THE EFFECTS OF FILM SHRINKAGE UPON THE MULTIPLEX MODEL

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THIS paper is the result of a series of investigations into the causes of certain anomalies which occur in the scaling of flights by the multiplex methods. At one point in the investigations, the possibility of changes occurring in the dimensions of film were considered, and in anticipation of finding such changes, the author inquired into the consequent effects upon the multiplex model.

Let A and B be any two image points on the film before shrinkage has begun, such that AB is parallel to a given direction ξ. Let A' and B' be the corresponding image points on the distorted film. The ratio $A'B'/AB$ will be designated by ρ_ξ and is called the coefficient of shrinkage for the direction ξ. If $\rho_\xi > 1$ the film has expanded; if $\rho_\xi < 1$ the film has contracted. We shall use the word shrinkage to include expansion since the magnitude of ρ_ξ leaves no doubt as to which is meant in any specific instance. If $\rho_\xi = \rho_\eta$ for any two directions ξ and η then we shall call the distortion a uniform shrinkage and write the coefficient of shrinkage ρ without a subscript because it is independent of direction. We

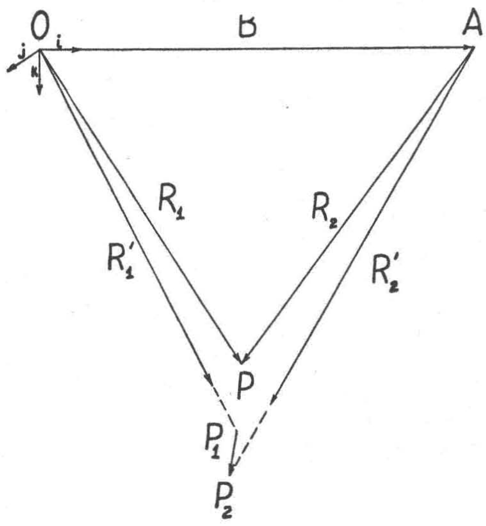

FIG. 1

shall also consider the case when $\rho_x \neq \rho_y$ where ρ_x is the coefficient of shrinkage in the x direction (direction of the line of flight) and ρ_y is the coefficient of shrinkage in the y direction (normal to the line of flight). This type of distortion will be called differential shrinkage. In every instance in this paper we shall assume tilt-free photographs.

In Figure 1, O, the lens of diapositive 1, is the origin of an $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ system of vectors. The coordinates of A, the lens of diapositive 2 are (a, O, c). Thus the air-base vector, $\boldsymbol{B}$, is given by $\boldsymbol{B} = a\boldsymbol{i} + c\boldsymbol{k}$, adopting the convention of writing all vector quantities in bold-faced type. Let P be any point in the oriented, distortionless model and have coordinates (x, y, z). Let $\boldsymbol{R}_1$ be the ray from projector 1 to P and $\boldsymbol{R}_2$ be the ray from projector 2 to P. Then $\boldsymbol{R}_1 = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $\boldsymbol{R}_2 = \boldsymbol{R}_1 - \boldsymbol{B} = (x-a)\boldsymbol{i} + y\boldsymbol{j} + (z-c)\boldsymbol{k}$.

Let us first consider a differential shrinkage such that the coefficient of shrinkage in the x direction is 1 in both diapositives. Let the coefficients of shrinkage in the y direction for diapositives 1 and 2 be ρ and σ respectively. Then due to the shrinkage the rays $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are changed to $\boldsymbol{R}_1'$ and $\boldsymbol{R}_2'$ where $\boldsymbol{R}_1'=x\boldsymbol{i}+\rho y\boldsymbol{j}+z\boldsymbol{k}$ and $\boldsymbol{R}_2'=(x-a)\boldsymbol{i}+\sigma y\boldsymbol{j}+(z-c)\boldsymbol{k}$. The rays $\boldsymbol{R}_1'$ and $\boldsymbol{R}_2'$ are extended to P_1 and P_2 such that P_1P_2 is a vector in the $\boldsymbol{j}$ direction. The length of P_1P_2 measures the Y-parallax which we shall designate by Δy. It is evident from the diagram that:

$$\boldsymbol{B} + \lambda \boldsymbol{R}_2' = \mu \boldsymbol{R}_1' + \Delta y \boldsymbol{j} \tag{1}$$

where λ and μ are constants to be determined. Substituting in equation 1 for $\boldsymbol{B}$, $\boldsymbol{R}_1'$ and $\boldsymbol{R}_2'$ their expressions in the $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ system we get:

$$a\boldsymbol{i} + c\boldsymbol{k} + \lambda[(x - a)\boldsymbol{i} + \sigma y\boldsymbol{j} + (z - c)\boldsymbol{k}] = \mu(x\boldsymbol{i} + \rho y\boldsymbol{j} + z\boldsymbol{k}) + \Delta y\boldsymbol{j}.$$

Equating the coefficients of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ and transposing we obtain the following:

$$\begin{cases} \lambda(x - a) - \mu x = -a \\ \lambda\sigma y - \mu\rho y - \Delta y = 0 \\ \lambda(z - c) - \mu z = -c \end{cases}$$

whose determinant, D, is given by

$$D = az - xc \neq 0.$$

These equations are linear in the unknown, λ, μ, and Δy and are readily solved to yield $\lambda=\mu=1$ and:

$$\Delta y = y(\sigma - \rho). \tag{2}$$

Thus the elevations in the model are completely unaffected. The model is parallax free along the line of flight. Furthermore the model can be cleared of parallax by using the BZ motion of either projector. However if the shrinkage is identical in both diapositives, i.e., $\rho=\sigma$ then no parallax at all results from the shrinkage. But even if $\rho\neq\sigma$ the model may be cleared of parallax and then horizontalized by using the bar. Of course the scale of the model in the Y direction will be in error. This error can be corrected to a certain extent by adjusting the air-base of the model so that there will be a scale error in both X and Y directions of smaller magnitude.

Let us now consider the case of uniform shrinkage in diapositives 1 and 2 with coefficients of shrinkage ρ and σ respectively. Using the same notation as above, $\boldsymbol{R}_1'=\rho x\boldsymbol{i}+\rho y\boldsymbol{j}+z\boldsymbol{k}$ and $\boldsymbol{R}_2'=\sigma(x-a)\boldsymbol{i}+\sigma y\boldsymbol{j}+(z-c)\boldsymbol{k}$. Substituting these expressions in equation (1) and proceeding as before:

$$a\boldsymbol{i} + c\boldsymbol{k} + \lambda[\sigma(x - a)\boldsymbol{i} + \sigma y\boldsymbol{j} + (z - c)\boldsymbol{k}] = \mu(\rho x\boldsymbol{i} + \rho y\boldsymbol{j} + z\boldsymbol{k}) + \Delta y\boldsymbol{j}.$$

From which we obtain the following three equations:

$$\begin{aligned} \lambda\sigma(x - a) - \mu\rho x &= -a \\ \lambda\sigma y - \mu\rho y - \Delta y &= 0 \\ \lambda(z - c) - \mu z &= -c. \end{aligned}$$

whose determinant, D, is given by:

$$D = xz(\rho - \sigma) + \sigma az - \rho xc \neq 0.$$

These equations are easily solved to yield the following values for λ, μ. and Δy:

$$\lambda = \frac{az - \rho xc}{-\sigma z(x - a) + \rho x(z - c)} \text{ equals approximately } \frac{a}{x(\rho - \sigma) + a\sigma} \tag{3}$$

$$\mu = \frac{-\sigma c(x - a) + a(z - c)}{-\sigma z(x - a) + \rho x(z - c)} \text{ equals approximately } \frac{a}{x(\rho - \sigma) + a\sigma} \tag{4}$$

$$\Delta y = \frac{y[az(\sigma - \rho) + \rho ac(1 - \sigma)]}{-\sigma z(x - a) + \rho x(z - c)}\,. \tag{5}$$

Thus the model is parallax-free along the line of flight. Elsewhere in the model the parallax is a complicated function of position (see equation 5) which can be closely approximated by:

$$\Delta y = y(\sigma - \rho).$$

As before the model can be cleared of parallax by using *BZ* motion.

From equations 3 and 4 it can be seen that there is a variation in the vertical scale depending upon x, which approximates a linear function of x since ρ and σ are close to unity. If $\rho = \sigma$ then there is a constant vertical scale which is equal to $1/\sigma$. Consequently it can be seen that the product of the magnification of the vertical scale by the magnification of the horizontal scale in the Y-direction is unity.

Any specified differential shrinkage may always be produced by combining a uniform shrinkage with a differential shrinkage where $\rho_x = 1$, so that it may be investigated as in the two cases above.

The investigation above suggests that distorted film be corrected in the diapositive printer so that there is no error in the X-direction. The model can then be cleared of parallax and leveled, and a compromise solution can be made in scale by adjusting the air-base of the model. It is evident that a change in the focal distance made in the diapositive printer has the same effect upon the model as a uniform shrinkage of the film. This illustrates the fact that if shrinkage is produced in the diapositives by causes other than film shrinkage in the negatives, the final effect upon the model is necessarily the same.