TILT ANALYSIS BASED UPON VERTICAL CONTROL DATA

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TILT analysis is an intriguing basic problem of photogrammetry for which several solutions have been proposed from time to time. Some of these solutions, although interesting or even instructive, are difficult and quite impracticable, while others appear to have some practical value.

Virtually every solution which has been advanced for the tilt analysis problem employs three control points whose images appear in the photograph being analyzed, points for which both the horizontal positions and the elevations must be pre-determined by geodetic methods. Regarding these control data requirements, note should be taken that the geodetic determination of the horizontal positions with the degree of precision which permits an accurate tilt determination, entails some difficulty; for a slight error in the shape of the control triangle introduces a considerable error into the resulting tilt. Although this difficulty may be largely, or even entirely, eliminated in the multiplex projector orientation of aerial photographs by "bridging," it is doubtless frequently encountered in the tilt analysis of individual photographs.

Distinction must be made between three types of tilt analysis problems. The first type might be designated as the determination of "rectification data," the problem of determining for each photograph only the magnitude of the resultant tilt and its direction on the photograph, often called the "swing." Instead of the tilt and the swing, the same data for a photograph might be specified by means of two tilt components called "tip and tilt" or x tilt and y tilt. Or in fact the same data might be given by means of the rectangular coordinates of the nadir point of the photograph, referred to the geometric axes of the photograph. Regardless of this detail, the data supplied suffice for the precise rectification of the photograph. The second type of tilt analysis problem might be called the determination of "scale data," the problem of determining for each photograph the exact altitude of the exposure station, together with the tilt and swing, or the x tilt and y tilt, or the photographic coordinates of the nadir point. In our teaching of photogrammetry we define the "scale data" for an aerial photograph as the quantities whose values must be known for the photograph in order that exactly correct horizontal distances between ground points of known but unequal elevations may be found from image measurements upon the photograph; and for a tilted photograph the "scale data" comprise the tilt, the swing, and the exposure station altitude, but not the azimuthal orientation of the photograph. The third type of tilt analysis problem is the complete space resection problem, wherein the actual space coordinates of the exposure station are determined, based upon the survey rectangular coordinate system, and therein the complete space orientation of the photograph is found, including the tilt, the swing, and the survey azimuth of the principal plane. The data from the last type of problem permit finding from photographic measurements not only the correct horizontal distances between any pairs of ground points whose elevations are known, but also the correct survey coordinates of the individual points.

For the tilt analysis problem of the first type, the determination of rectification data only, a solution is possible based upon control data consisting of *elevations only*. This point is of the utmost importance; for the determination of the control elevations demands only such a comparatively simple field procedure that the element of uncertainty regarding the reliability of the control data upon which tilt determinations depend, is completely eliminated, and the tilt determinations are entirely free from the errors often introduced by small errors in the horizontal control data.

This problem of determining rectification data from vertical control only, is the objective of this paper. The solution given here probably furnishes the most precise method for computing tilts of aerial photographs.

The problem is solved for two photographs simultaneously, by means of a few simple principles of the so-called direction-cosine system of analysis. The control data comprise only the elevations of four points, so distributed that the image of one of the points appears near each one of the four corners of the field of overlap of the two photographs.

Statement of the Problem

The illustrative problem, for which the complete solution is to be shown here, is stated as follows:

Two aerial photographs taken with a camera whose focal length is 150.00 mm., contain the images of four ground points A, B, C, and D, for which the elevations above sea-level are given as

A	1,000 ft.	C	400 ft.
B	200 ft.	D	400 ft.

The coordinates of the images, measured on the photographs, are

	First Photograph		*Second Photograph*	
	x	y	x	y
a	+ 3.72 mm.	+43.26 mm.	−79.52 mm.	+43.70 mm.
b	+ 3.69	−33.79	−75.28	−33.73
c	+81.38	+42.56	0.00	+42.48
d	+80.38	−34.62	0.00	−34.11

All exposures in the flight strip were made at an altitude of approximately 20,100 feet above sea level. Find precisely the tilt and swing of each of the two photographs.

Note:—The approximate common exposure altitude to be used for all of the photographs in an entire flight project, is but a rough value, which is of course always available from preliminary approximate scale calculations before the photography or from altimeter readings during the flight. The precision of the tilt determinations is not dependent upon a close approximation in the value selected.

The data for this problem were taken from those computed by the Army Engineers at Fort Belvoir for a series of synthetic photographs established for test purposes. These data were chosen here in order to provide a check upon the accuracy of the results obtained. The test conditions imposed by this problem are rather severe, in that there are large differences in elevation between the control points, and in that the images have not been selected at the most advantageously critical positions at the extreme margins of the area of overlap of the two photographs.

Fundamental Principles Employed

The explanation of the method will be much clearer after a reading of numbers 15 and 18 in the series of bulletins published by the Department of Photogrammetry at Syracuse University. Since the names of all members of the American Society of Photogrammetry are on the mailing list at Syracuse, any repetition here of material contained in those bulletins would seem inadvisable. However, for the benefit of any readers who may not have the above mentioned publications available, or for any readers who may wish to use the procedure described herein without following out in detail all of the basic theory, there are first stated here without proof the principles involved in this method.

First, if an aerial photograph is perfectly vertical or absolutely free from tilt, the simple formulas

$$X = \frac{H - h}{f} x, \qquad Y = \frac{H - h}{f} y \tag{1}$$

give correct relative coordinates of ground points. In these formulas, x and y are the plane rectangular coordinates of the image of some point measured on the photograph, h is the elevation of this point on the ground, H is the exposure station altitude, and f is the focal length of the aerial camera. These relative coordinates of the ground points are obviously free from errors caused by the topographic relief displacements of images, and horizontal distances between pairs of ground points computed from these coordinates are correct.

Second, the simplest way to arrive at correct relationships between image coordinates on a tilted photograph and relative coordinates of the corresponding ground points, is explained in the following statements. (a) If an aerial photograph is so tilted that the photographic coordinates of the nadir point are x_v and y_v, a table of direction-cosine elements of orientation of the photograph can be quickly prepared by means of the following formulas:

	m	n	k	
X	$\cos mX = L_{v1}/L_v$	$\cos nX = -x_v y_v/L_v \cdot L_{v1}$	$\cos kX = f x_v/L_v \cdot L_{v1}$	
Y	$\cos mY = 0$	$\cos nY = f/L_{v1}$	$\cos kY = y_v/L_{v1}$	(2)
Z	$\cos mZ = -x_v/L_v$	$\cos nZ = -y_v/L_v$	$\cos kZ = f/L_v$	

where $L_{v1} = \sqrt{f^2 + y_v^2}$ and $L_v = \sqrt{x_v^2 + y_v^2 + f^2}$. (b) A table of *direction-numbers* for lines from photographic images to the emergent node of the camera lens, can be made up for all of the points to be used on a photograph, merely by tabulating the measured image coordinates in the following manner:

(Direction-numbers)

	m	n	k	
aL	$-x_a$	$-y_a$	$+f$	
bL	$-x_b$	$-y_b$	$+f$	(3)
			etc.	

(c) Then a table of direction-numbers for the lines in space from each of the ground points to the exposure station of the photograph, in the form

(Direction-numbers)

	M	N	K	
AL				(4)
BL				
			etc.,	

can be prepared very easily by means of a calculating machine, by substituting for the direction-cosines in the following formulas the quantities in table (2) and the direction-numbers in table (3):

$$\begin{aligned}\cos MAL &= \cos maL \cos mX + \cos naL \cos nX + \cos kaL \cos kX \\ \cos NAL &= \cos maL \cos mY + \cos naL \cos nY + \cos kaL \cos kY \\ \cos KAL &= \cos maL \cos mZ + \cos naL \cos nZ + \cos kaL \cos kZ\end{aligned} \quad (5)$$

etc.

(d) Finally correct relative coordinates of each ground point are given by the simple relations

$$X = (-M/K)(H - h), \qquad Y = (-N/K)(H - h) \quad (6)$$

and horizontal lengths of lines joining pairs of ground points, computed from these relative coordinates are rigorously correct, regardless of either topographic relief displacements or images or any distortion in the photograph caused by the tilt.

The latter group of simple basic principles of photogrammetry serve to facilitate greatly any analytical work with tilted photographs. Perhaps many readers will appreciate these easy manipulations, which eliminate calculating any so-called tilt displacements of images, and eliminate any transformations of pho o-graphic coordinates to nadir points or principal lines.

These foregoing principles suffice for the complete solution of the problem in hand by means of the operations now to be described step by step.

Solution of the Problem

1. At first a temporary assumption is made that the first photograph was absolutely vertical or free from tilt. By the use of the approximate value 20,100 feet for H and the control elevations of the four points A, B, C, and D given in the problem, $H-h$ in feet and $(H-h)/f$ in feet per millimeter are found for each of the four points. Formulas (1) then give relative coordinates for the four ground points, from which corresponding horizontal lengths are computed for the lines AB, BC, CA, BD, and DA by the usual calculating machine method, using

$$AB = \sqrt{(X_A - X_B)^2 + (Y_A - Y_B)^2}, \quad \text{etc.}$$

Values are then found for ratios between these distances, designated as $p=AB/BC$, $q=AB/CA$, $r=AB/BD$, and $s=AB/DA$.

	$H-h$	$(H-h)/f$	X	Y		$X-diff.$	$Y-diff.$	Length	Ratios
A	19,100	127.333	+ 474	+5,508	AB	16	9,991	9,991	
B	19,900	132.667	+ 490	−4,483	BC	10,199	10,073	14,335	$p=.69697$
C	19,700	131.333	+10,689	+5,590	CA	10,215	82	10,215	$q=.97807$
D	19,700	131.333	+10,557	−4,547	BD	10,067	64	10,067	$r=.99245$
					DA	10,083	10,055	14,240	$s=.70162$

2. Next, a temporary assumption is made that the first photograph was so tilted that the nadir point is situated on the photographic x-axis at the point $x_v=+2.00$ mm., $y_v=0.000$. In this case $L_{v1}=150.00$ and $L_v=150.013$; and the orientation table given by formula (2) above, when expressed as direction-numbers obtained by multiplying all direction-cosines by L_v, becomes simply

	m	n	k
X	+150	0	+ 2
Y	0	+150.013	0
Z	− 2	0	+150

Direction-numbers for aL, bL, cL, and dL, by formula (3), are tabulated directly from the measured image coordinates thus

	m	n	k
aL	$-$ 3.72	-43.26	$+150$
bL	$-$ 3.69	$+33.79$	$+150$
cL	-81.39	-42.56	$+150$
dL	-80.38	$+34.62$	$+150$

Then the table of direction-numbers for AL, BL, CL, and DL, indicated by (4) above is computed by formula (5) as follows:

	M	N	K	$-M/K$	$-N/K$
AL	$-$ 258.0	$-6,489.6$	$+22,507.4$	$+.011463$	$+.288332$
BL	$-$ 253.4	$+5,068.9$	$+22,507.4$	$+.011263$	$-.225210$
CL	$-11,908.5$	$-6,384.6$	$+22,662.8$	$+.525465$	$+.281722$
DL	$-11,757.0$	$+5,193.5$	$+22,660.8$	$+.511825$	$-.229184$

Relative ground coordinates for the four points A, B, C, and D are then computed by formula (6) above; and from these ground coordinates corresponding horizontal lengths are computed for AB, BC, CA, BD, and DA in the usual manner, and values are found for the ratios between the lengths of the same lines as before, this time designated as $p_1=AB/BC$, $q_1=AB/BA$, $r_1=AB/BD$, and $s_1=AB/DA$.

	X	Y		$X-diff.$	$Y-diff.$	*Length*	*Ratios*	
A	+ 219	+5,507	AB	5	9,989	9,989		
B	+ 224	−4,482	BC	10,128	10,032	14,255	$p_1=.70074$	$p_1-p=+.00377$
C	+10,352	+5,550	CA	10,133	43	10,133	$q_1=.98579$	$q_1-q=+.00772$
D	+10,221	−4,515	BD	9,997	33	9,997	$r_1=.99920$	$r_1-r=+.00675$
			DA	10,002	10,022	14,159	$s_1=.70549$	$s_1-s=+.00387$

3. Next, a temporary assumption is made that the first photograph was so tilted that the nadir point is situated on the photographic y-axis at the point $x_v=0.000$, $y_v=+2.00$ mm. In this case $L_v=L_{v_1}=150.013$; and the orientation table given by formula (2), when expressed as direction-numbers obtained by multiplying all direction-cosines by L_v, becomes simply

	m	n	k
X	$+150.013$	0	0
Y	0	$+150$	$+$ 2
Z	0	$-$ 2	$+150$

The same table of direction-numbers is used for aL, bL, cL, and dL as that used in the preceding paragraph. Another table of direction-numbers for AL, BL, CL, and DL is computed by formula (5) by the use of a calculating machine as before.

	M	N	K	$-M/K$	$-N/K$
AL	$-$ 558.0	$-6,189.0$	$+22,586.5$	$+.024705$	$+.274013$
BL	$-$ 553.5	$+5,368.5$	$+22,432.4$	$+.024674$	$-.239319$
CL	$-12,209.6$	$-6,084.0$	$+22,585.1$	$+.540604$	$+.269381$
DL	$-12,058.0$	$+5,493.0$	$+22,430.8$	$+.537564$	$-.244886$

Relative ground coordinates for the four points A, B, C, and D are again computed by formula (6), corresponding horizontal lengths are again computed for AB, BC, CA, BD, and DA, and values are once more found for the ratios between these computed distances, this time called $p_2=AB/BC$, $q_2=AB/CA$, $r_2=AB/BD$, and $s_2=AB/DA$.

	X	Y		$X-diff.$	$Y-diff.$	*Length*	*Ratios*	
A	+ 472	+5,234	AB	19	9,996	9,996		
B	+ 491	−4,762	BC	10,159	10,069	14,303	$p_2=.69887$	$p_2-p=+.00190$
C	+10,650	+5,307	CA	10,178	73	10,178	$q_2=.98212$	$q_2-q=+.00405$
D	+10,590	−4,824	BD	10,099	62	10,099	$r_2=.98980$	$r_2-r=-.00255$
			DA	10,118	10,058	14,267	$s_2=.70064$	$s_2-s=-.00098$

4. All operations described for the first photograph in the preceding three paragraphs are now followed out for the second photograph in exactly the same manner. Primes are used on the symbols for the ratios between the lengths of ground lines computed from measurements on this second photograph.

	$(H-h)/f$	X	Y		$X-diff.$	$Y-diff.$	Length	Ratios
A	127.333	−10,126	+5,564	AB	139	10,039	10,040	
B	132.667	− 9,987	−4,475	BC	9,987	10,054	14,171	$p'=$ .70849
C	131.333	0	+5,579	CA	10,126	15	10,126	$q'=$ .99151
D	131.333	0	−4,480	BD	9,987	5	9,987	$r'=$1.00531
				DA	10,126	10,044	14,262	$s'=$ .70397

	m	n	k		m	n	k
X	+150	0	+ 2	aL	+79.52	−43.70	+150
Y	0	+150.213	0	bL	+75.28	+33.73	+150
Z	− 2	0	+150	cL	0.00	−42.48	+150
				dL	0.00	+34.11	+150

	M	N	K	$-M/K$	$-N/K$
AL	+12,228.0	−6,555.6	+22,341.0	−.547334	+.293434
BL	+11,592.0	+5,050.0	+22,349.4	−.518672	−.226404
CL	+ 300.0	−6,372.6	+22,500.0	−.013333	+.283227
DL	+ 300.0	+5,116.9	+22,500.0	−.013333	−.227418

	X	Y		$X-diff.$	$Y-diff.$	Length	Ratios	
A	−10,454	+5,605	AB	132	10,110	10,111		
B	−10,322	−4,505	BC	10,059	10,085	14,244	$p_1'=$ .70984	$p_1'-p'=+.00135$
C	− 263	+5,580	CA	10,191	25	10,191	$q_1'=$ .99215	$q_1'-q'=+.00064$
D	− 263	−4,480	BD	10,059	25	10,059	$r_1'=$1.00517	$r_1'-r'=-.00014$
			DA	10,191	10,085	14,337	$s_1'=$ .70524	$s_1'-s'=+.00127$

	m	n	k		M	N	K	$-M/K$	$-N/K$
X	+150.013	0	0	AL	+11,929.0	−6,255.0	+22,587.4	−.528126	+.276924
Y	0	+150	+ 2	BL	+11,293.0	+5,359.5	+22,432.5	−.503421	−.238917
Z	0	− 2	+150	CL	0	−6,072.0	+22,585.0	0	+.268851
				DL	0	+5,416.5	+22,431.8	0	−.241465

	X	Y		$X-diff.$	$Y-diff.$	Length	Ratios	
A	−10,087	+5,289	AB	69	10,043	10,043		
B	−10,018	−4,754	BC	10,018	10,050	14,190	$p_2'=$ .70775	$p_2'-p'=-.00074$
C	0	+5,296	CA	10,087	7	10,087	$q_2'=$ .99564	$q_2'-q'=+.00413$
D	0	−4,757	BD	10,018	3	10,018	$r_2'=$1.00250	$r_2'-r'=-.00281$
			DA	10,087	10,046	14,236	$s_2'=$ .70547	$s_2'-s'=+.00150$

5. Obviously $\frac{1}{2}(p_1-p)$, $\frac{1}{2}(q_1-q)$, $\frac{1}{2}(r_1-r)$, and $\frac{1}{2}(s_1-s)$ are approximately equal to the changes produced in the computed ratios AB/BC, AB/CA, AB/BD, and AB/DA, respectively, by tilting the first photograph of the pair in such a manner as to cause the nadir point to move one millimeter in the positive direction along the x-axis of the photograph; and similarly $\frac{1}{2}(p_2-p)$, $\frac{1}{2}(q_2-q)$, $\frac{1}{2}(r_2-r)$, and $\frac{1}{2}(s_2-s)$ are approximately equal respectively to the changes produced in these same computed ratios by tilting the first photograph in such a manner as to cause the nadir point to move one millimeter in the positive direction along the y-axis of the photograph. Therefore if the first photograph is so tilted that the nadir point moves to a position designated by the photographic coordinates (x_v, y_v), these four computed ratios which are actually functions of x_v and y_v, will assume values approximately expressed in terms of x_v and y_v as

$$p + \tfrac{1}{2}(p_1 - p)x_v + \tfrac{1}{2}(p_2 - p)y_v, \quad \text{etc.}$$

Similarly, if the second photograph is so tilted that its nadir point moves to a position designated by (x_v', y_v'), the four ratios as computed from the second photograph will assume values approximately expressed in terms of x_v' and y_v' as

$$p' + \tfrac{1}{2}(p_1' - p')x_v' + \tfrac{1}{2}(p_2' - p')y_v', \quad \text{etc.}$$

But of course when both photographs assume the tilts which actually existed at the instant of exposures, the geometric figures assumed by the horizontal projections of the ground quadrilateral *ABCD*, as computed from the two photographs, must be exactly similar, and the values of the ratios between corresponding pairs of horizontal distances become exactly equal. Therefore four equations, which might be expressed in conventional symbols in the forms

$$p + \frac{\partial p}{\partial x}\Delta x + \frac{\partial p}{\partial y}\Delta y = p' + \frac{\partial p'}{\partial x'}\Delta x' + \frac{\partial p'}{\partial y'}\Delta y', \quad \text{etc.},$$

may be written with approximate values for the partial differential coefficients taken from the foregoing computations, as follows:

$$69{,}697 + \tfrac{1}{2}(+\ 377)x_v + \tfrac{1}{2}(+\ 190)y_v = 70{,}849 + \tfrac{1}{2}(+\ 135)x_v' + \tfrac{1}{2}(-\ 74)y_v'$$
$$97{,}807 + \tfrac{1}{2}(+\ 772)x_v + \tfrac{1}{2}(+\ 405)y_v = 99{,}151 + \tfrac{1}{2}(+\ 64)x_v' + \tfrac{1}{2}(+\ 413)y_v'$$
$$99{,}245 + \tfrac{1}{2}(+\ 675)x_v + \tfrac{1}{2}(-\ 265)y_v = 100{,}531 + \tfrac{1}{2}(-\ 14)x_v' + \tfrac{1}{2}(-\ 281)y_v'$$
$$70{,}162 + \tfrac{1}{2}(+\ 387)x_v + \tfrac{1}{2}(-\ 98)y_v = 70{,}397 + \tfrac{1}{2}(+\ 127(x_v' + \tfrac{1}{2}(+\ 150)y_v'$$

or

$$+\ 188.5x_v +\ 95.0y_v -\ 67.5x_v' +\ 37.0y_v' = +\ 1152$$
$$+\ 386.0x_v +\ 202.5y_v -\ 32.0x_v' -\ 206.5y_v' = +\ 1344$$
$$+\ 337.5x_v -\ 132.5y_v +\ 7.0x_v' +\ 140.5y_v' = +\ 1286$$
$$+\ 193.5x_v -\ 49.0y_v -\ 63.5x_v' -\ 75.0y_v' = +\ 235$$

The solution of these four simple linear equations gives the following results:

$$x_v = +\ 3.640 \text{ mm.} \qquad x_v' = +\ 0.187 \text{ mm.}$$
$$y_v = +\ 3.570 \qquad y_v' = +\ 3.767$$

Therefore

$$\tan t = \frac{\sqrt{3.640^2 + 3.570^2}}{150} = .03399 \qquad \tan t' = \frac{\sqrt{0.187^2 + 3.767^2}}{150} = .02514$$
$$t = 1^\circ\ 57' \qquad t' = 1^\circ\ 26'$$
$$\tan s = 3.640/3.570 = 1.01961 \qquad \tan s' = 0.187/3.767 = .04964$$
$$s = 45^\circ\ 33' \qquad s' = 2^\circ\ 50'$$

Checking the Computation

The foregoing solution of this problem of determining the rectification data for the two photographs should now be checked. A brief statement of the actual steps in the verification is essential.

6. With the values just computed for the photographic coordinates x_v and y_v of the nadir point of the first photograph, formula (2) is used to set up a new table of direction-cosine orientation elements for this photograph.

Then the same table of direction-numbers for the lines *aL*, *bL*, *cL*, and *dL* from the respective photographic images a, b, c, and d to the emergent node of the camera lens, as that originally set up by formula (3) in operation 2 and used again in operation 3, is used once more in connection with the new table of direction-cosine orientation elements, to compute with a calculating machine by means of formula (5) a revised table of direction-numbers for the lines AL, BL, CL, and DL joining the respective ground points to the exposure station, as before.

Then with the same values as before of $(H-h)$ for each of the four points, formula (6) is again used to revise the relative ground coordinates of the four points. Horizontal distances AB, BC, CA, and DA are again computed from these revised coordinates, and values are again found for the ratios $p=AB/BC$, $q=AB/CA$, $r=AB/BD$, and $s=AB/DA$.

		m	n	k
$X_v=+$	3.640	X +.99970	−.00058	+.02425
$y_v=+$	3.570	Y 0	+.99972	+.02379
$L_{v1}=$	150.042	Z −.02425	−.02379	+.99942
$L_v=$	150.087			

	M	N	K	−M/K	−N/K
AL	− 0.056	−39.679	+151.032	+.000373	+.262721
BL	− 0.071	+37.349	+149.199	+.000476	−.250331
CL	−77.703	−38.980	+152.899	+.508200	+.254936
DL	−76.738	+38.179	+151.039	+.508072	−.252775

	X	Y		X−diff.	Y−diff.	Length	Ratios
A	+ 7	+5,018	AB	2	10,000	10,000	
B	+ 9	−4,982	BC	10,003	10,004	14,147	$p=$ .70686
C	+10,012	+5,022	CA	10,005	4	10,005	$q=$ .99950
D	+10,009	−4,980	BD	10,000	2	10,000	$r=$1.00000
			AD	10,002	9,998	14,142	$s=$ .70711

7. The same procedure as that described in the preceding paragraph for the first photograph, is carried out for the second, starting with the values x_v' and y_v' already found in the foregoing tilt computations for the photographic coordinates of the nadir point of the second photograph, and concluding with another set of values for the ratios p', q', r', and s' between the calculated horizontal lengths of the ground lines AB, BC, CA, BD, and DA.

		m	n	K
$x_v=+$	0.187	X +1.00000	−.00003	+.00125
$y_v=+$	3.767	Y 0	+.99969	+.02511
$L_{v1}=$	150.047	Z − .00125	−.02511	+.99969
$L_v=$	150.047			

	M	N	K	−M/K	−N/K
AL	+79.709	−39.920	+150.951	−.528043	+.264456
BL	+75.466	+37.486	+149.012	−.506444	−.251563
CL	+ 0.189	−38.700	+151.020	−.001250	+.256259
DL	+ 0.186	+37.866	+149.097	−.001251	−.253968

	X	Y		X−diff.	Y−diff.	Length	Ratios
A	−10,086	+5,051	AB	8	10,057	10,057	
B	−10,078	−5,006	BC	10,053	10,054	14,218	$p'=$ .70734
C	− 25	+5,048	CA	10,061	3	10,061	$q'=$ .99960
D	− 25	−5,003	BD	10,053	3	10,053	$r'=$1.00040
			DA	10,061	10,054	14,223	$s'=$ .70709

8. Identity between p, q, r, and s and p', q', r', and s', respectively, would positively establish the correctness of the tilt and swing computations for the two photographs.

In case of discrepancies remaining between these sets of ratios, a second solution of the problem is required. This is easily accomplished, however, merely by inserting the remaining discrepancies between the respective ratios found in operations 6 and 7 above, as new constant terms in the equations of operation 5 of the solution without re-computing the coefficients of x_v, y_v, x_v', and y_v', and then repeating the computations only from this point. The second solution then gives increments in x_v, y_v, x_v', and y_v' which are to be added algebraicly to the first values obtained.

As a matter of fact, small discrepancies will be noticed between the respective values of the four ratios in the specimen problem solved here. See operations 6 and 7 above. In this case these are too small to indicate any appreciable errors in the results of the computations. However, a second solution of the problem in hand is shown below for illustrative purposes and as a matter of interest.

Second Solution

$$+188.5\Delta x_v + 95.0\Delta y_v - 67.5\Delta x_v' + 37.0\Delta y_v' = 70{,}734 - 70{,}686 = +48$$
$$+386.0\Delta x_v + 202.5\Delta y_v - 32.0\Delta x_v' - 206.5\Delta y_v' = 99{,}960 - 99{,}950 = +10$$
$$+337.5\Delta x_v - 132.5\Delta y_v + 7.0\Delta x_v' + 140.5\Delta y_v' = 100{,}040 - 100{,}000 = +40$$
$$+193.5\Delta x_v - 49.0\Delta y_v - 63.5\Delta x_v' - 75.0\Delta y_v' = 70{,}709 - 70{,}711 = -2$$

$\Delta x_v = +.073 \quad \Delta y_v = +.146 \qquad \Delta x_v' = -.162 \quad \Delta y_v' = +.256$

$x_v = +3.640 + .073 = +3.713 \qquad x_v' = +.187 - .162 = +0.025$

$y_v = +3.570 + .146 = +3.716 \qquad y_v' = +3.767 + .256 = +4.023$

$$\tan t = \frac{\sqrt{3.713^2 + 3.716^2}}{150} = .03502 \qquad \tan t' = \frac{\sqrt{0.025^2 + 4.023^2}}{150} = .02682$$

$t = 2°\ 00.3' \qquad t' = 1°\ 32.2'$

$\tan s = 3.713/3.716 = .99919 \qquad \tan s' = 0.025/4.023 = .00621$

$s = 44°\ 59' \qquad s' = 0°\ 21'$

Final Verification

First Photograph

$L_{v1} = 150.046 \quad L_v = 150.092$

	m	n	k		M	N	K	−M/K	−N/K
				AL	+0.017	−39.531	+151.072	−.000113	+.261670
X	+.99969	−.00061	−.02473	BL	0.000	+37.495	+149.163	.000000	−.251369
Y	0	+.99969	+.02477	CL	−77.629	−38.831	+152.976	+.507459	+.253837
Z	−.02474	−.02476	+.99939	DL	76.667	+38.325	+151.040	+.507594	−.253741

	X	Y		X−diff.	Y−diff.	Length	Ratios
A	− 2	+4,998	AB	2	10,000	10,000	
B	0	−5,002	BC	9,997	10,003	14,142	p = .70711
C	+ 9,997	+5,001	CA	9,999	3	9,999	q = 1.00010
D	+10,000	−4,999	BD	10,000	3	10,000	r = 1.00000
			DA	10,002	9,997	14,141	s = .70716

Second Photograph

$L_{v1} = 150.054 \quad L_v = 150.054$

	m	n	k		M	N	K	−M/K	−N/K
				AL	+79.545	−39.663	+151.104	−.526426	+.262488
X	+1.00000	−.00000	+.00017	BL	+75.305	+37.739	+149.029	−.505304	−.253233
Y	0	+.99964	+.02681	CL	+0.025	−38.443	+151.085	−.000165	+.254446
Z	− .00017	−.02681	+.99964	DL	+0.025	+38.119	+149.032	−.000168	−.255777

	X	Y		X−diff.	Y−diff.	Length	Ratios
A	−10,055	+5,014	AB	1	10,053	10,053	
B	−10,056	−5,039	BC	10,053	10,052	14,216	$p' = .70716$
C	− 3	+5,013	CA	10,052	1	10,052	$q' = 1.00010$
D	− 3	−5,039	BD	10,053	0	10,053	$r' = 1.00000$
			DA	10,052	10,053	14,216	$s' = .70716$

Tabulation of Results

Computed from Vertical Control		*Theoretically Correct Answers*	
	First Photograph		
$t = 2°00'$	$x_v = +3.71$ mm.	$t = 2°00'$	$x_v = +3.70$ mm.
$s = 44°59'$	$y_v = +3.72$ mm.	$s = 45°00'$	$y_v = +3.70$ mm.
	Second Photograph		
$t' = 1°32'$	$x_v' = +0.02$ mm.	$t' = 1°30'$	$x_v' = 0.00$ mm.
$s' = 0°21'$	$y_v' = +4.02$ mm.	$s' = 0°00'$	$y_v' = +3.93$ mm.

Rectification Data for a Strip of Photographs

If the rectification data are to be computed from vertical control for all the photographs in a strip or for all the photographs in a complete project, either one of two procedures may be followed.

After the computations for the initial pair of photographs have been completed, the work can proceed directly to the third and fourth photographs, then to the fifth and sixth, and so on, the computations being carried out independently for each pair in exactly the same manner as before, with the control consisting of the elevations of four new points appearing in the area of overlap of each pair of photographs. The control data requirements for this procedure then amount to the field determinations of two elevations per photograph.

An alternate method is to advance with the computations for only one photograph at a time after the first pair. This procedure requires for control one additional elevation to advance to the third photograph, two additional ones to advance to the fourth, one again to the fifth, and so on, or an average of but one and a half elevations per photograph to be determined in the field. Each single computation actually requires three known elevations within the overlap of the preceding photograph and the new one whose tilt and swing are being determined; but if the control points whose elevations are supplied, are always selected near the margins so that they appear in three photographs, this demand is always met by the amount of control data just specified.

The final table of direction-cosine orientation elements for the second photograph, as computed by means of formula (2) in the final verification of the foregoing calculations, is used again in the preparatory work before proceeding to the third photograph. A table of direction-numbers for the lines cL, dL, and eL to the emergent node of the camera lens from each of the images c, d, and e of the three control points common to the second and third photographs, is set up by formula (3) merely by tabulating the image coordinates measured on the second photograph. The three ground points whose elevations constitute the control data, are called C, D, and E; the C and D represent two of the points previously used in the computations for the first pair of photographs, but situated sufficiently close to the forward margin of the overlap of those two photographs for their images to appear in the third photographs, and E is the additional point common to the second and third photographs. The table of direction numbers for the lines CL, DL, and EL in space are computed by formula (5), relative ground coordinates for C, D, and E by formula (6), horizontal lengths for the lines CD, DE, and EC from these coordinates, and two ratios $p = CD/DE$ and $q = CD/EC$ from these computed lengths.

Then by following out for the third photograph operations exactly analogous to those previously described in paragraphs 1, 2, and 3, values are found for the corresponding ratios p', q', p_1', q_1', p_2', and q_2' between lengths for these three lines computed from the third photograph under the same tilt assumptions as before.

Then two linear equations are set up as follows:

$$p = p' + \tfrac{1}{2}(p_1' - p')x_v + \tfrac{1}{2}(p_2' - p')y_v$$

$$q = q' + \tfrac{1}{2}(q_1' - q')x_v + \tfrac{1}{2}(q_2' - q')y_v$$

wherein x_v and y_v this time represent the coordinates of the nadir point on the third photograph.

These equations are solved for the values of x_v and y_v, and the tilt and swing of the third photograph are then found in the usual manner. The computation is then verified in the same manner as before, but much more easily than before because this time only three control points and two length ratios are employed instead of four points and four ratios.

As a matter of fact the computations are not difficult in either one of these two procedures for computing rectification data through strips of aerial photographs by means of vertical control.

In conclusion to this discussion, two statements are made for the benefit of any readers interested in the analytical problem of progressing with determinations of rectification data through photographic strips, without resorting to any control data whatever beyond the initial four elevations, except for checks. First, in any progressive computations, elevations must be carried forward as well as the tilts themselves, for either absolute or even relative tilts determined otherwise are apparently of little value. Second, a reasonably easy method has been devised for extending the determinations of both the rectification data and the required elevations of ground points by photogrammetric methods. This will probably constitute the subject of a bulletin to be prepared at some later date by the Photogrammetry Department at Syracuse University.

Lieutenant Henry A. Harrington, Jr., 91st Reconnaissance Squadron, Howard Field, Canal Zone is the fifth member to become eligible for the Ford Bartlett Membership Award and will be presented a gold Society emblem by courtesy of Lockwood, Kessler & Bartlett, Inc. The effort of Lt. Harrington and others is resulting in a sizeable segment of new members from the AAF units stationed in the Canal Zone.