AFFINE TRANSFORMATION IN STEREOPHOTOGRAMMETRY

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(1) Introduction

STRICT solutions in the stereoplotting problems are usually found by keeping the inner orientation of the projecting cameras identical with that of the photographing camera. Or, in other words, the pencil of rays in the plotting should be made identical with that obtained during the picture taking. In actual practice, however, the projection center is sometimes located at a different distance from its respective photograph in the plotting as compared with the corresponding distance in the picture taking. In this case, the pencil of rays has undergone an affine transformation. Theoretical investigations in this case are necessary, particularly in view of the American development, where mirror stereoscopes are improved and used in the works of precision. The instruments thus developed are, for example, the K. E. K. Plotter, the Multiscope, the Mahan Plotter and the different Lyon Models. Mr. Lyon, as mentioned in his article "Automatic Map Plotting Instruments,"[1] was trying to use those devices in carrying out part of the aerial triangulation and was in favor of adopting viewing distances much longer than the focal distance of the photographing camera.

Take for instance, the focal distance of the photographic camera f equal to 6 inches, while the photographs are viewed from substitute perspective centers that are 15 inches from the respective photographs, then we have:

$$k = \frac{f}{d} = \frac{6}{15} = 0.4.$$

k is called the ratio of the affine transformation. The angles α of the rays measured from their respective optical axes in the plotting differ from the corresponding angles $\bar{\alpha}$ in the photographying camera by the following relation:

$$\tan \alpha = k \tan \bar{\alpha}. \tag{1}$$

Because of this kind of transformation, the angular settings of the picture frames in the process of the reciprocal orientation and the absolute orientation are not the same as the corresponding values of the camera position in the air. It is the object of this article to make an investigation on these differences and their effects on the stereoscopic models. This investigation is limited to vertical photographs.

The practical value of these investigation is extended also to the case where, at one time, people were trying to make use of the multiplex projectors in plotting the transformed photographs taken with the German nine lens cameras.

(2) Fundamental Formulas

In this investigation, the pairs of photographs are reciprocally oriented by means of their five elements of outer orientation starting from their normal (zero) positions. These five elements are $d\phi_1$, $d\phi_2$, $d\kappa_1$, $d\kappa_2$, $d\omega_2$ where ϕ, κ and ω designate respectively the angle of horizontal swing, the angle of swing, and the angle of tilt; d designates a differential small value, while the suffixes 1 and

[1] Photogrammetric Engineering, Vol. XII, No. 3, pp. 332, 335.

2 designate respectively the left and right photographs. The differential relations between the vertical parallax p_y, the point coordinates x, y, z and the elements of outer orientation are represented by the following formulas:

$$dx = -\frac{\{z^2 + (x+b)^2\}x}{bz} d\phi_1 + \frac{(z^2 + x^2)(x+b)}{bz} d\phi_2 - \frac{xy}{b} d\kappa_1 + \frac{y(x+b)}{b} d\kappa_2 + \frac{yx(x+b)}{bz} d\omega_2. \tag{2}$$

$$dy = -\frac{y}{bz}(z^2 + x(x+b))d\phi_1 + \frac{y}{bz}(z^2 + x^2)d\phi_2 - \left\{(x+b) + \frac{y^2}{b}\right\} d\kappa_1 + \frac{y^2}{b} d\kappa_2 + \frac{xy}{bz} d\omega_2. \tag{3}$$

$$dz = -\frac{z^2 + (x+b)^2}{b} d\phi_1 + \frac{z^2 + x^2}{b} d\phi_2 - \frac{yz}{b} d\kappa_1 + \frac{yz}{b} d\kappa_2 + \frac{xy}{b} d\omega_2. \tag{4}$$

$$p_y = -\frac{(x+b)y}{z} d\phi_1 + \frac{xy}{z} d\phi_2 + (x+b)d\kappa_1 - x d\kappa_2 + \left(z + \frac{y^2}{z}\right) d\omega_2. \tag{5}$$

The positive signs of the linear and angular values are shown by the arrows in Figure 1, the perspective center of the second photograph being taken as the origin of all the linear measurements. b designates the length of the air base. The above formulas are comparable with the formulas (51), (52), (53), (54) derived by Baeschlin and Zeller in their book "Lehrbuch der Stereophotogrammetrie" pp. 47–48.

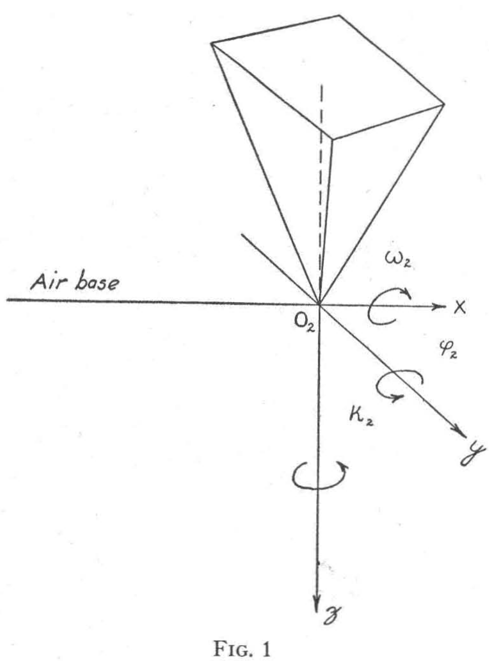

FIG. 1

(3) RECIPROCAL ORIENTATION

Starting from the normal position of both photographs, where all the elements of outer orientation are in their zero position, let us designate the true values of these elements which should be added to their zero values, by $\overline{d\phi_1}$, $\overline{d\phi_2}$, $\overline{d\kappa_1}$, $\overline{d\kappa_2}$, $\overline{d\omega_2}$. Take formula (5), let b be the length of the base measured on the photograph instead of being the length of the actual air base. In this case all the x and y values represent the coordinates of image points on the right photograph, z is approximately equal to the principal distance of the projection system, which should be equal to the focal distance f of the photographing camera and p_y represents the difference in the y-coordinates between the right and the left photographs of the image points. Formula (5) may now be written as:

$$p_y = -\frac{(x+b)y}{f}\overline{d\phi_1} + \frac{xy}{f}\overline{d\phi_2} + (x+b)\overline{d\kappa_1} - x\,\overline{d\kappa_2} + \left(f + \frac{y^2}{f}\right)d\omega_2. \tag{6}$$

Select five properly distributed image points, measure their picture coordinates and establish five of the above simultaneous equations, we can then calculate the values of $d\phi_1$, $d\phi_2$, $d\kappa_1$, $d\kappa_2$, $d\omega_2$ which can be treated as constants for any particular pair of photographs.

Now, in the actual plotting, suppose we use a principal distance of $d = f/k$ instead of f, then the movements of the swings and tilts of the photographs which are necessary to bring all the vertical parallaxes equal to zero, are not necessarily all equal to the respective values of $d\phi_1$, $d\phi_2$, $d\kappa_1$, $d\kappa_2$, $d\omega_2$. Let us denote the respective values in this second case by $d\phi_1'$, $d\phi_2'$, $d\kappa_1'$, $d\kappa_2'$, $d\omega_2'$ then their values should be those obtained by solving the five simultaneous equations of Formula (7).

$$p_y = -k\frac{(x+b)y}{f}d\phi_1' + k\frac{xy}{f}d\phi_2' + (x+b)d\kappa_1' - x\,d\kappa_2' + \left(\frac{f}{k} + \frac{ky^2}{f}\right)d\omega_2'. \tag{7}$$

In order that the solving of the simultaneous equations formed by any set of five image points should also lead to one and only one solution, the unknowns in equation (7) must be able to express in some constant functions of the unknowns in equation (6). The values of x and y are here considered as variables, while f is treated as constant, under the assumption that the ground undulations are not very excessive. By comparing equations (6) and (7) we obtain the values of the angular settings in this second case as follows:

$$\begin{aligned} d\phi_1' &\equiv \frac{\overline{d\phi_1}}{k}, \qquad d\phi_2' \equiv \frac{\overline{d\phi_2}}{k}, \qquad d\kappa_1' \equiv \overline{d\kappa_1} + \frac{f}{b}\left(1 - \frac{1}{k^2}\right)\overline{d\omega_2} \\ d\kappa_2' &\equiv \overline{d\kappa_2} + \frac{f}{b}\left(1 - \frac{1}{k^2}\right)\overline{d\omega_2}, \qquad d\omega_2' \equiv \frac{\overline{d\omega_2}}{k}. \end{aligned} \tag{8}$$

From the relations represented by the equation (8), we know that a definite solution can be obtained by such kind of affine transformation of the pencil of rays, so far as equation (5) holds true. Equation (5) will hold true only when the elements of outer orientation are small values (ordinarily less than 2 or 3 degrees) as were represented by their differentials.

(4) Effect on the Stereoscopic Model

Equations (2), (3), (4) can be used for investigating the displacements of the stereoscopic model through the change of the elements of outer orientation. By making the same substitutions as before, we obtain the displacements in space due to the elements $\overline{d\phi_1}$, $\overline{d\phi_2}$, $\overline{d\kappa_1}$, $\overline{d\kappa_2}$, $\overline{d\omega_2}$ as:

$$dx = -\frac{\{f^2 + (x+b)^2\}x}{bf}\overline{d\phi_1} + \frac{(f^2 + x^2)(x+b)}{bf}\overline{d\phi_2} - \frac{xy}{b}\overline{d\kappa_1}$$

$$+ \frac{y(x+b)}{b}\overline{d\kappa_2} + \frac{yx(x+b)}{bf}\overline{d\omega_2}. \qquad (9)$$

$$dy = -\frac{y}{bf}\{f^2 + x(x+b)\}\overline{d\phi_1} + \frac{y}{bf}(f^2 + x^2)\overline{d\phi_2} - \left((x+b) + \frac{y^2}{b}\right)\overline{d\kappa_1}$$

$$+ \frac{y^2}{b}\overline{d\kappa_2} + \frac{xy}{bf}\overline{d\omega_2}. \qquad (10)$$

$$dz = -\frac{f^2 + (x+b)^2}{b}\overline{d\phi_1} + \frac{f^2 + x^2}{b}\overline{d\phi_2} - \frac{yf}{b}\overline{d\kappa_1} + \frac{yf}{b}\overline{d\kappa_2} + \frac{xy}{b}\overline{d\omega_2}. \qquad (11)$$

In the second case where f is changed into $d = f/k$ and $\overline{d\phi_1}$, $\overline{d\phi_2}$, $\overline{d\kappa_1}$, $\overline{d\kappa_2}$, $\overline{d\omega_2}$ are changed into $d\phi_1'$, $d\phi_2'$, $d\kappa_1'$, $d\kappa_2'$, $d\omega_2'$ we get different values, say dx', dy', dz', of the spatial point displacement. Referring to the equations (9), (10), (11), and substituting the relations represented by equation (8) we obtain the differences of the respective point displacements as follows:

$$\Delta x = dx' - dx' = -\frac{\{(f/k)^2 + (x+b)^2\}x}{b(f/k)}\left(\frac{\overline{d\phi_1}}{k}\right)$$

$$+ \frac{\{(f/k)^2 + x^2\}(x+b)}{b(f/k)}\left(\frac{\overline{d\phi_2}}{k}\right) - \frac{xy}{b}\left\{\overline{d\kappa_1} + \frac{f}{b}2\left(1 - \frac{1}{k^2}\right)\overline{d\omega_2}\right\}$$

$$+ \frac{y(x+b)}{b}\left\{\overline{d\kappa_2} + \frac{f}{b}\left(1 - \frac{1}{k^2}\right)\overline{d\omega_2}\right\}$$

$$+ \frac{yx(x+b)}{b(f/k)}\left(\frac{\overline{d\omega_2}}{k}\right) + \frac{\{f^2 + (x+b)^2\}x}{bf}\overline{d\phi_1}$$

$$- \frac{(f^2 + x^2)(x+b)}{bf}\overline{d\phi_2} + \frac{xy}{b}\overline{d\kappa_1} - \frac{y(x+b)}{b}\overline{d\kappa_2} - \frac{yx(x+b)}{bf}\overline{d\omega_2}$$

$$= \frac{(x+b)f}{b}(\overline{d\phi_2} - \overline{d\phi_1})\left(\frac{1}{k^2} - 1\right) - \frac{yf}{b}\left(\frac{1}{k^2} - 1\right)\overline{d\omega_2}. \qquad (12)$$

$$\Delta y = dy' - dy = -\frac{y}{b(f/k)}\left\{\frac{f^2}{k^2} + x(x+b)\right\}\left(\frac{\overline{d\phi_1}}{k}\right) + \frac{y}{b(f/k)}\left(\frac{f^2}{k^2} + x^2\right)\left(\frac{\overline{d\phi_2}}{k}\right)$$

$$- \left\{(x+b) + \frac{y^2}{b}\right\}\left\{\overline{d\kappa_1} + \frac{f}{b}\left(1 - \frac{1}{k^2}\right)\overline{d\omega_2}\right\}$$

$$+ \frac{y^2}{b}\left\{d\kappa_2 + \frac{f}{b}\left(1 - \frac{1}{k^2}\right)\overline{d\omega_2}\right\} + \frac{xy}{b(f/k)}\left(\frac{\overline{d\omega_2}}{k}\right)$$

$$+ \frac{y}{bf}\{f^2 + x(x+b)\}\overline{d\phi_1} - \frac{y}{bf}(f^2 + x^2)\overline{d\phi_2}$$

$$+ \left\{(x+b) + \frac{y^2}{b}\right\}\overline{d\kappa_1} - \frac{y^2}{b}\overline{d\kappa_2} - \frac{xy}{bf}\overline{d\omega_2}$$

$$= \frac{yf}{b}(\overline{d\phi_2} - \overline{d\phi_1})\left(\frac{1}{k^2} - 1\right) + \frac{(x+b)f}{b}\left(\frac{1}{k^2} - 1\right)\overline{d\omega_2}. \qquad (13)$$

$$\Delta z = dz' - dz = -\frac{(f/k)^2 + (x+b)^2}{b}\left(\frac{\overline{d\phi_1}}{k}\right) + \frac{(f/k)^2 + x^2}{b}\left(\frac{\overline{d\phi_2}}{k}\right)$$

$$-\frac{y(f/k)}{b}\left\{\overline{d\kappa_1} + \frac{f}{b}\left(1 - \frac{1}{k^2}\right)\overline{d\omega_2}\right\} + \frac{y(f/k)}{b}\left\{\overline{d\kappa_2} + \frac{f}{b}\left(1 - \frac{1}{k^2}\right)\overline{d\omega_2}\right\}$$

$$+\frac{xy}{b}\overline{d\omega_2} + \frac{f^2 + (x+b)^2}{b}\overline{d\phi_1} - \frac{f^2 + x^2}{b}\overline{d\phi_2} + \frac{yf}{b}\overline{d\kappa_1} - \frac{yf}{b}\overline{d\kappa_2} - \frac{xy}{b}\overline{d\omega_2}$$

$$= \frac{f^2}{kb}(\overline{d\phi_2} - \overline{d\phi_1})\left(\frac{1}{k^2} - 1\right) + \left(\frac{1}{k} - 1\right)dz. \tag{14}$$

Let:

$$m = \frac{f}{b}(\overline{d\phi_2} - \overline{d\phi_1})\left(\frac{1}{k^2} - 1\right) \tag{15}$$

$$\theta = \frac{f}{b}\left(\frac{1}{k^2} - 1\right)\overline{d\omega_2}. \tag{16}$$

Substituting equations (15), (16) into equations (12), (13), (14) we have:

$$\Delta x = (x+b)m - y\theta. \tag{17}$$

$$\Delta y = ym + (x+b)\theta. \tag{18}$$

$$\Delta z = (f/k)m + \left(\frac{1}{k} - 1\right)(dz). \tag{19}$$

From the coefficients of m and θ in the equations (17) and (18), we know that m is a scale change while θ is the rotation of the stereoscopic model with the vertical axis through the left station as axis. The difference in the vertical displacements, as judged from equation (19), consists of that due to a scale change by the amount of m/k in z (represented as f in equation 19) and a multiplier $(1/k-1)$ of the amount dz, dz being the displacement of the spatial point in the first case. In the actual plotting, the effects of m and θ are eliminated by the process of absolute orientation, while the effect of Δz is considered by utilizing a different vertical scale.

COMMENTS ON "AFFINE TRANSFORMATIONS IN STEREOPHOTOGRAMMETRY"

Everett L. Merritt
Chairman, Nomenclature Committee

Dr. Wang's paper treats of an important subject in a concise manner. Frequently there is not a one-to-one correspondence between the f distance of the taking camera and the f distance of American plotting instruments used. Dr. Wang concludes that the errors introduced in plotting when there is not a one-to-one correspondence are negligible if the tilt of the photographs does not exceed 3° and if a constant is used.

The Society's *Manual of Photogrammetry* contains no symbols such as those used by Dr. Wang. However, C. A. Hart gives the below listed definitions in his book *Air Photography Applied to Surveying.*

1. *Swing,* designated κ_1 and κ_2 for each photograph of a stereopair, is their respective rotation about a plumb line passing through the perspective centers.
2. *Cant,* designated ϕ_1 and ϕ_2 for each photograph is their respective rotation in a vertical plane that includes the air-base.
3. *Differential tilt,* designated ω, is the resultant difference between the rotation of the two photographs about a line defined by the air-base.
(Kappa); κ (Phi); ϕ (Omega); ω

The definitions conform exactly with the application Dr. Wang gives them.