THE DETERMINATION OF TILT FROM SCALE CHECK LINES

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IN THE "Manual of Photogrammetry," beginning on page 274, is an article by Jack Rihn on an improved method of determinating tilt from scale check lines, a modification of the original system devised by R. O. Anderson. In the opinion of the writer this can be still further improved by additional modifications. This change will result in increased accuracy, especially when the elevations of the control points differ considerably one from another.

The scheme is to find by Rihn's method, or by a modification which is more accurate, the tilt and swing of the photograph relative to a plane through the

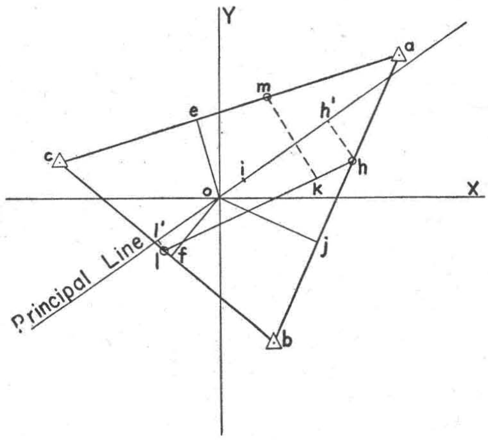

FIG. 1

three control points and then to find from these by taking into account the tilt and swing of the plane through the control points the corresponding quantities for the picture plane with respect to the horizontal, i.e., these elements of orientation for the photograph as they are usually defined. As an intermediate step the tilt of the plane through the control points must be found. This may easily be done. Also it will be shown how scale points may be located with some precision.

The problem may be stated as follows: Given are three suitably located control points A, B and C whose X, Y, Z coordinates are known on the ground system of control. The horizontal and inclined lengths of the sides of the triangle may respectively be calculated from the formulas

$$\text{horizontal length} = G' = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$$

and

$$\text{inclined length} = G = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}$$

where X_1, Y_1, Z_1 are coordinates of one end of the line and X_2, Y_2, and Z_2 are the coordinates of the other end.

In Figure 1 the points a, b and c are the images of the three control points as located on the photograph. Since we are going to use the triangle ABC as formed in the plane of the control points A, B and C, no correction for relief is to be made in the positions of a, b and c on the photograph. Thus one step in Rihn's method is omitted. The lengths of the sides of the triangle abc are scaled or they may be computed from the coordinates of the vertices, if these coordinates have been determined. (While the construction work might be carried out on the photograph, it will be assumed that the graphical solution is made on drawing paper to which the necessary transfers have been made. There is some advantage in doubling the scale in transferring to reduce the errors in the subsequent graphical work.) The scale ratios are found by dividing the length of each line on the photograph (P) by the corresponding inclined length on the ground (G). The solution proceeds as in Rihn's method or Anderson's by next finding the scale ratio points by dropping perpendiculars from the principal point o to each of the three sides of the triangle abc and then laying off the shorter segment on the longer of each of these lines to get the three scale ratio points l, m and h. (Here l is the point where the scale ratio is the smallest, h where it is the largest and m where the value is the intermediate between the other two.) As in Rihn's method l and h are connected and an intermediate point k interpolated, the scale varying uniformly from l to h. The ratio at k is the same as that at m. Let the scale ratios at h, l and m be designated respectively S_h, S_l and S_m. Then the distance

$$lk = lh\left(\frac{S_m - S_l}{S_h - S_l}\right).$$

The principal line is drawn through o perpendicular to mk. Perpendiculars ll' and hh' are then dropped from l and h to the principal line $l'oh'$. Then the change of scale ratio per unit length along the principal line (dS), the scale ratio at o, (S_o), and the tilt of the photograph with respect to the plane of the control points (t) may be found from the formulae

$$dS = \frac{S_h - S_l}{l'H'}, \qquad S_0 = (S_l + l'o)dS, \quad \text{and} \quad \sin t' = \frac{f(dS)}{S_0}\cdot$$

As in Rihn's method it is not necessary that the distances on the photograph should be expressed in the same units as those on the ground. Thus millimeters may be used for the one and feet for the other. Moreover after finding the scale ratios, which are small quantities, the decimal point may be arbitrarily shifted to the right a certain number of places, the same, of course, for each ratio, to give whole numbers which are more convenient for use in the subsequent work.

Unless the value of t is small, the solution should be repeated after determining the location of the isocenter by dropping perpendiculars from this point to the sides of the triangle to get the segments of the latter which by laying off the shorter on the longer are used to get the scale points. This solution should be considerably more precise than the first. This solution is illustrated by Figure 2, where points are marked similarly to those in Figure 1. The distance io from the principal point o to the isocenter is equal to f tan $\frac{1}{2}t'$ and this distance is laid off on the principal line as found in the first solution to locate the isocenter i. After dropping the perpendicular ie, if, and ig, the three scale points h, l and m are found as previously indicated by laying off the shorter segment of each side upon the longer. Thus the scale point h is found by laying off the shorter segment be of the side ab upon the longer segment ae. The solution then proceeds as before except that in computing sin t', the scale ratio Si for the isocenter is used instead of So. The solution is shown in Figure 2 where $l'oh'$ is the new principal line. Using a convenient scale for the purpose, like one millimeter is equal to two minutes of arc, a tilt circle is drawn with its center at s on the principal line, the diameter being equal to the value of t'. This circle is drawn upon the opposite side of o from i. The chords of this tilt circle, radiating from o, give the elements of tilt in their particular directions with respect to the plane of A, B and C. The tilt is up in the direction away from o.

Another tilt circle is now to be drawn to show approximately the elements of tilt for the plane ABC with respect to the horizontal. This is done by getting the slopes of the sides of the triangle ABC and considering these to be elements along their respective directions. The slope of the line AB, for example, is ob-

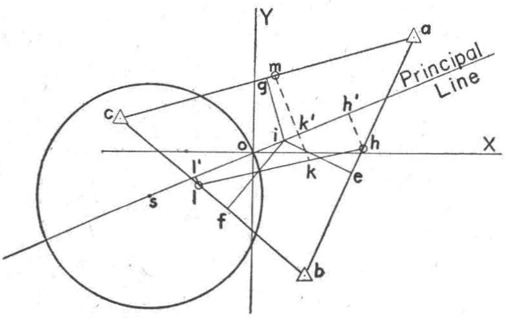

FIG. 2

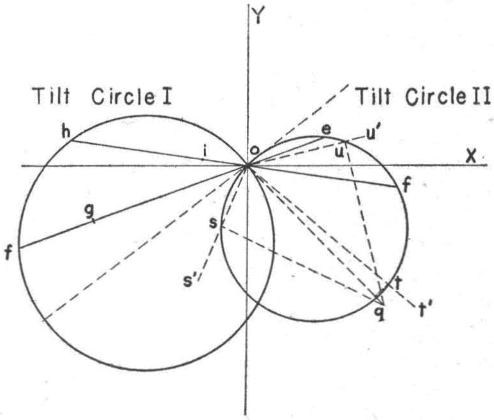

FIG. 3

tained from its tangent, which is found by dividing the difference in elevation between A and B by the horizontal distance between these points. Through o, lines os', ot' and ou', are drawn parallel to AB, BC, and CA respectively, (or in other words, parallel to ab, bc and ca of Figure 2) and upon these to the same scale as used for the tilt circle previously drawn, lengths are laid off from o equal to the slopes of the respective sides of the triangle ABC to give the points s, t, and u. (This construction is carried out in Figure 3, which is an extension of Figure 2, with certain lines omitted.) These lines are drawn in the direction of the upward slope of the corresponding sides of the triangle. At the points s, t and u perpendiculars to os, ot and ou are drawn. If the theory of tilt circles were precisely accurate and these perpendiculars had been drawn from the ends of lines perfectly representing the elements of tilt in their respective directions, the perpendiculars would all have met at a point. As it is, they fail to meet perfectly, giving a "triangle of error" through their three points of intersection. By inspection a point q is chosen as an averaging point and on oq as a diameter a circle is drawn to represent the tilt circle for the plane of the points A, B and C, relative to the horizontal. Call the original tilt circle, Circle I, and this one Circle II.

If now a line is drawn through o to cross Circles I and II, the chords thus formed represent by their lengths the elements of slope of the picture with respect to the plane of the control points and of the latter plane with respect to the horizontal. By combining these two elements of tilt, the element of tilt of the picture, with respect to the horizontal is obtained in the direction of the chord. This combining is done by laying off the length of the chord of Circle II upon the corresponding chord of Circle I, each time in the direction of the point of intersection o. Thus fg is laid off equal to oe, and hi equal to of. The points o, i and g and other points found similarly to i and g should lie on a circle. The center of this circle is found by getting the intersection of the perpendicular bisectors of og and oi respectively. The diameter of this circle gives by its scaled length the value of the tilt angle t of the photograph, and by its direction, the angle of swing.

The diameter of the circle may also be found by drawing perpendiculars to oi, og and os at u, t and s respectively. These should meet at a point, which should lie at the opposite end of the diameter from o. Due to error, they probably will not meet exactly, and an averaging point can be chosen to define the diameter of tilt Circle II.

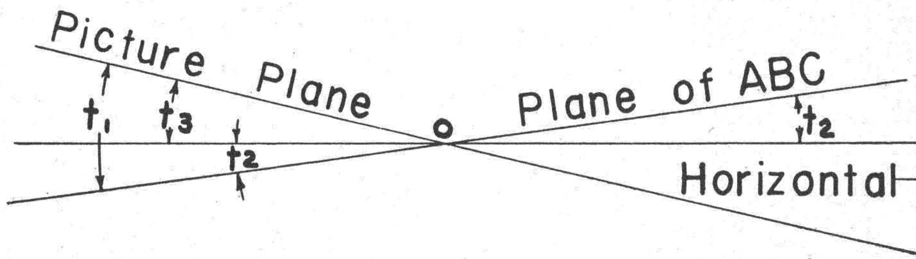

FIG. 4

The solution will be only approximate, since neither tilt Circle I nor tilt Circle II is exact.

Attention must be paid to the direction of tilt which is fixed by consideration of the directions represented by Tilt Circles I and II. Thus in finding $o'g$ as an element of tilt, oe running to the right shows that the tilt of the plane ABC is upward in that direction (see Figure 4 where $t_2 = oe$ in arc). The direction of of to the left shows that the element of tilt of the picture is upward toward the left with respect to the plane of A, B and C. The section, Figure 4, shows this angle t_1 which is equal to oe. Evidently the angle of tilt of the photograph with respect to the horizontal represented by angle t_3, is equal to $t_1 - t_2$ and thus to $of - oe$. It is assumed in this solution that all angles of tilt are measured in the same plane. This is not true but the error introduced is not great for small angles of tilt, and not so large as to be important even for angles of tilt of several degrees.

This completes the solution unless one wishes to go to the trouble of locating the scale points more precisely, which as will be shown, can be done. To develop the method we will go back to the determination of scale points along the principal line as given in Rihn's article.

Figure 5 shows a section through the principal plane of the tilted photograph. Let L be the perspective center, o the principal point, i the isocenter, v the vertical point of the picture, V the vertical point in the datum plane, f the focal length $= Lo = Lv'$, and $H = LV$ the altitude of the camera station above the datum. Extend the line Li to meet the datum plane at I.

Let A be any point in the datum plane. Connect L and A. Then a is the image of A in the tilted photograph and a' its image in the "equivalent" vertical photograph. The scale ratio Sa at a is equal to the ratio of La to LA or to $\frac{f + x \sin t}{H}$. This relationship may be derived as follows: Through a draw a vertical line am and a horizontal line an. Then $am = ia \sin t = x \sin t$, $v'n = ma = x \sin t$ and $Ln = f + x \sin t$. From the similar triangles Lan and LAV,

$$\frac{La}{LA} = \frac{Ln}{LV} = \frac{f + x \sin t}{H} = Sa.$$

The equation $S = \dfrac{f + x \sin t}{H}$ will hold for finding the scale ratio at any point on the principal line if x is considered to be plus for points on the low side of the picture and negative for points on the high side.

The scale ratio, S_{ia} of the line ia is equal to ai/AI or in other words to its length on the photograph divided by its length in the datum plane. Let the

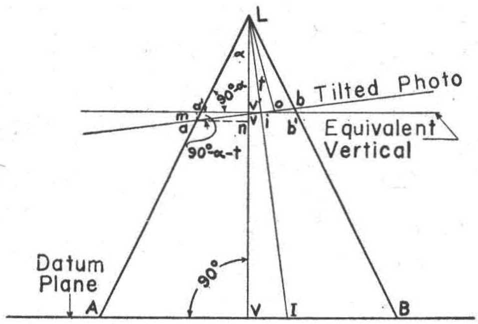

FIG. 5

angle $VLa = \alpha$. Then the angle $ia'a = 90 + \alpha$ and the angle $a'ai = 90 - \alpha - t$. From the triangle $aa'i$ by the sine law

$$\frac{ai}{a'i} = \frac{\cos \alpha}{\cos (\alpha + t)} \quad \text{or} \quad ai = \frac{a'i \cos \alpha}{\cos (\alpha + t)}.$$

Since $Lv' = Lo = f$, we have from the right triangles $Lv'a'$ and Loa,

$$La' = \frac{f}{\cos \alpha} \quad \text{and} \quad La = \frac{f}{\cos (\alpha + t)}.$$

From the right triangle LVA, $LA = LV/\cos \alpha$. But $LV = H$. Therefore $LA = H/\cos \alpha$. But $Sa = La/LA$. By substituting values of La and LA,

$$Sa = \frac{f}{\cos (\alpha + t)} \cdot \frac{\cos \alpha}{H},$$

$Sia = ai/AI$ and from the similar triangles Lia' and LIA,

$$\frac{AI}{a'i} = \frac{H}{f} \quad \text{or} \quad AI = \frac{H}{f} \cdot a'i.$$

Whence

$$Sia = \frac{a'i \cos \alpha}{\cos (\alpha + t)} \cdot \frac{f}{Ha'i} = \frac{f \cos \alpha}{\cos (\alpha + t) \cdot H} = Sa.$$

Thus the scale ratio for a segment of the principal line one end of which is at the isocenter is the same as the scale ratio at the other end of the segment. If the distance ia is designated by x, then the scale ratio at a is equal to $\frac{f+x\sin t}{H}$ which is also the scale ratio of the line ia.

Let b be any other point on the principal line, corresponding to the point B in the datum plane. Let the distance ib be designated by x'', which in this case is a negative quantity, and let the distance ia be redesignated x'. It is now proposed to find the location of the point along the principal line where the scale ratio is equal to that of the line ab. Let the distance of this point from the isocenter be equal to x, which is at yet unknown. The scale ratio at this point is then equal to $\frac{f+x\sin t}{H}$, which by hypothesis is to be equal to the scale ratio of the line ab or to ab/AB.

Therefore

$$\frac{f + x \sin t}{H} = \frac{x' - x''}{AB}.$$

Then $AB = AI + IB$. But as has previously been shown the scale of ia is equal to $\frac{f+x' \sin t}{H}$ and that of ib to $\frac{f+x'' \sin t}{H}$. But the scale of ia is equal to ia/AI and that of ib to ib/IB. Therefore

$$AI = \frac{x'H}{f + x' \sin t} \quad \text{and} \quad IB = \frac{-x''H}{f + x'' \sin t}.$$

(It is to be noted that the negative sign must be inserted before x'' since this is a negative quantity, whereas IB is a positive one.)

By substitution

$$AB = \frac{x'H}{f + x' \sin t} - \frac{x''H}{f + x'' \sin t} = \frac{fH(x' - x'')}{(f + x' \sin t)(f + x'' \sin t)}$$

and

$$\frac{f + x \sin t}{H} = \frac{(x' - x'')(f + x' \sin t)(f + x'' \sin t)}{fH(x' - x'')}$$

which may be reduced to the simple form

$$x = x' + x'' + \frac{x'x'' \sin t}{f}.$$

This is the same except for the notation as the equation given on p. 284 of the Manual of Photogrammetry.

It is now proposed to show how to apply the formula just derived to the general case of finding the scale point of any line on the photograph, i.e., a line other than the principal line. Let ab (Figure 6) be any such line and let the corresponding control points on the ground be A and B. We will suppose that an approximate solution has been made by methods previously outlined to determine the tilt of the photograph. A tilt circle has been drawn. This indicates the tilt of the photograph with respect to the plane through the three control points, two of which are A and B. Drop a perpendicular from the principal point o of the photograph to ab at h. The line oh may then be considered to represent the intersection of the plane of the picture with a plane passed through the axis Lo of the camera lens perpendicular to the line ab. This latter plane is, of course, perpendicular to the plane of the photograph. The distance Lh may be found graphically by constructing the triangle Loh or it may be found by scaling oh and then calculating the value Lh by the formula

$$Lh = \sqrt{f^2 + (oh)^2}.$$

The distances La and Lb may also be determined in a similar manner.

The construction of Figure 7 is now begun by laying off ahb taken from Figure 6, in an approximately horizontal position and then on a perpendicular to

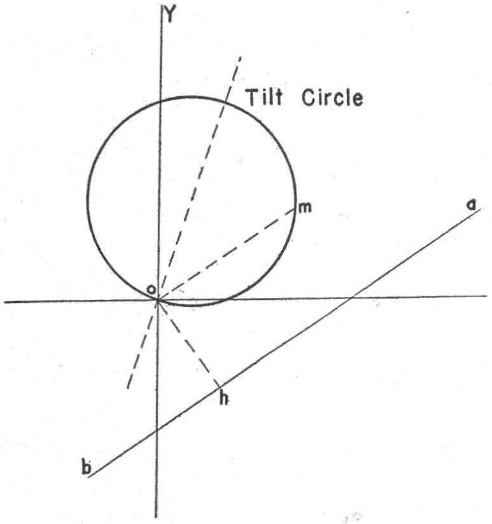

FIG. 6

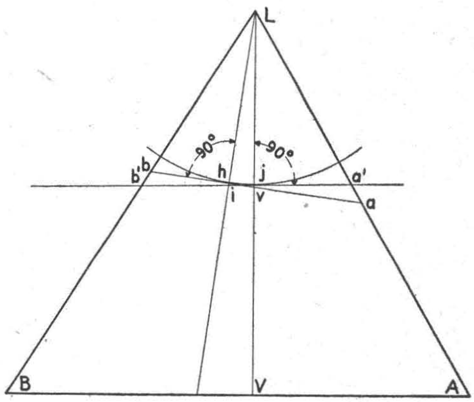

FIG. 7

ab at h, laying off the distance Lh as previously determined. The point L is then connected with a and b and the lines La and Lb extended. The scaled lengths of La and Lb should check with the values previously determined. It is known that the control point A will fall upon La extended and control point B upon Lb extended, but the exact locations of A and B are unknown, although the length of AB is known. The actual position of AB does not need to be known in order to obtain, at least approximately, the location of the scale point of the line ab; but we do need to know the angle t between Lh and LvV. This angle, since Lh is perpendicular to ab and LvV is perpendicular to AB is equal to the angle between the lines ab and AB extended. This angle t may be obtained approximately by calculation from certain data as will now be shown.

To show the relationships between the quantities involved, use will be made of a sphere whose center is at h (See Fig. 8). It is immaterial what radius is assumed for this sphere, since only angular relationships are involved. Let $QRSQ'$ be the great circle cut from this sphere by a plane parallel to the reference plane ABC and let $NMRN'$ be the great circle cut by a plane through ab perpendicular to the plane ABC. Let another plane be passed through the camera station L and the line ab, cutting from the sphere the great circle $TMST'$. These intersecting circles form the triangle MRS, which is right angled at R. Since ab lies in the plane of each of the great circles $NMRN'$ and $TMST'$, it must lie in their line of intersection, which is the radius hm of the sphere. The arc MR measures the slope of the line hM with respect to the plane $QRSQ'$ and therefore is equal to the slope of the line ab with respect to the plane ABC. Let this angle be equal to t'. An approximate value for it may be obtained from the tilt circle, which shows the slope of the plane of the photograph with respect to the plane ABC, by drawing the chord of this circle which is parallel to ab.

The spherical angle RMS is equal to the complement of the dihedral angle between the plane Lab and the plane of the photograph or to the complement of the angle at h in the right triangle Loh. Let this angle be called γ. It may be found from the equation $\tan \gamma = f/oh$.

But the arc Ms is equal to t. This is true because the radius hs must be parallel to AB, since AB and hs both lie in the plane of the great circle $TMST'$ and are also in parallel planes cut by this plane. Then by one of Napier's analogies $\tan t = \tan t'/\sin \gamma$. Thus when t' and γ are known, t may be found by use of this equation.

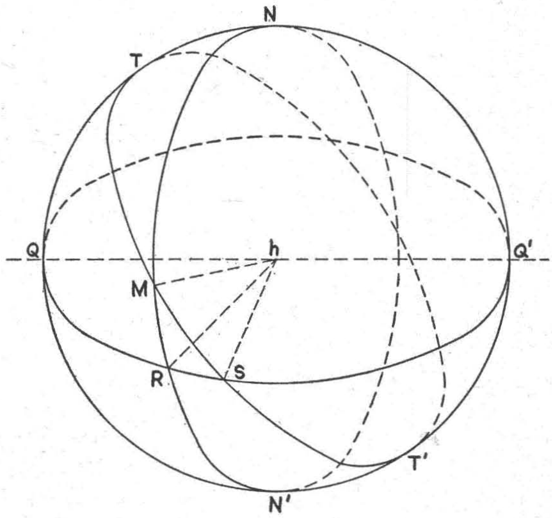

FIG. 8

Returning now to Figure 7 we draw an arc of a circle with L as center and Lh as radius and draw LV by laying off the angle t at L from Lh, to the right as here, if the a or right hand end of this line ab is high or to the left or counterclockwise if the left end is high. At j where the line LV cuts the circular arc draw a tangent, which is, of course, perpendicular to LV, cutting La at a' and Lb extended at b'. This line $a'b'$ is parallel to AB. Let i be the point of intersection of ab and $a'b'$. The line AB may now be drawn to any assumed scale. It must be parallel to $a'b'$ and fit between the lines La and Lb or between these lines extended, according to the scale.

The relationships shown in Figure 7 are analogous now to those found in the case where the scale line lies in the principal plane of the photograph where it was found that the distance from the isocenter to the scale point was given by the quantity x in the equation $x = x' + x'' + (x'x'' \sin t/f)$. The point i in Figure 7 at the intersection of ab and $a'b'$ is analogous to the isocenter and therefore must be used as the origin for measuring the x-values. The distance Lh is used for f.

We are now in a position to give explanations of the ordinary methods of locating scale points, to see why these methods are approximately correct and to make some appraisal of their precision. Thus in Figure 9 let o be the principal point of a photograph, QQ' the principal line to show the direction of the tilt t_0 with respect to the datum plane, i' the isocenter of the photograph and f its focal length. Also let ab be one of the scale lines. From o and i' drop perpendiculars oh and $i'i''$ to ab. Let i be the isocenter point of the line ab, when the section through L and ab is taken. This point should lie in the immediate vicinity of i'' as will now be shown.

Prolong ba to meet QQ' and let α be the angle between QQ' and ba prolonged. Then hi'' is equal to $oi'\cos\alpha$. From the well known relationships between tilt, focal length, principal point and isocenter, $oi'=f\cdot\tan\frac{1}{2}t_0$. The distance hi is

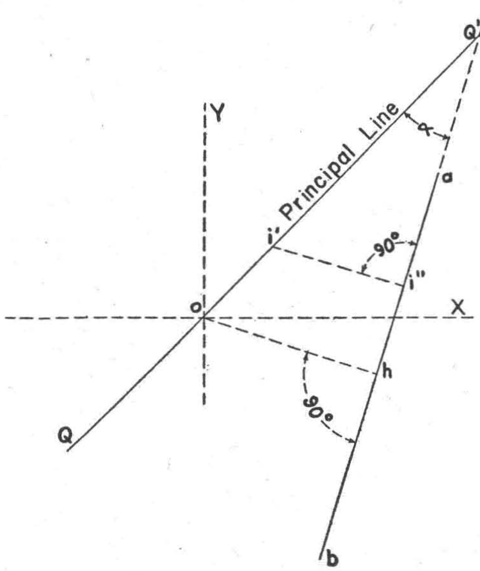

FIG. 9

likewise equal to $Lh\cdot\tan\frac{1}{2}t$. Also the tilt $t=t_0\cos\alpha$ approximately. (See Bagley's "Aero-Photography and Aero-Surveying p. 129.) When the tilt is small we may write $oi'=f\cdot\frac{1}{2}t_0\cdot\tan l'$, $hi=Lh\cdot\frac{1}{2}t\cdot\tan l'$ where the tilt is expressed in minutes of arc. Thus by substitution

$$hi = oi'\,\frac{Lh}{f}\cdot\frac{t}{t_0} = oi'\,\frac{Lh}{f}\cdot\cos\alpha.$$

But $hi''=oi'\cos\alpha$ and therefore $hi=hi''(Lh/f)$.

Since Lh is necessarily greater than f the point i must be on the opposite side of i'' from h, and much nearer the former than the latter.

Thus the most accurate method of locating the scale point of a line involves locating its "isocenter point" by the method previously described. Then the distance x' and x'' from this point to the ends of the line are scaled. These quantities are plus when measured toward the low side of the photograph and minus when measured toward the high side. The value of x in the formula $x=x'+x''+(x'x''\sin t)/Lh$ is then computed and laid off along the line from the isocenter point, toward the low side of the photograph if plus or toward the high side if minus, to give the scale point.

If in the above process the $x'x''\sin t/Lh$ term is neglected (for small values of the tilt angle it is small), then $x=x'+x''$ and the scale point may be found by laying off the shorter segment of the line upon the longer, provided the isocenter point of the line lies on the line, or by laying off the distance to the nearer point beyond the farther point if the isocenter lies on a prolongation of the line.

It is thus evident that the usual methods of dropping perpendiculars from the principal point or the isocenter to divide the line into segments corresponds to using the foot of the perpendicular in each case as the isocenter point and neglecting the $x'x''\sin t/Lh$ term. Since as was shown the isocenter point lies near the foot of the perpendicular dropped from the isocenter of the photograph, it is generally more accurate to use this point than it is to use the foot of the perpendicular dropped from the principal point.

That the term $x'x''\sin t/Lh$ is small may be seen by computing it from assumed data which will give a larger value than would usually be encountered. Thus if $x'=-x''=100$ mm., $t=3°$ and $Lh=250$ mm.

$$\frac{x'x''\sin t}{Lh} = -\frac{100 \times 100 \times 0.051}{250} = -2.0 \text{ mm.}$$

Evidently to neglect corrections like this would considerably weaken a solution.

The following numerical example will illustrate the method. One of the Agriculture Adjustment Administration photographs, known as Photo ART-3-14 which covers an area where control data are known, will be used. On this photograph three control points A, B, and C were chosen and their x and y coordinates referred to axes, whose origin was at the principal point, were carefully measured. The survey coordinates on a rectangular coordinate system for these points, X and Y, are also known as well as the elevations to give the Z coordinates. The coordinates are tabulated below as well as the computed distances, horizontal and slope, between consecutive points on the ground and the distances between the like points on the photograph.

TABLE I

Control Point		A	B	C
X	(feet)	35,900.2	27,420.5	36,541.0
Y	(feet)	5,890.9	15,924.9	16,481.3
Z	(feet)	1,238.4	1,384.6	1,348.9
x	(mm.)	−26.88	+102.69	−33.11
y	(mm.)	+74.57	−70.70	−83.16
Horizontal Distance	(feet)	13,137.23	9,137.46	10,609.77
Slope Distance $=G$	(feet)	13,138.03	9,137.53	10,610.34
Distance on Photograph $=P$	(mm.)	194.655	136.370	157.853
P/G		1,481.62	1,492.42	1,487.73
Tangent of Slope of Control Line		0.011129	0.003907	0.01045
Angle of Slope		0°38′15″	0°13′26″	0°35′48
ho	(mm.)	29.58	79.68	29.75
Lh	(mm.)	211.62	224.19	211.65
$\cot r$		0.1411596	0.3802434	0.141971
r		81°57′55″	69°10′52″	81°55′11″
$\sin r$		0.9901836	0.9347086	0.9900721
t'		0°42′30″	0°54′10″	1°38′10″
$\tan t'$		0.1236336	0.0157575	0.02856329
$\tan t$		0.01248595	0.0168846	0.02884971
t		0°42′55″	0°57′57″	1°39′09″
hi	(mm.)	1.32	1.89	3.05

TABLE II

		A	B	C
x'	(mm.)	-74.55	-97.85	$+81.20$
x''	(mm.)	$+120.0$	$+38.6$	-76.35
$x'x'' \sin t/Lh$	(mm.)	-0.53	-0.28	-0.84
x	(mm.)	$+44.97$	-59.53	$+4.01$

Solution from scale points found by dropping perpendiculars from o.

$$lk = 59.4 \times \frac{6.11}{10.80} = 33.60 \text{ mm. (See Fig. 10)}$$

$$ds = \frac{10.80}{l'h'} = \frac{10.80}{50.35} = 0.2145$$

$$So = 1481.62 + 7.25 \times 0.2145 = 1483.18 \text{ (Scale ratio at Prin. Pt. } o)$$

$$\sin t = \frac{f \times ds}{So} = \frac{209.55 \times 0.2145}{1483.18} = 0.030305$$

$$t = 1°44'12'', \qquad oi = f \times \tan \tfrac{1}{2}t = 3.18 \text{ mm.}$$

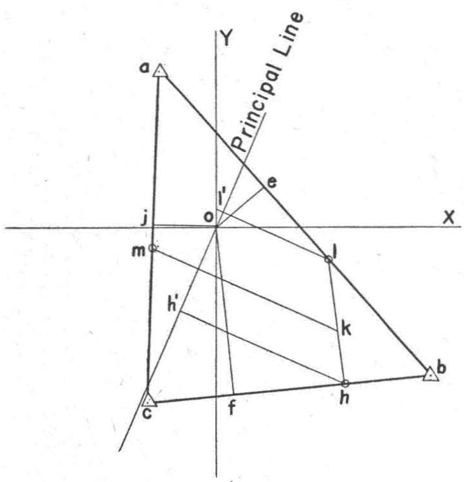

FIG. 10

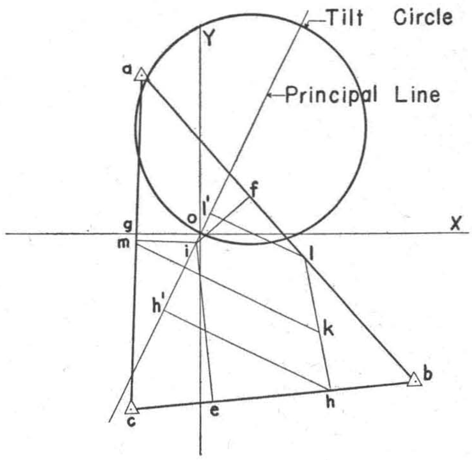

FIG. 11

The isocenter i is now located and the solution repeated with perpendiculars dropped from i (see Fig. 11).

$$lk = 60.85 \times \frac{6.11}{10.80} = 34.42 \text{ mm.}$$

$$ds = \frac{10.80}{49.15} = 0.21974$$

$$Si = 1481.62 + 12.5 \times 0.21974 = 1484.37$$

$$\sin t = \frac{209.55 \times 0.21974}{1484.37} = 0.031021$$

$$t = 1°46'40''.$$

On the principal line resulting from this solution a tilt circle (see Fig. 11) is drawn with the diameter equal to the tilt. (In the original drawing a scale of 1 mm. = 2 minutes of arc was used for the diameter). The slopes of the lines AB, BC and CA are now found by dividing in the case of each the difference in elevation between the ends by the horizontal length of the line to get the tangent of the angle of slope. These angles are respectively 0°38′15″, 0°15′26″ and 0°35′48″ as shown in the table.

The principal line and the tilt circle I are copied in Figure 12 and tilt circle II constructed as previously explained by use of slope angles. As also has been previously described tilt circles I and II are combined to give a third tilt circle which shows the tilt by its diameter, and the swing by the angle which the

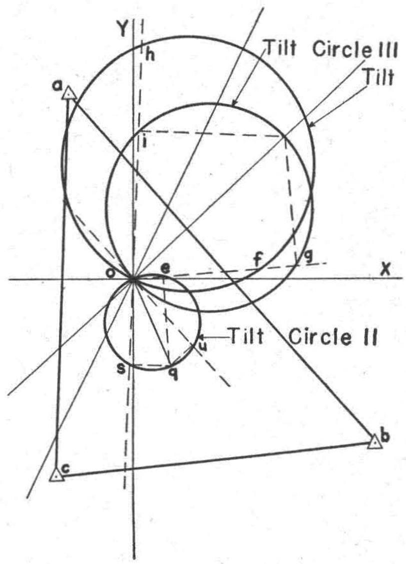

FIG. 12

diameter through o, representing the principal line, makes with the x-axis. The tilt was thus found to be 1°26′7″ and the swing to be 226°15′. From an analytical solution in which the same basic data were used the tilt was found to be 1°26′47″ and the swing to be 225°59′. Thus the results check within a small fraction of a minute for tilt and within a quarter of a degree for swing.

While little increase in accuracy can be expected from a further solution, the theoretically more precise one by use of "isocenter points" will be made. The results are shown in the tabulation. The lengths of the perpendiculars from the principal point to the lines ab, and bc, and ca are scaled to give the quantities in the ho column. Values of Lh and of the angle γ are computed as previously explained. The values of t', representing the slopes of the lines ab, bc and ca with respect to the plane ABC, are found from tilt circle I by drawing chords through the principal point parallel to those lines and scaling the lengths of these chords. From the values of t' the values of t are found by the relationship that $\tan t = (\tan t'/\sin\gamma)$ or more briefly and easily and to a high degree of precision, if t small, by the approximate equation $t = (t'/\sin\gamma)$. The hi distances are now computed and laid off on line from h toward the low side of the picture to locate the isocenter points. Values of x' and x'' are now scaled on each line from i to the ends of the line, x' toward the initial end and x'' toward the forward end and from these by applying the formula $x = x' + x'' + \dfrac{x'x'' \sin t}{Lh}$ the values of x are computed. (The respective values of the term $x'x'' \sin t/Lh$ give closely the distances from the scale points located by the perpendiculars dropped from the isocenter to the more nearly correct scale points located by use of the x distances. In this case the error in the location by the first method is not likely to be greater than 1 millimeter.) The solution by use of the x values is as follows:

$$LK = 60.5 \times \frac{6.11}{10.80} = 34.23 \text{ mm.}$$

$$ds = \frac{10.80}{48.80} = 0.22131$$

$$Si = 1481.62 + 12.5 \times 0.22131 = 1481.62 + 2.77 = 1484.39$$

$$\sin t = \frac{209.55 \times 0.22131}{1484.39} = 0.0312423$$

$$t = 1°47'25'' \; (t = \text{tilt with respect to plane } A, B, C).$$

Reconstruction of tilt Circle I and a new solution by combining with tilt Circle II gives as a final result for the tilt of the photograph 1°27′2″ and for the swing 226°25′. As it happens these values are not quite as exact in this case as those found from the use of the perpendiculars from the isocenter (the preceding solution).

In these solutions it is not necessary to actually construct tilt Circles I and II. Thus for the solution just made after t and the direction of the principal line with respect to the plane of A, B and C have been determined, by use of a protractor the angles between the principal line and the lines ab, bc, and ca may be found. When t, expressed say in minutes, is multiplied by the cosines of these angles, the components of tilt along the three control lines are found. These components, which are with respect to the plane ABC, are then combined with the slope angles of the line AB, BC, and CA to give the components of slope of the control lines with respect to the horizontal plane of reference. From these latter components tilt Circle III may be found.

A few words may be added as to the accuracy of the method. The example given involves a tilt slightly larger than the average. The control points were all at about the same elevation. The results were accurate to a fairly high degree of precision. As the tilt increases the precision diminishes, but not rapidly for the range of angles encountered in practice. For photographs of rough terrain where the three control points are at considerably different elevations the precision may be considerably less than where the control points are more nearly at the same level, especially if the photograph has much tilt. A favorable case is where both photograph and control plane ABC are tilted in the same direction and an unfavorable case is where they are tilted in opposite directions.