SYMPOSIUM ON THE ANDERSON TILT LAWS

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THE Greek letter τ (*tau*) has become a "classical" symbol and a working tool in the equations derived by Mr. R. O. Anderson for tilt displacement.

The ratio $(\sin t)/f$ (sine of the tilt angle divided by the focal length) has been so often encountered in photogrammetric formulas that Mr. R. O. Anderson has denoted this expression with the symbol τ. The practicability of this designation rests in the ease of writing a symbol for the frequently encountered ratio, $(\sin t)/f$.

It is therefore suggested that the American Society of Photogrammetry standardize the symbol τ to denote the expression $(\sin t)/f$. Since τ has not been used as a photogrammetric symbol, there should be no conflict in its usage. One may ask why so much importance should be placed on the expression τ. Around this expression the tilt laws revolve as it represents the ratio change per unit induced by tilt. An endeavor will be made to demonstrate briefly the full significance of τ and at the same time clarify other principles.

Mr. M. G. Misulia has outlined in the article "A Derivation for the Image Desplacement Due to Tilt"[1] a new derivation for tilt displacement. Mr. Misulia vanishes his equation into the same formula as originally derived by Mr. Anderson.[2] This new derivation, with the many others by different methods, conclusively proves the exactness of the tilt displacement laws. Nevertheless, it is stated in this article that the new derivation is shorter and more direct than the Anderson method. This infers that there is but one Anderson derivation. As a matter of fact, Mr. Anderson outlines five different derivations of the tilt displacement formulas for academic purposes.[2] One of these derivations is simplicity itself. This method can be called the Reciprocal τ or Vanishing Point Method and definitely fulfills the requirements of shortness and directness. As a means of easier comparison, symbols as used in the reference article will be again used whenever possible:

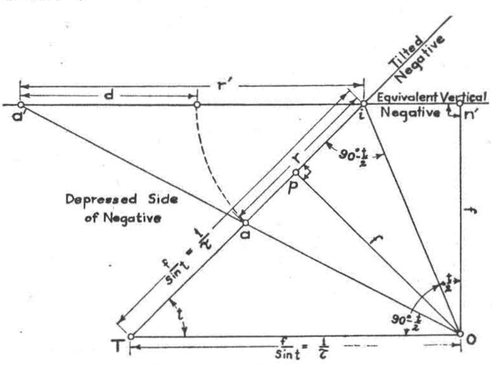

FIG. 1. Sectional view of the tilted negative in the principal plane.

Symbols used in the figure are defined as follows:

t = angle of tilt
p = principal point
i = isocenter

[1] PHOTOGRAMMETRIC ENGINEERING, Vol. XII, No. 4, Dec. 1946, pp. 461–463.
[2] Applied Photogrammetry, Edwards Bros., Ann Arbor, Mich.

n' = nadir point (map)
O = perspective center (lens)
f = focal length
$\tau = (\sin t)/f$
a = image of ground point in the principal plane of the negative
a' = corresponding point on the equivalent vertical negative
r = distance ia
d = the displacement for an image, parallel to the principal line induced by tilt, on the depressed side of negative.
T = vanishing point.

By condition,

$$\angle OTi = t.$$

Since,

$$Op = On' = f,$$
$$\angle iOn' = t/2.$$

Therefore,

$$\angle TOi = 90° - t/2,$$

and

$$\angle TiO = 180° - (t + 90° - t/2) = 90° - t/2.$$

Therefore,

$$\Delta TOi \text{ is isosceles.}$$

By trigonometry and definition,

$$Ti = TO = f/\sin t = 1/\tau.$$

From similarity of the triangles, $\Delta aia'$ and ΔaTO,

$$\frac{r'}{r} = \frac{1/\tau}{(1/\tau) - r} = \frac{1}{1 - r\tau}. \tag{A}$$

Solving Eq. (A)

$$r' = \frac{r}{1 - r\tau}. \tag{B}$$

By condition,

$$d = r' - r. \tag{C}$$

Substituting Eq. (B) in Eq. (C),

$$d = \frac{r}{1 - r\tau} - r. \tag{D}$$

Reducing Eq. (D) to a common denominator,

$$d = \frac{r^2\tau}{1 - r\tau}. \tag{E}$$

Dividing Eq. (E) by τ,

$$d = \frac{r^2}{\dfrac{f}{\sin t} - r} \tag{F}$$

which is identical to Mr. Misulia's equation (5). This form of the equation applies to the depressed side of the negative. When the displacement is on the raised side, r is positive and the displacement is:

$$d = \frac{r^2}{1/\tau + r}.$$

This particular derivation of tilt displacement is of interest since it portrays an additional principle of photogrammetry.[3] Point T, Fig. 1, lies on the horizon trace (the intersection of the horizontal plane through the exposure station O and the tilted negative plane). Point T is the intersection of the horizon trace and the principal line; therefore all lines in the datum plane parallel to the principal line will vanish at the point T. It can be seen from Fig. 1 that the vanishing point T can be located by laying off $1/\tau$ or $f/\sin t$[4] from the isocenter, i, along the principal line in the direction of the depressed side of the negative.

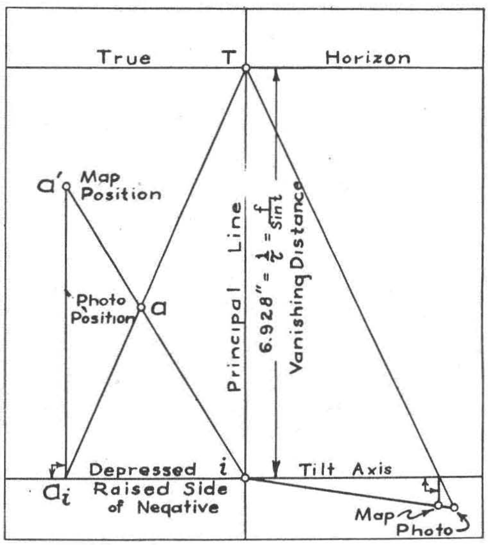

FIG. 2. Tilt Grid.

This principle has been used in oblique photography to locate graphically the map position of a photographic image in tilt only.

Focal Length	t	$\sin t$	$f/\sin t = 1/\tau$
6.000″	3°00′	0.0523360	114.644″
6.000″	60°00′	0.8660254	6.928″

[3] Air Photography Applied to Surveying, C. A. Hart, Longmans, Green and Co. New York.

[4] Major C. A. Hart, Royal Engineers, uses the expression f cosec θ for $f/\sin t$ which is a trigonometric identity. As an additional constructive comment, Major Hart does not note or differentiate between the raised and depressed side of his "tilt distortion" formula.

It can be seen that the ratio $f/\sin t$ for a 3°00′ tilt and a focal length of 6.000″ is 114.644″ which is at impractical distance to handle as a vanishing point. However within the range of trimetrogon photography the distance 6.928″ can be handled practically. Refer to Fig. 2. Construct a vanishing ray from T through the photographic image to intersect the tilt axis at a_i. Erect a perpendicular a_ia' at this intersection to intersect the iso-radial ia extended. This intersection defines the map position a' of the photographic image a.

The tilt formulas derived by Mr. Anderson have four-way action:

Photo to Map Positions

(1) $$e_r = \frac{E^2\tau}{1 + E\tau} \text{ (Raised)}$$

(2) $$e_d = \frac{E_{11}{}^2\tau}{1 - E_{11}\tau} \text{ (Depressed)}$$

Map to Photo Positions

(3) $$e_r = \frac{E_1{}^2\tau}{1 - E_1\tau} \text{ (Raised)}$$

(4) $$e_d = \frac{E_{111}{}^2\tau}{1 + E_{111}\tau} \text{ (Depressed).}$$

The position of the "E's" in relation to the tilt axis are shown in Fig. 3.

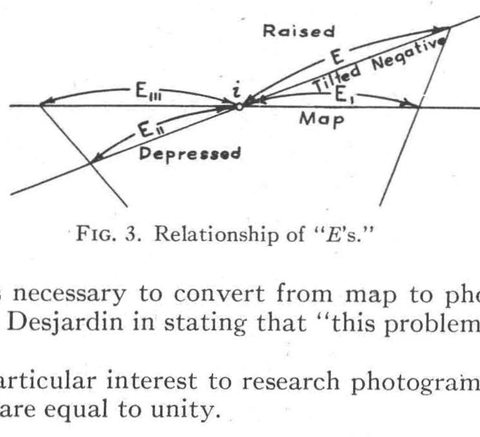

FIG. 3. Relationship of "E's."

It is sometimes necessary to convert from map to photo positions as outlined by Mr. Louis Desjardin in stating that "this problem confronts the structural geologist."[5]

It may be of particular interest to research photogrammetists that the following expressions are equal to unity.

Anderson Constants

$$(1 + E\tau)(1 - E_1\tau) = 1$$

$$(1 - E_{11}\tau)(1 + E_{111}\tau) = 1.$$

The Anderson Scale Point Method employs the following formula as a means to determine the tilt:

$$\sin t = \Delta Rf.$$

This can be rewritten,

$$\tau = \Delta R$$

[5] PHOTOGRAMMETRIC ENGINEERING, Vol. VI, No. 4, Oct., Nov., Dec., 1940, pp. 163–165.

where ΔR is the map-unit-ratio-change induced by tilt.

When relief is present, h_{ec} is multiplied by $(1 \pm M\tau)$ to compensate for the inherent relief displacement. This means that when this factor is applied, the resultant ΔR is definitely expressed as a function of the true position without the effects of tilt or relief.

It can readily be seen that τ is one of the most significant tilt symbols in the science of photogrammetry. Further significance of τ will be realized as photogrammetric research progresses.

COMMENTS ON "SYMPOSIUM ON THE ANDERSON TILT LAWS"

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1. General Comments

Generally speaking, the paper is well written and is of interest to the readers of the Photogrammetric Engineering, as Mr. McNeil concisely and simply summarizes Anderson's tilt laws. He clearly demonstrates that Mr. Misulia's "Derivation of Image Displacement Due to Tilt" is not new at all, but rather a duplication of one of Anderson's original tilt formulas. Mr. McNeil's paper further demonstrates that the articles in Photogrammetric Engineering are stimulating judicious and critical interest in the Journal. Of equal interest, however, is the fact that Mr. McNeil's description of a method of graphically locating the map position of a photo image is also not new. This method of graphically locating the map position by use of the fundamental laws of perspective was described by M. P. Bridgland of Canada in 1924 (Photographic Surveying, by M. P. Bridgland. Dept. of Interior, Canada, Bulletin no. 56. 1924. p. 28, 29).

2. Comments on the Symbol τ

With a full appreciation of the tremendous contribution that R. O. Anderson has made to analytical photogrammetry, it is felt that his repeated use of the Greek symbol τ for the expression Sin t/f is not sufficient justification for adopting it as a standard symbol. What amounts to a convenience for anyone familiar with, or using, Anderson's method is a limitation for one more familiar with, and using, other methods. In lengthy analytical equations it is quite easy to confuse English letters symbolizing sides with English letters symbolizing angles. This is particularly true of Prof. Church's space resection formulas. It is suggested that all Greek characters be reserved to symbolize "any angle," as expressed by Mr. Tewinkel, so that "any angle" is not confused with any linear value in photogrammetric formulas.

CORRECTIONS TO AUTOMATIC MAP PLOTTING INSTRUMENTS*

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The above entitled article, published in the September 1946 issue (Vol. XII, No. 3) of Photogrammetric Engineering, is incomplete and contains several errors. The significant corrections to that article, with which the author is acquainted, are described in the following paragraphs.

A plotting instrument erroneously called the "Stereotopograph" was illustrated by Figure 3 on page 319, and described in the accompanying text. Figure

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