TILT BY THE GRAPHICAL PYRAMID METHOD

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THE Graphical Pyramid Method is a direct graphical solution of the tilt problem. This implies that successive determinations are not required as the only errors inherent to the method are those of drafting. If it were possible to draft with precision, resulting values would also be precise. Without reservation, it can be stated that successive determinations are not required and better values cannot be accomplished in a second determination than could have been obtained in the original solution.

Although the method functions through all ranges of tilt and relief in one determination, it has been referred to as "the missing link" in that it can be applied in the range where vertical or horizon methods would be impractical or impossible.

Photogrammetric nomenclature has been held to a minimum, and an endeavor has been made to clarify the problem by correlating the method in terms of an elementary "strike and dip" problem in geology. A background of basic trigonometry will fulfill the mathematical requirements.

Initial data,[1] to illustrate the Graphical Pyramid Method, are as follows:

Photographic Coordinates

Point	*X*	*Y*
a	−4.000″	4.000″
b	4.000″	4.000″
c	0.000″	−4.000″
o=PP	0.000″	0.000″

Focal Length
$F=10.000''$

Inasmuch as a comparator may not be available, photographic distances are measured to the nearest 0.001″. Refer to Figure 1.

Photographic Measurements

Line	*Length*	*Line*	*Length*
ab	8.000″	*oa*	5.657″
bc	8.944″	*ob*	5.657″
ca	8.944″	*oc*	4.000″

Ground Control

Horizontal	*Slope*	*Vertical*
$A'B'=6,409.49'$	$AB=6,487'$	$AA'=1,000'$
$B'C'=8,621.25'$	$BC=8,850'$	$BB'=2,000'$
$C'A'=8,919.71'$	$CA=8,976'$	$CC'=0'$

The exact tilt conforming to this tabulated data is 12° 00′. A tilt of 12° 00′ and relief of 2,000′ were taken to prove that the method will function under so-called "adverse conditions" of excessive tilt and/or relief. The procedure is similar whether the tilt is 3° 00′ with 100′ of relief or the tilt is 60° 00′ with 10,000′ of relief.

Errors caused by incorrect measurements and the identification of control points on the photograph, shrinkage, lens distortion and earth curvature will not be considered. These errors are present in all methods. The purpose of this discussion is to prove the reasoning, accuracy and application of the method. It can then be stated that the resulting errors are inherent only to the particular application of the method and are so isolated.

[1] A different solution of the same example appears in *Applied Photogrammetry*—Anderson, page 349, Edwards Bros., Inc., Ann Arbor, Michigan.

The first step is to compute the photo pyramid lateral edges using the following formulae:

$$La = \sqrt{(oa)^2 + f^2} = \sqrt{(5.657)^2 + (10.000)^2} = 11.489''$$

$$Lb = \sqrt{(ob)^2 + f^2} = \sqrt{(5.657)^2 + (10.000)^2} = 11.489''$$

$$Lc = \sqrt{(oc)^2 + f^2} = \sqrt{(4.000)^2 + (10.000)^2} = 10.770''.$$

All the lateral and base edges of the photo pyramid have now been determined. It is then necessary to lay out the developed pyramid, Figure 2. Obtain a sheet of tracing paper about 18″×24″. Prick a point near the top center of the sheet and designate this point as L. Lay off La from L, arc ab from a and Lb from L. This determines the photo pyramid lateral face Lab. The procedure is similar for the two remaining lateral faces. Arc ac from a and Lc from L to determine Lac. Arc bc from b and Lc from L to determine Lbc. The photo pyramid is now developed.

Extend the edges La, Lb, and Lc. Obtain another sheet of tracing paper about 6″×18″ and lay off in the middle of the sheet the slope distance AB to the scale of 1″=1,000′. Arc the slope distances AC and BC to the same scale.[2] Line up the blade of a drafting machine or adjustable head T-square to coincide with cc. Orient the latter sheet so that A falls along ray La, B along ray Lb and the intersections of both arcs with rays Lc fall along the blade. It can readily be seen that the reason for this last procedure of maintaining the intersections to fall along the blade is that LC must equal LC since it is the common edge of the developed ground pyramid. When the orientation is complete, prick through the control points A, B and C onto their respective pyramid lateral edges. This completes the graphical determination of the ground pyramid lateral edges. The actual lengths of these edges are measured using a scale of 1″=1,000′.

A tabulation of the drafted ground pyramid lateral edges is as follows:

Edge	*Drafted*	*Exact*	*Error*
LA	9,740′	9,743′	−3′
LB	8,658′	8,660′	−2′
LC	12,028′	12,034′	−6′

Extreme care in drafting was performed to accomplish this degreee of accuracy. It will be demonstrated later that this degree of accuracy is not necessary, for practical purposes, where the time of operation is the essence. Various degrees of error will be intentionally introduced in later examples to indicate the range of accuracy of the final product.

It should be mentioned at this point that an instrument could be constructed to determine the ground pyramid lateral edges if the graphical pyramid method is placed on a production basis. The instrument would consist of slotted calibrated arms coupled with calibrated adjustable cross bars representing the ground slope distances. The apex angles of the photo pyramid would be calculated by elementary analytical geometry. An index mark on the extremities of the radial arms would be set to this angle over a degree scale attached to the base board. The radial arms would subtend the same angle as the pyramid lateral edges did in space at the instant of exposure. The cross bars would then be adjusted so that the radial arms on each side would be equal. The length of each ground pyramid lateral edge would then be determined and read along the calibration of each radial arm. Only a production basis would warrant the construction cost of this instrument, but would eliminate a large portion of the

[2] *Aerophotography and Aerosurveying*—Bagley, McGraw-Hill.

drafting. The operation of the instrument would be routine requiring unskilled personnel.

The next operation is to find the altitude of the ground pyramid, $LABC$, realizing that the base of this pyramid is the relief plane determined by the control points ABC in elevation. This altitude is designated as LQ in Figure 1. Q is the foot of the dropped perpendicular from the exposure station, L, to the relief plane ABC. This is not a plumb line relative to the horizontal inasmuch as relief is present. If there were no relief, Q would coincide with the ground nadir point. Q may fall beyond the limits of ABC if the relief is excessive as in this problem. Nevertheless Q lies in the relief plane ABC extended and presents no difficulty as will be demonstrated in the developed pyramid layout.

Q is located graphically by dropping a perpendicular from L to the three developed ground slope lines. These lines, LD, LE and LF, in terms of solid geometry, are the slant heights. The developed slope lines AC and BC are arced to form the ground slope triangle ABC. The perpendiculars from this triangle at the points D, E and F intersect at the point Q.

The measured lengths to the nearest 0.005″ or 5′ were as follows:

$LD = 8{,}490'$ $\quad$ $LE = 8{,}650'$ $\quad$ $LF = 9{,}585'$

$QD = 55'$ $\quad$ $QE = 1{,}655'$ $\quad$ $QF = 4{,}445'$

Since the above lengths are the hypothenuse and base of a right triangle, the altitude is computed as follows:

$LQ = 8{,}489.8'$ $\quad$ $LQ = 8{,}490.2'$ $\quad$ $LQ = 8{,}492.0'$

$$\text{Average } LQ = 8{,}490.7'$$

It would be necessary only to compute one triangle but the average of the three is taken. This also serves as a check, as it will indicate any gross error in any one computation.

It is now required to find the angle that the relief plane makes with the horizontal. This is commonly referred to in geology as the "dip."

Since the elevation of C is zero feet and B 2,000′, prorate along CB to determine elevation 1,000′ at G. Connect A with G which defines a line of constant relief, or a horizontal line in the relief plane 1,000′ above the datum plane. A line parallel to the line of constant relief through C is the datum relief axis. This is simply the intersection of the relief plane determined by the three control points and datum, or a horizontal line in the relief plane zero feet above the datum plane.

Draw JB perpendicular to the datum relief axis. The measured length of JB was found to be 8,335′. If ϕ equals the angle of dip

$$\sin \phi = \frac{BB'}{JB} = \frac{2{,}000'}{8{,}335'} = 0.2399520$$

$$\cos \phi = 0.9707852$$

The relief plane therefore makes an angle of 13° 53′ with the datum plane.

The datum elevation of Q in the relief plane is found by measuring KQ which is also perpendicular to the datum relief axis.

If QQ' is the elevation of Q above the datum:

$$QQ' = (\sin \phi)(KQ)$$

$$QQ' = (0.2399520)(7{,}310') = 1{,}754.0'$$

Inasmuch as the altitude of the ground pyramid subtends an angle of ϕ with the vertical, the flying height, H, can be computed as follows:

$$H = (\cos\phi)(LQ) + QQ'$$

$$H = (.9707852)(8{,}490.7') + 1{,}754.0' = 9{,}997'$$

Exact H	*Drafted H*	*Error*
10,000′	9,997′	−3′

The tilt of the photograph will also be found by means of the "strike and dip" method. In other words, the vertical distance of points a, b, and c from a horizontal plane will be computed. This horizontal plane is perpendicular to the plumb line and, for convenience, will be passed through one of the photo points so that the vertical distance will be zero.

Refer to Figure 1. The intersection of the plumb line LV and the horizontal line from photo point c will be denoted as k. There will be three positions of k along the plumb line when three control points are given. There will therefore be three values of Lk, one for each control point. If the photograph were truly horizontal, all three positions of k would coincide, resulting in three identical values of Lk. In this special case, Lk is equal to the focal length and k the position of the principal point. It can therefore be seen that the difference in the values of Lk is a function of the tilt. It is this principle that is utilized in determining the tilt and swing by the graphical pyramid method. For convenience in handling symbols, Lk for photo points a, b and c will be designated as f_a, f_b and f_c, respectively. From Fig. 1 the following relationship exists from similarity of triangles:

$$f_a = \frac{(La)(H - AA')}{LA} \qquad f_b = \frac{(Lb)(H - BB')}{LB} \qquad f_c = \frac{(Lc)(H - CC')}{LC}.$$

Using computed numbers and drafted measurements:

$$f_a = \frac{(11.489'')(8{,}997')}{9{,}740'} \qquad f_b = \frac{(11.489'')(7{,}997')}{8{,}658'} \qquad f_c = \frac{(10.770'')(9{,}997')}{12{,}028'}$$

$$f_a = 10.6126'' \qquad f_b = 10.6119'' \qquad f_c = 8.9514''$$

$$f_c = 8.9514'' \qquad f_c = 8.9514'' \qquad f_c = 8.9514''$$

$$\Delta f_a = 1.6612'' \qquad \Delta f_b = 1.6605'' \qquad \Delta f_c = 0.0000''$$

The Δf_a and Δf_b are the incremental changes in respect to Δf_c. This indicates that photo point a is 1.661″ (nearest 0.001″) and photo point b is 1.661″ vertically above photo point c. Since photo points a and b are the same vertical distance above photo point c, photographic line ab is parallel to the tilt axis and is of constant scale.

Refer to Figure 2 where the developed photographic lengths are arced to reform the photographic control triangle. Since Δf_a and Δf_b are equal, it is not necessary to prorate for a line of constant scale. This will seldom be the case but presents no difficulty since a line of constant scale can be graphically determined by proration as previously described in locating the line of constant relief. A perpendicular from the principal point o to line ab defines the principal lin . This line so happens to coincide with the $+y$ axis; therefore the swing s is equal to

$$\tan(s) = \frac{x}{y} = \frac{0.000}{4.000} = 0.000$$

$$s = 0° \, 00''$$

The "dip" angle of the photographic plane is equal to the tilt angle. If the dip angle or the tilt angle is denoted as t and the perpendicular distance to the line of constant scale from the datum photo point c is denoted as l:

$$\sin t = \frac{\Delta f_a}{l}$$

$$\sin t = \frac{1.661''}{8.000''} = 0.20763$$

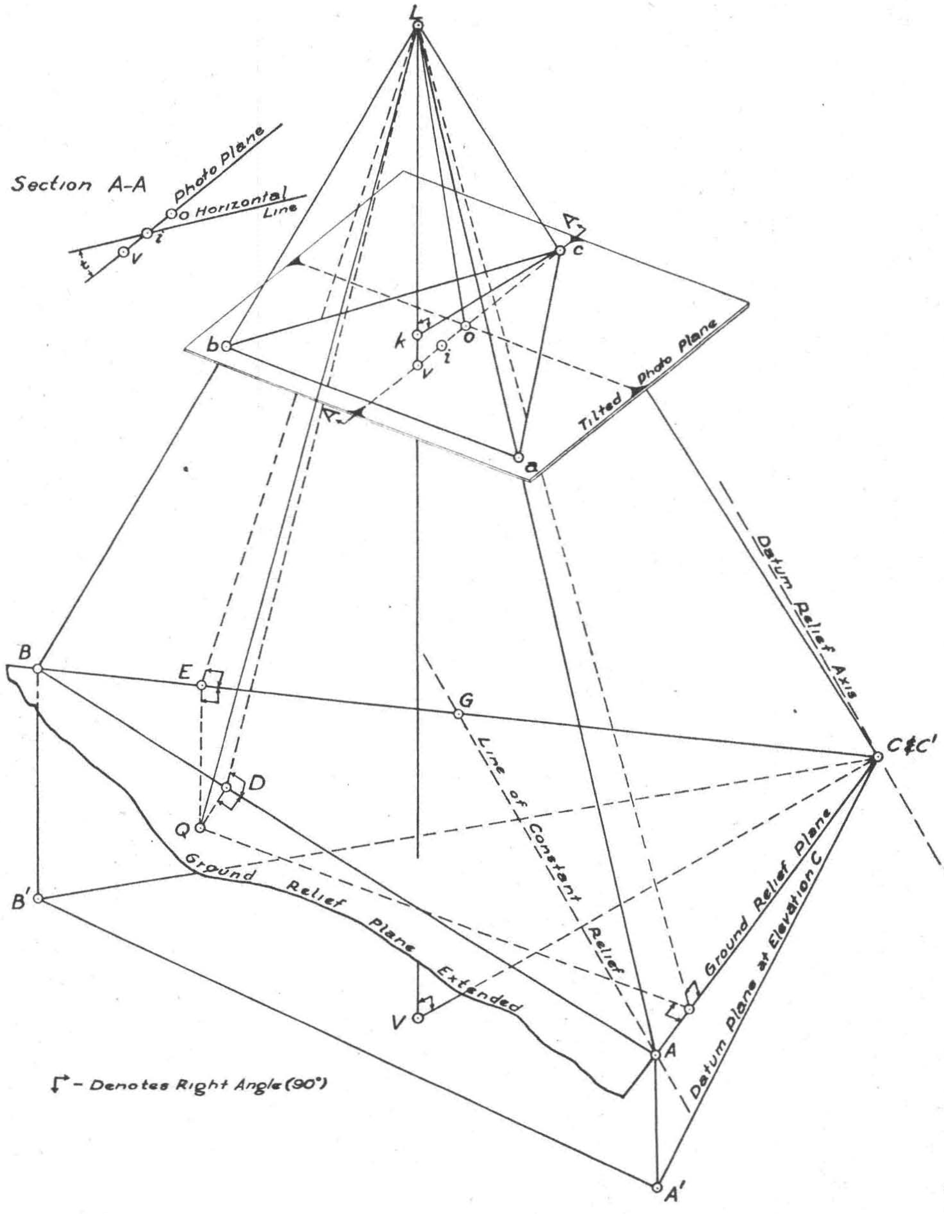

FIG. 1. Notation diagram.

$$t = 11°59'$$

$$ov = f \tan t = (10.000'')(0.2124) = 2.124''$$

$$oi = f \tan \frac{t}{2} = (10.000'')(0.1050) = 1.050''$$

These distances are laid off from *o* along the principal line in the direction "from low to high Δf."

The problem is now complete with the error of tilt and swing tabulated as follows:

	Drafted	*Exact*	*Error*
Tilt	11°59	12°00′	−0°01′
Swing	0°00′	0°00′	0°00′

A work sheet, Form 1, will facilitate the listing of given data, measurements and calculations. The work sheet is reproduced with the symbols only, leaving

GRAPHICAL PYRAMID METHOD

WORK SHEET

$f = 10.000''$

Code:
* indicates drafted measurements

Horizontal	*Vertical*	*Slope*	*Photo Measurements*		*Photo Pyramid Edges*
$A'B' = 6{,}409.49'$	$AA' = 1{,}000'$	$AB = 6{,}487'$	$ab = 8.000''$	$oa = 5{,}657''$	$La = 11.489''$
$B'C' = 8{,}621.25'$	$BB' = 2{,}000'$	$BC = 8{,}850'$	$bc = 8.944''$	$ob = 5.657''$	$Lb = 11.489''$
$C'A' = 8{,}919.71'$	$CC' = 0'$	$CA = 8{,}976'$	$ca = 8.944''$	$oc = 4.000''$	$Lc = 10.770''$

Ground Pyramid Edges		*Ground Pyramid Altitude*	
$LA^* = 9{,}740'$	$LD^* = 8{,}490'$	$LE^* = 8{,}650'$	$LF^* = 9{,}585'$
$LB^* = 8{,}658'$	$QD^* = 55'$	$QE^* = 1{,}655'$	$QF^* = 4{,}445'$
$LC^* = 12{,}028'$	$LQ = 8{,}489.8'$	$LQ = 8{,}490.2'$	$LQ = 8{,}492.0'$
	Average	$LQ = 8{,}490.7'$	

Dip Angle of Relief Plane	*Datum Elevation of Q*	*Flying Height*
$\sin\phi = BB'/JB^*$	$QQ' = (\sin\phi)\ (KQ^*)$	$H = (\cos\phi)\ (LQ) + QQ'$
$\sin\phi = 2{,}000'/8{,}335'$	$QQ' = (0.2399520)\ (7{,}310')$	$H = (0.970\ 7852)\ (8{,}490.7') + 1{,}754.0'$
$\sin\phi = 0.2399520$	$QQ' = 1{,}754.0'$	$H = 9{,}997'$
$\cos\phi = 0.9707852$		

Effective Focal Lengths

$f_a = \frac{(La)(H\text{-}AA')}{La}$	$f_b = \frac{(Lb)(H\text{-}BB')}{LB}$	$f_c = \frac{(Lc)(H\text{-}CC')}{LC}$
$f_a = 10.6126''$	$f_b = 10.6119''$	$f_c = 8{,}9514''$
$f_c = 8.9514''$	$f_c = 8.9514''$	$f_c = 8.9514''$
$\Delta f_a = 1.6612''$	$\Delta f_b = 1.6605''$	$\Delta f_c = 0.0000''$
$\Delta f_a = 1.661''$	$\Delta f_b = 1.661''$	$\Delta f_c = 0.000''$

Tilt	*Swing*
$\sin(t) = \Delta f_a/l$	$\tan(s) = x/y$
$\sin(t) = 1.661''/8.000''$	$\tan(s) = 0.000''/4.000''$
$\sin(t) = 0.20763$	$\tan(s) = 0.00000$
$t = 11°\ 59'$	$s = 0°\ 00'$

FORM 1

blank spaces for numbers. This will expedite a production system. Operator One can list the data in the first row of the work sheet. The sheet is then turned over to Operator Two and has all the data to accomplish the drafting. The drafted measurements are entered on the work sheet opposite the asterisks and

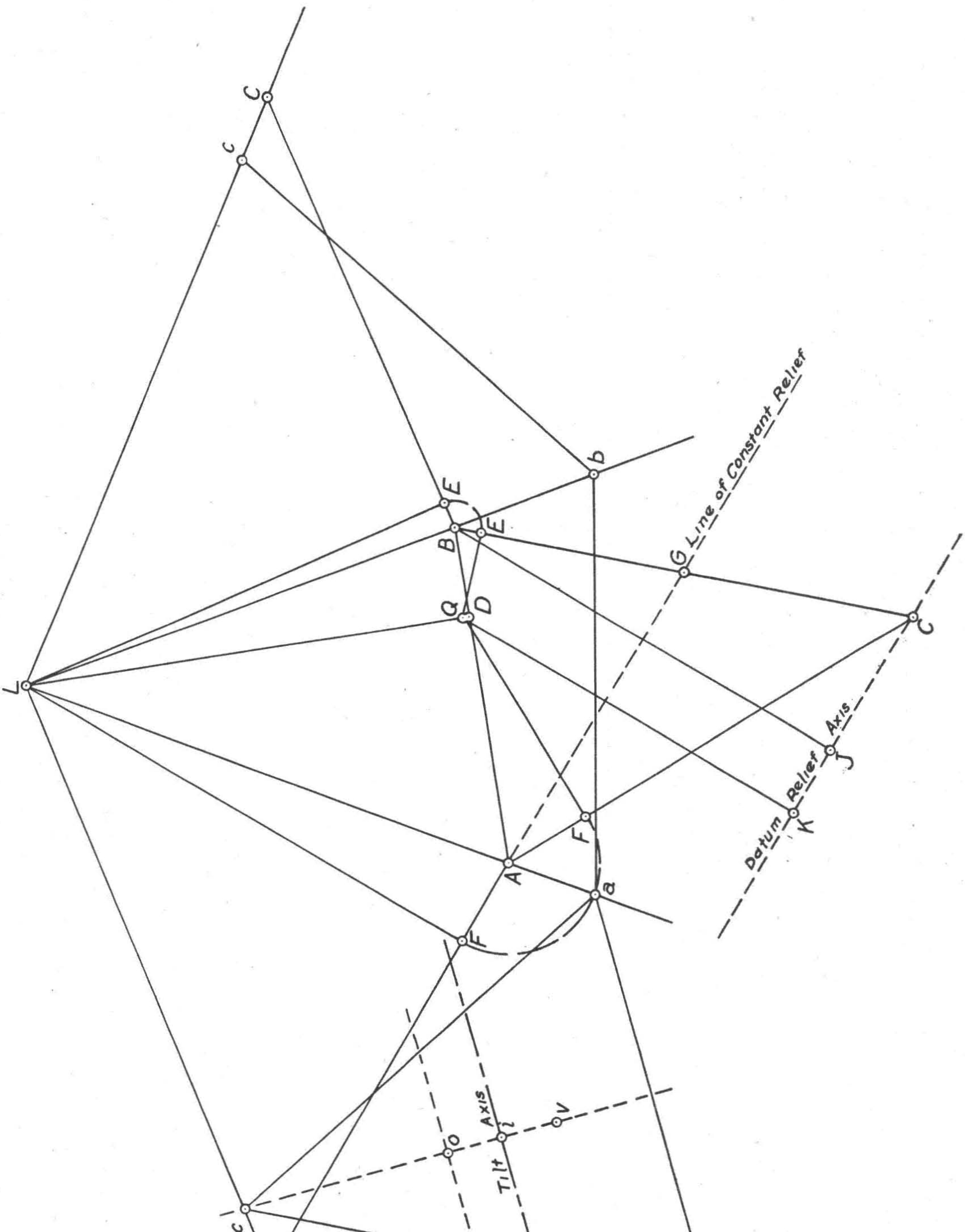

FIG. 2. Developed Pyramid Layout.

turned back to Operator One who completes the necessary calculations. The extent of the mathematics consists of solving for the altitude of a right triangle and the recovery of the cosine from a known sine.

Although the swing and tilt were determined by the "strike and dip" method, it can be correlated to photogrammetrical theory. The "strike" differs from swing by 90°. It can also be stated that f_a, f_b and f_c are the "effective focal lengths,"[3] induced by tilt, for their respective photo points. If the focal length (10.000″) of the camera were subtracted from f_a, f_b and f_c, the difference would represent the increase or decrease of focal length induced by tilt.

It is important to discuss the magnitude of the tilt and swing error caused by errors in drafting. The practical approach is to determine the allowance or tolerance in drafting that will produce an error in tilt and swing within a range that is acceptable for the photogrammetrical purpose under consideration. An increase in the drafting tolerance may decrease the time of drafting considerably, and yet the slight increase in the tilt and swing error may be within the allowable range for practical purposes. In this method, the only final drafted elements that are a function of the tilt and swing are:

(a) Ground pyramid lateral edges.

(b) Flying height.

The ratio of the ground lateral pyramid edges to the flying height is near unity; therefore if the drafting error of the ground pyramid lateral edges and flying height are in the *same direction*, there is a tendency to cancel the drafting error. Since the flying height is a drafted function of the drafted pyramid lateral edges, there is a greater probability that the drafting errors will be in the same direction.

For example, an error of +0.02″ or +20′ on the ground pyramid lateral edges and flying height in the *same direction* resulted in a tilt of 12° 00′ or an error of 0° 00′. This is due to the aforementioned ratio being near unity.

To explain this advantageous condition in greater detail consider the portion of the formula for f_a that is a function of the drafting error:

$$\text{(Exact)} \qquad \frac{H - AA'}{LA} = \frac{9{,}000'}{9{,}743'} = 0.92374$$

$$(+20' \text{ Error}) \qquad \frac{H - AA'}{LA} = \frac{9{,}020'}{9{,}763'} = 0.92390$$

Difference of ratio due to +20′ error = 0.00016.

The difference of ratio is in error 0.017 of 1% of the exact ratio. This numerical condition can be termed a "mechanical advantage of numbers."

Although the probability is that the drafting errors will be in the same direction, this of course will not always be the case. For a condition of opposite drafting errors, assume the pyramid lateral edges are in error +0.02″ or +20′ and the flying height −0.02″ or −20′ to introduce the maximum tilt error for the given drafting tolerance. The tilt is then computed to be 11° 54′ or an error of −0° 06′.

If the flying height is known with any degree of accuracy it is possible to eliminate a great portion of the drafting. The only unknowns in the "$f_{a,b,c}$" formula are the ground pyramid lateral edges and the flying height. The ground pyramid lateral edges are determined in the first stage of drafting. The drafting

[3] This is not to be confused with the effective focal length as applied to lens calibration.

may be terminated at this point if a fairly reliable flying height can be obtained from an altimeter reading.

To illustrate this condition, assume an altimeter reading is in error −1% or −100′ of the exact flying height. The length of the ground pyramid lateral edges, which are slightly in error, as determined graphically will be used. The resultant tilt was found to be 11° 44′ or an error of −0° 16′.

It can readily be seen that if the ground pyramid lateral edges can be obtained by an instrument and used in conjunction with a reliable altimeter reading, the drafting layout is entirely eliminated and the computations consist only of solving for the effective focal lengths, swing and tilt.

If a method of tilt determination is claimed to perform through vertical and oblique ranges of tilt, the "acid test" would consist of working the problem under specifications of trimetrogon photography.

A fictitious photograph will be constructed so as to incorporate, intentionally, elements that adversely affect the solution.

PHOTOGRAPHIC COORDINATES

Point	*x*	*y*
a	−2.000″	2.000″
b	2.000″	−2.000″
c	−2.000″	−2.000″
p.p.	0.000″	0.000″

Trimetrogon photographs are 9″×9″ in size so the positions of the photographic coordinates are within less than the middle half of the photograph. If the Dropped Perpendicular Method[4] of tilt determination were used in conjunction with these photographic coordinates, the scale points would fall nearly in a straight line and this condition is too sensitive for a close solution.

A flying height of 20,000′ above datum, 60° 00′ tilt and 180° 00′ swing will be assumed with vertical elevations of the ground control points as follows:

Point	*Datum Elevation*
A	0′
B	10.000′
C	5,000′

Visualize control point *A* as lying on a beach at sea level. Points *B* and *C* located on a mountain range. It would be noted that the elevation of control point *B* is one-half the flying height. It can then be stated that the problem contains excessive tilt and relief.

For research purposes, the assumed data are used to solve for the horizontal ground distances. Then to simulate actual conditions, only the ground control, photographic positions and focal length are used as the initial data. This enables one to check the accuracy of the method for it is known that the tilt should be 60° 00′, swing 180° 00′ and the flying height 20,000′.

The drafted pyramid lay-out is not illustrated but is similar to the one previously drawn. The scale for laying out the photo pyramid was 1″=3″ and the ground pyramid 1″=3,000′. It can be seen from the work sheet that the ground pyramid lateral edge *LA* is 104,650′. This means that the distance from the aircraft, at the time of exposure, to control point *A* is approximately 20 miles. The developed pyramid is therefore elongated but presents no difficulty in solution and maintains the same principles of procedure as previously outlined.

Results transcribed from the work sheet Form 2 are as follows:

	Drafted	*Exact*	*Error*
Tilt	59° 48′	60° 00′	−0° 12′
Swing	180° 07′	180° 00′	+0° 07′

[4] Anderson, *loc. cit.*

GRAPHICAL PYRAMID METHOD

WORK SHEET

$f=6.000''$

Code:
* indicates drafted measurements

Horizontal	*Vertical*	*Slope*	*Photo Measurements*		*Photo Pyramid Edges*
$A'B'=95,797.67'$	$AA'=0'$	$AB=96,318'$	$ab=5.657''$	$oa=2.828''$	$La=6.633''$
$B'C'=11,458.78''$	$BB'=10,000'$	$BC=12,502'$	$bc=4.000''$	$ob=2.828''$	$Lb=6.633''$
$C'A'=88,116.28'$	$CC'=5,000'$	$CA=88,258'$	$ca=4.000''$	$oc=2.828''$	$Lc=6.633''$

Ground Pyramid Edges		*Ground Pyramid Altitude*	
$LA^*=104,650'$	$LD^*=11,800'$	$LE^*=13,600'$	$LF^*=14,375'$
$LB^*=14,050'$	$QD^*=2,300'$	$QE^*=7,200'$	$QF^*=8,400'$
$LC^*=21,050'$	$LQ=11,574'$	$LQ=11,538'$	$LQ=11,665'$
	Average	$LQ=11,592$	

Dip Angle of Relief Plane	*Datum Elevation of Q*	*Flying Height*
$\sin\phi=BB'/JB^*$	$QQ'=(\sin\phi)(KQ^*)$	$H=(\cos\phi)(LQ)+QQ'$
$\sin\phi=10,000'/21,475'$	$QQ'=(0.4656577)(20,900')$	$H=(0.8849649)(11,592)+9,732$
$\sin\phi=0.4656577$	$QQ'=9,732'$	$H=19,991'$
$\cos\phi=0.8849649$		

Effective Focal Lengths

$f_a=\dfrac{(La)(H\text{-}AA')}{LA}$	$f_b=\dfrac{Lb)\ H\text{-}BB')}{LB}$	$f_c=\dfrac{(Lc)(H\text{-}CC')}{LC}$
$f_a=1.2671''$	$f_b=4.7167''$	$f_c=4.7238''$
$f_a=1.2671''$	$f_a=1.2671''$	$f_a=1.2671''$
$\Delta f_a=0.0000''$	$\Delta f_b=3.4496''$	$\Delta f_c=3.4567''$
$\Delta f_a=0.000''$	$\Delta f_b=3.450''$	$\Delta f_c=3.457''$

Tilt	*Swing*
$\sin(t)=\Delta f_b/l$	$\tan(s)=x/y$
$\sin(t)=3.450''/3.992''$	$\tan(s)=0.008''/-4.000''$
$\sin(t)=0.86423$	$\tan(s)=0.00200$
$t=59°\ 48'$	$s=180°\ 07'$

FORM 2

If overlapping photographs were obscured by cloud coverage, it would be possible to locate the apparent horizon line since the drafted tilt and flying height are known. By using these drafted values, the position of the apparent horizon line was in error 0.024″. This line can be transferred to the overlapping photograph and the tilt computed by the horizon method should ground control be lacking.

When relatively high degrees of tilt and relief adversely affect the methods of Professor Earl Church, Syracuse University and Professor P. H. Underwood, Cornell University, a fairly accurate position of the exposure station by the Graphical Pyramid Method would expedite their solution immensely. Their present method of locating an approximate nadir point on the ground consists of raying in the principal point. This approximate position of the ground nadir point will be in error considerably, depending upon the magnitudes of tilt and relief. "In solutions of this kind, approximate methods must be used for some of the steps."[5] If these methods were given fairly accurate coordinates of the exposure station as initial data, their solutions would be greatly facilitated. Higher

[5] "Space Resection Problems in Photogrammetry"—Professor P. H. Underwood, Proceedings, American Society of Civil Engineers, Sept. 1946.

ranges of tilt could then be determined analytically by these methods. Viewing it from an economical standpoint, it would not be practical to better a graphical value of 11° 59′, in this case, by another method to obtain a value of 12° 00′, for initial measurements may be in error to induce a 0° 01′ tilt. This suggestion is therefore offered in the light of academic discussion.

To obtain the ground coordinates of the exposure station, Figure 2 is not complete. Construct a perpendicular line from A to the datum relief axis. Designate this intersection as M. Then:

$$MA' = \quad A(\cos\phi)$$

MA is measured and $\cos\phi$ has been previously listed on the work sheet. MA' is laid off from M in the direction of A. The datum or horizontal position of control point A is A'.

The same procedure exists for the displacement of control point B.

$$JB' = JB(\cos\phi).$$

The final step is to find the datum position of the ground nadir point V which is the vertical projection of exposure station L to datum. Visualize the ground pyramid altitude perpendicular to the ground relief plane at Q. The upper extremity of the ground pyramid altitude is L. The datum position of L in terms of the position of Q is computed as follows:

$$KV = \frac{KQ - H\sin\phi}{\cos\phi}.$$

The value of each term of the right side of the equation can be obtained from the work sheet. KV is layed off from K in the direction of Q and so locates the position of the ground nadir point V. Since the coordinates of the control points would be given as initial data, it is a simple drafting procedure to determine the X and Y coordinates of V since its position is known in relation to the three control points.

It was assumed that a rough but quick value of flying height, swing and tilt was desired for the 12° 00′ problem. A sheet of tablet paper, sheet of vellum, pencil, two triangles, scale and dividers were the working tools. The scale used for the lay-out was 1″ = 3,000′ or $\frac{1}{3}$ of the original scale. Extreme care was not maintained in the construction of the lines. Results were as follows:

	Rough Drafting	*Exact*	*Error*
LA	9,775′	9,743′	+32
LB	8,625′	8,660′	−35
LC	12,000′	12,034′	−34
H (Average)	9,996′	10,000′	−4
Tilt	11° 33′	12° 00′	−0° 27′
Swing	0° 18′	0° 00′	+0° 18′

Exposure station coordinates determined from these data would facilitate the solution of the analytical space resection methods.

The effective focal length formula of the graphical pyramid method could be used to advantage for tilt determination in the method presented by Professor Underwood.[6] This formula is exact and therefore conforms to the accuracy desired in an analytical space resection solution. The Δf's are effective at the control points, thereby maintaining the same size triangle for strength of proration as the original photographic control triangle. Although scale is not directly

[6] Underwood, *loc. cit.*

solved for in the Graphical Pyramid Method, it can be considered in the light of scale checking along the ground pyramid lateral edges. In lieu of placing a scale, La/LA, at the control point, the effective focal length induced by tilt, $La(H-AA')/LA$, is used.

It is not to be presumed that usage of the Graphical Pyramid Method is recommended for all tilt solutions regardless of the magnitudes of tilt and relief. When tilt and relief are fairly moderate, the "Dropped Perpendicular Scale Point"[7] method is recommended for economical reasons or, more to the point, the Graphical Pyramid Method is recommended for all cases requiring more than two or three successive determinations by the Dropped Perpendicular Method.

The Graphical Pyramid Method may be used to good advantage in cases of relatively high tilt and relief, and in cases of low tilt and relief when the scale points are extremely close to each other or when they fall on or nearly in a straight line.

In discussing a graphical pyramid method of tilt determination it is appropriate to refer to a similar method by Lt. Col. James W. Bagley.[8] In this method the ground pyramid is developed using the horizontal ground distances in place of the slope ground distances. This accounts for the author's statement "The method is affected adversely by excessive tilt and relief." The method referred to also requires successive determinations when relatively high degrees of tilt and/or relief are present.

As a final comment it has often been stated that anything that can be drafted can be computed. The entire graphical pyramid method can be computed analytically to as many decimal places required and will be precise as the trigonometric functions used. If this were accomplished for academic purposes, it is suggested that the ground pyramid lateral edges be computed by use of the cosines of the base angles as presented by Professor Underwood.[9] These analytical computations would be lengthy indeed, and for practical purposes the graphical method is suggested.

COMMENTS ON "TILT BY THE GRAPHICAL PYRAMID METHOD"

Everett L. Merritt

Mr. McNeil's paper, "Tilt by the Graphical Pyramid Method" is clearly written, the numerical example is concisely demonstrated, and the basic principles are graphically illustrated. The paper is of interest to members of the American Society of Photogrammetry.

I was impressed with the nearly identical development of the five elements of exterior orientation in Mr. McNeil's paper with those in my "Space Resection," paper. He determines X, Y, Z semigraphically, while I develop X, Y, Z analytically. The principle underlying the determination is the same in both papers. Mr. McNeil's method of determining tilt is identical with the fourth method described in my paper, both in principle and solution. In all other parts the papers are radically different.

Since two persons have developed two identical ideas independent of each other, I feel that in the interest of the Society and in fairness to both Mr. McNeil and myself, both papers should be not only published in the same issue but also placed next to each other for ready comparison.

[7] Anderson, *loc. cit.*
[8] Bagley, *loc. cit.*
[9] Underwood, *loc. cit.*