, **k** system. When projector II is rotated, the **i**, **j**, **k** base vectors, if rigidly attached to the projector, would have rotated into the position **i**, **j**, **k**. If the base vectors are not allowed to rotate, then the transformation must be considered a rotation of projector II with respect to the **i**, **j**, **k** system.

In order to ensure that ray I and ray II still intersect, we must place a condition on the displacement. Thus, one of the elements of relative orientation, say Δy, shall be considered a dependent variable. Collecting terms,

$$[a + \Delta x - (1+\lambda)x_1 - (1+\mu)A]\boldsymbol{i}$$
$$+ [b + \Delta y - (1+\lambda)y_1 - (1+\mu)B]\boldsymbol{j}$$
$$+ [c + \Delta z - (1+\lambda)z_1 - (1+\mu)C]\boldsymbol{k} = \boldsymbol{o},$$

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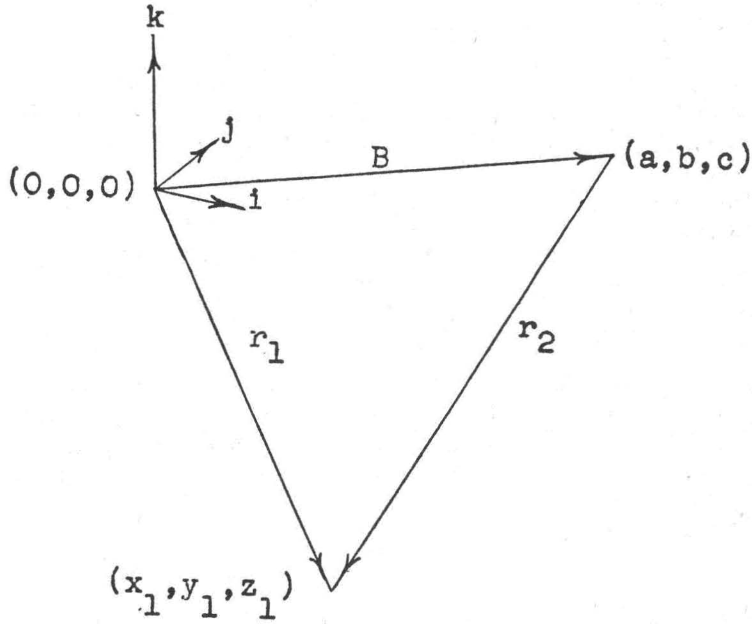
k
j
B
(0,0,0)
i
(a,b,c)
r1
r2
(x1,y1,z1)

FIG. 1.

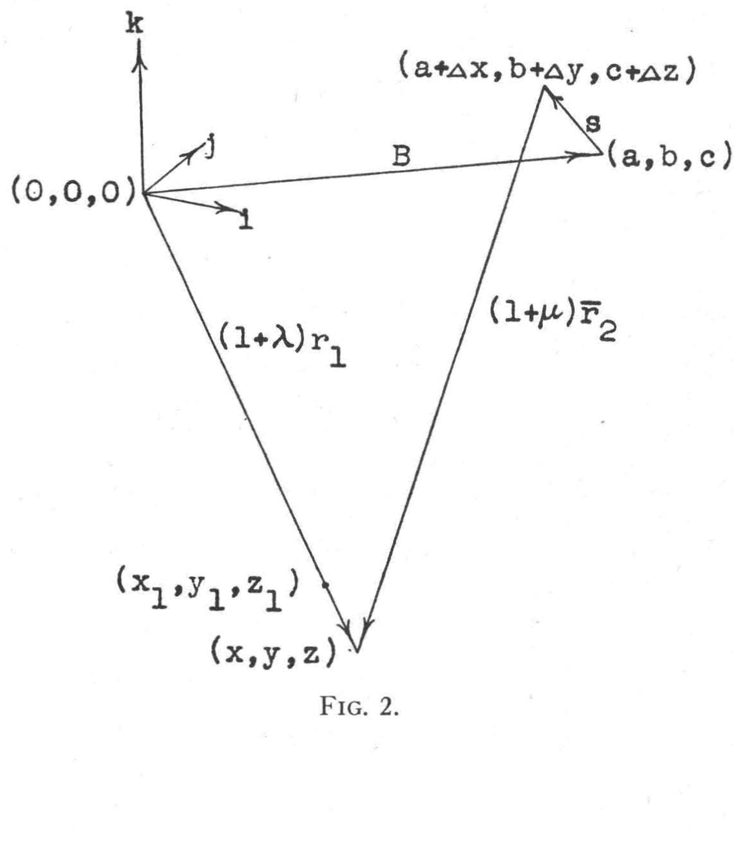
k
(a+Δx,b+Δy,c+Δz)
s
j
B
(a,b,c)
(0,0,0)
i
(1+μ)r̄2
(1+λ)r1
(x1,y1,z1)
(x,y,z)

FIG. 2.

where

$$\begin{aligned} A &= \alpha_{11}(a - x_1) + \alpha_{21}(b - y_1) + \alpha_{31}(c - z_1), \\ B &= \alpha_{12}(a - x_1) + \alpha_{22}(b - y_1) + \alpha_{32}(c - z_1), \\ C &= \alpha_{13}(a - x_1) + \alpha_{23}(b - y_1) + \alpha_{33}(c - z_1). \end{aligned}$$

Since i, j, k are linearly independent, their coefficients can be set equal to zero and we obtain,

$$\begin{aligned} x_1\lambda + A\mu \quad\quad\quad &= a - x_1 + \Delta x + A \\ y_1\lambda + B\mu - \Delta y &= b - y_1 \quad\quad\quad + B \\ z_1\lambda + C\mu \quad\quad\quad &= c - z_1 + \Delta z + C. \end{aligned}$$

Solving by Cramer's rule,

$$\begin{aligned} \lambda &= \frac{\begin{vmatrix} (a - x_1) + \Delta x & A \\ (c - z_1) + \Delta z & C \end{vmatrix}}{\begin{vmatrix} x_1 & A \\ z_1 & C \end{vmatrix}}, \\ \Delta y &= \frac{\begin{vmatrix} a + \Delta x & x_1 & A \\ b & y_1 & B \\ c + \Delta z & z_1 & C \end{vmatrix}}{\begin{vmatrix} x_1 & A \\ z_1 & C \end{vmatrix}}. \end{aligned} \tag{3}$$

In terms of the variables x, y, z, ray I can be expressed

$$(1 + \lambda)(x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k}) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$$

Collecting like terms

$$[(1 - \lambda)x_1 - x]\boldsymbol{i} + [(1 + \lambda)y_1 - y]\boldsymbol{j} + [(1 + \lambda)z_1 - z]\boldsymbol{k} = \boldsymbol{o}.$$

Since **i**, **j**, **k** are linearly independent, their coefficients can be set equal to zero. Solving these three equations,

$$\lambda = \frac{x - x_1}{x_1} = \frac{y - y_1}{y_1} = \frac{z - z_1}{z_1}.$$

To examine the effect of translation substitute in (3)

$$\alpha_{\sigma\rho} = \delta_\sigma^{\ \rho}$$

where the symbol $\delta_\sigma^{\ \rho}$, known as the *Kronecker delta*, is defined as

$$\delta_\sigma^{\ \rho} = \begin{cases} 1 & \text{for} \quad \sigma = \rho \\ 0 & \text{for} \quad \sigma \neq \rho. \end{cases}$$

Solving,

$$\lambda = \frac{x - x_1}{x} = \frac{y - y_1}{y} = \frac{z - z_1}{z} = \frac{\begin{vmatrix} \Delta x & (a - x_1) \\ \Delta z & (c - z_1) \end{vmatrix}}{\begin{vmatrix} x_1 & a \\ z_1 & c \end{vmatrix}},$$

$$\Delta y = \frac{\begin{vmatrix} \Delta x & x_1 & a \\ 0 & y_1 & b \\ \Delta z & z_1 & c \end{vmatrix}}{\begin{vmatrix} x_1 & a \\ z_1 & c \end{vmatrix}}. \tag{4}$$

The effect of an x-movement of projector II can be obtained by setting $\Delta z = 0$ in (4),

$$\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1} = \frac{z - z_1}{z_1} = \frac{(c - z_1)\Delta x}{cx_1 - az_1},$$

$$\Delta y = \frac{(cy_1 - bz_1)\Delta x}{cx_1 - az_1}. \tag{5}$$

The effect of a z-movement of projector II can be obtained by setting $\Delta x = 0$ in (4),

$$\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1} = \frac{z - z_1}{z_1} = \frac{-(a - x_1)\Delta z}{cx_1 - az_1},$$

$$\Delta y = \frac{(bx_1 - ay_1)\Delta z}{cx_1 - az_1}. \tag{6}$$

Under certain conditions projector II may be translated and no y-parallax will be introduced in the model. Assume that the original model is parallax-free. If projector II is translated in the epipolar plane determined by (x_1, y_1, z_1), no y-parallax will be introduced at (x_1, y_1, z_1). If projector II is translated in another epipolar plane determined by some other point, no y-parallax will be introduced at that point. If projector II is translated along the intersection of the epipolar planes, a line defined as the *air-base*, no y-parallax will be introduced in the model.

The equation of the air-base is

$$\boldsymbol{B} = ai + bj + ck.$$

Since $(a+\Delta x, b+\Delta y, c+\Delta z)$ lies on this vector or its extension,

$$\nu(ai + bj + ck) = (a + \Delta x)i - (b + \Delta y)j + (c + \Delta z)k$$

$$\nu = \frac{a + \Delta x}{a} = \frac{b + \Delta y}{b} = \frac{c + \Delta z}{c}, \tag{7}$$

$$\frac{\Delta x}{a} = \frac{\Delta y}{b} = \frac{\Delta z}{c}.$$

Substituting in (4),

$$\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1} = \frac{z - z_1}{z_1} = \frac{\Delta x}{a}. \tag{8}$$

Thus, (7) specifies the relations among the translatory movements in order to achieve a scale change in the model, and (8) determines the effect of such a change.

To examine the effect of x-tilt substitute in (3)

$$\begin{array}{lll} \Delta x = 0, & \Delta z = 0, & \\ \alpha_{11} = 1, & \alpha_{12} = 0, & \alpha_{13} = 0, \\ \alpha_{21} = 0, & \alpha_{22} = \cos\alpha, & \alpha_{23} = \sin\alpha, \\ \alpha_{31} = 0, & \alpha_{32} = -\sin\alpha, & \alpha_{33} = \cos\alpha. \end{array}$$

Solving,

$$\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1} = \frac{z - z_1}{z_1} = \frac{(a - x_1)\begin{vmatrix} (b - y_1) & (1 - \cos\alpha) \\ (c - z_1) & \sin\alpha \end{vmatrix}}{\begin{vmatrix} x_1 & a \\ z_1 & c \end{vmatrix} + x_1 \begin{vmatrix} (b - y_1) & (a - \cos\alpha) \\ (c - z_1) & \sin\alpha \end{vmatrix}},$$

$$\Delta y = \frac{\begin{vmatrix} a & x_1 & +(a - x_1) & (1 - \cos\alpha) \\ b & y_1 & -(c - z_1) & \sin\alpha \\ c & z_1 & +(b - y_1) & \sin\alpha \end{vmatrix}}{\begin{vmatrix} x_1 & a \\ z_1 & c \end{vmatrix} + x_1 \begin{vmatrix} (b - y_1) & (1 - \cos\alpha) \\ (c - z_1) & \sin\alpha \end{vmatrix}}. \tag{9}$$

To examine the effect of y-tilt substitute in (3)

$$\begin{array}{lll} \Delta x = 0, & \Delta z = 0, & \\ \alpha_{11} = \cos\beta, & \alpha_{12} = 0, & \alpha_{13} = \sin\beta, \\ \alpha_{21} = 0, & \alpha_{22} = 1, & \alpha_{23} = 0, \\ \alpha_{31} = -\sin\beta, & \alpha_{32} = 0, & \alpha_{33} = \cos\beta. \end{array}$$

Solving,

$$\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1} = \frac{z - z_1}{z_1} = \frac{\begin{vmatrix} (a - x_1) & -(c - z_1) \\ (c - z_1) & +(a - x_1) \end{vmatrix} \tan\beta}{\begin{vmatrix} x_1 & a \\ z_1 & c \end{vmatrix} + \begin{vmatrix} x_1 & -(c - z_1) \\ z_1 & +(a - x_1) \end{vmatrix} \tan\beta},$$

$$\Delta y = \frac{\begin{vmatrix} a & x_1 & -(c - z_1)\tan\beta \\ b & y_1 & +(b - y_1)(\sec\beta - 1) \\ c & z_1 & +(a - x_1)\tan\beta \end{vmatrix}}{\begin{vmatrix} x_1 & a \\ z_1 & c \end{vmatrix} + \begin{vmatrix} x_1 & -(c - z_1) \\ z_1 & +(a - x_1) \end{vmatrix} \tan\beta}. \tag{10}$$

To examine the effect of swing substitute in (3)

$$\Delta x = 0, \qquad \Delta z = 0,$$
$$\alpha_{11} = \cos\gamma, \qquad \alpha_{12} = \sin\gamma, \qquad \alpha_{13} = 0,$$
$$\alpha_{21} = -\sin\gamma, \qquad \alpha_{22} = \cos\gamma, \qquad \alpha_{23} = 0,$$
$$\alpha_{31} = 0, \qquad \alpha_{32} = 0, \qquad \alpha_{33} = 1.$$

Solving,

$$\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1} = \frac{z - z_1}{z_1} = \frac{(c - z_1)\begin{vmatrix} (a - x_1) & -\sin\gamma \\ (b - y_1) & (1 - \cos\gamma) \end{vmatrix}}{\begin{vmatrix} x_1 & a \\ z_1 & c \end{vmatrix} + z_1 \begin{vmatrix} (a - x_1) & -\sin\gamma \\ (b - y_1) & (1 - \cos\gamma) \end{vmatrix}},$$

$$\Delta y = \frac{\begin{vmatrix} a & x_1 & -(b - y_1)\sin\gamma \\ b & y_1 & +(a - x_1)\sin\gamma \\ c & z_1 & +(c - z_1)(1 - \cos\gamma) \end{vmatrix}}{\begin{vmatrix} x_1 & a \\ z_1 & c \end{vmatrix} + z_1 \begin{vmatrix} (a - x_1) & -\sin\gamma \\ (b - y_1) & (1 - \cos\gamma) \end{vmatrix}}. \tag{11}$$

At this point we have accomplished our objective. Equations (3) determine, for any specified displacement of the movable projector, the altered conditions of any image point. Equations (5), (6), (9), (10), and (11) determine the specific effects of x-movement, z-movement, x-tilt, y-tilt, and swing, where the rotation movements are defined as being around the coordinate axes. The effect of a y-movement of the movable projector is obvious, and has not been considered. If the rotation is specified as a finite rotation about an arbitrary line, the transformation is known immediately, (*Analytical Dynamics*, E. T. Whittaker, page 8). This case occurs frequently in the practical operation of the multiplex, but shall be deferred because of its somewhat greater difficulty.

EDUCATION

THE Society has been informed that courses in or allied closely to Photogrammetry are given at the schools listed below. PHOTOGRAMMETRIC ENGINEERING will carry additional information on this subject as it becomes available.

- University of Chicago
- University of Cincinnati
- Harvard University:
  - Harvard Forest
  - Institute of Geographical Exploration
- University of Kansas
- Michigan State College
- Rensselaer Polytechnic Institute
- Syracuse University