AN APPROACH TO EXTERIOR ORIENTATION

O. M. MILLER
American Geographical Society

INTRODUCTION

NOW that precision stereoscopic measuring instruments and rapid graphical methods are available for the controlling operations necessary in order to make maps from aerial photographs, the use of precise computational techniques for doing the same thing might seem both unwarranted and undesirable. One may, however, regard a precision stereoscopic instrument as a specialized computing machine which combines means of measuring and of computing towards a particular end. The modern unspecialized electrical computing machine, which is now normally used in surveying and mapping organizations, does not, of course, make measurements. But if properly fed with measurements, it is capable of doing even more precisely and certainly with less capital expenditure, some of the tasks for which its more specialized brother has been designed. Indeed, the computational approach, provided a non-specialized computing machine is available, is not so far removed from the instrumental approach as many are inclined to think. It is after all a question of degree.

Although one can be fairly sure that the fully instrumental approach is in general the most efficient, it is well to keep an open mind on the subject and to consider the possible operations in photogrammetry where simple linear measurements on the photograph combined with ordinary machine computation may be advantageous. An operation that comes immediately to mind is when photorectification is used as part of the mapping process and it is necessary as a preliminary to determine the tilt and swing. Another is as an aid in extending control by graphical radial line methods in mountainous country where it is essential to know accurately the positions of the nadir points on the photographs. Computational methods in general, however, assume their greatest practical importance as a means of testing instruments and as alternatives for checking purposes in cases of difficulty. Then, of course, as expressed by Professor Church[1] "Mathematical analysis . . . provides an effective means of teaching the fundamental theory of photogrammetry."

In this article the subject matter is limited to the special problem of determining the angular exterior orientation of a single photograph at the moment of exposure. There are two fundamental methods of attacking this problem. One is first to make the space resection—i.e., the determination of the position of the air station— and then determine the angular exterior orientation; the other is to obtain the angular orientation directly, regardless of the position of the air station.

Concerning the former, which may be called the classical approach, von Gruber[2], first and last an advocate of instrumentation, has this to say: "The calculation of resection in space, either by the direct or the differential method, is merely waste of time and is of minor practical importance." He then proceeds with typical Teutonic thoroughness to give references to twenty-eight papers on the subject. In spite of von Gruber's disparaging remarks many have continued

[1] Church, Earl. *Analytical Computations in Aerial Photogrammetry.* Manual of Photogrammetry, p. 536.

[2] Von Gruber, O. *Photogrammetry. Collected Letters and Essays.* Pp. 9–10. Translated from the German original by G. T. McCaw and F. A. Cazolet. American Photographic Publishing Co., Boston, Mass., 1932.

to investigate this approach, and in this country Professor Church[3] and others have fully justified their developments of the subject.

Professor Church has not ignored other methods of attacking the problem. One method[4] proposed by him, in which the position of the nadir point on the photograph is obtained before that of the air station in space, is really a compromise between the two approaches as it involves trial determinations of the horizontal position of the air station in order to locate the nadir point.

Of those methods that are completely independent of the position of the air station perhaps the best known is Mr. Ralph O. Anderson's[5] scale-point method. This uses differential scale relationships and semigraphic techniques, though it can be transformed so that the work is purely arithmetical after the original measurements have been made.

In 1941 Professor Marston Morse showed that there is an explicit method of determining the exterior orientation of a photograph with n control points, if n is greater than 3, in which area relationships are used, but the method is truly rigorous only when the control points all lie in a plane. The computations involved, however, are easy and straightforward.

Recently Technical Sergeant George C. Hagedorn of the Corps of Engineers,[6] in developing a method of tilt determination, has given a fundamental expression involving line directions; this expression is independent of the position of the air station. As he considers the general solution too lengthy and complex for use, he proceeds to develop from it a simplified formula, which, however, is practicable only when the tilt is small and the ground nearly flat.

Many others, including the present writer, have tried their hand at one time or another at developing practicable methods in respect to the second approach, but none of these have really succeeded in giving a general solution, that is to say a method consistently successful under all conditions and comparable to that afforded by the classical approach. Some methods are efficient for near vertical photography provided the ground imaged is nearly flat. Others are adapted for use with high obliques. Most require at least a first good approximation of the position of the air station as a preliminary and are therefore compromises between the two approaches. Even Anderson's scale-point method becomes more involved when the simplifying conditions of low relief and small tilt are not present. Perhaps the trouble has been that each worker has had his own particular problem to solve and hence the completely general approach has been neglected.

In this paper it is found convenient to consider the problem as essentially that of transforming one system of rectangular coordinates to another and a system of notation is utilized which by its very nature makes it possible quickly

[3] Church, Earl. *Analytical Computations in Aerial Photogrammetry.* Edward Brothers, Inc. Ann Arbor, Michigan, 1936. See also articles in PHOTOGRAMMETRIC ENGINEERING, Vol. XIII, No. 3, 1947, by G. T. McNeil and Everett L. Merritt.

[4] Church, Earl. *Two New Analytical Methods of Space Resection in Aerial Photogrammetry.* Bulletin No. 12. Syracuse University, Syracuse, N. Y. 1941.

[5] Anderson, R. O. *Applied Photogrammetry.* 2nd edition, 1937. Much has been written since on the scale point method. See for instance the article in PHOTOGRAMMETRIC ENGINEERING, Vol. XIII, No. 1, 1947, by P. H. Underwood. It is of interest also to note that P. de Vanssay De Blavous in "*The Use of Aviation in Surveying,*" Hydrographic Review, Vol. XII, No. 1, May, 1935, uses the graphically found "scale point" in a similar method of tilt determination. He, however, calls it the "characteristic point" of a line image and points out that it was first discussed by him in the Annales Hydrographique for 1917, pp. 72–76.

[6] Hagedorn, George C. *Tilt Determination by Comparison of Line Directions.* PHOTOGRAMMETRIC ENGINEERING, Vol. XI, No. 4, pp. 315–324, 1945.

and simply to develop in terms of coordinate differences the general perspective relationships that are independent of the position of the air station.

The remainder of the paper is devoted to practical methods of using the relationships in numerical computation. These are particularly adapted for machine working.

One explicit and general method is developed for computing the azimuth of the principal plane, the tilt and the swing. This method requires that six control points be imaged on the photograph and that the elevations of at least two of these points be not the same as the others. The method also makes possible a field determination of the principal distance. There are also given two iterative methods derived from the same fundamental formulas used in the explicit method, one for obliques and the other for near verticals. These both require not more than three control points, and their efficiency is not dependent in any way on the nature of the terrain imaged.

The present writer believes that these methods are new. At least they have been arrived at independently. He has not, however, had the opportunity to study von Gruber's references or recent European publications on the subject.

System of Notation

(See Figures 1 to 4)

The positive rather than the negative position of the photograph will always be considered in this paper. The symbols used are those recently proposed by the Nomenclature Committee of the American Society of Photogrammetry, except that A, instead of the Greek letter alpha, will be used to denote the azimuth of the principal plane. In addition, in order to describe in simple steps the whole perspective transformation between photograph and ground positions, additional systems of coordinates are introduced.

Point symbols

- G: a ground point
- g: the image of G on the photograph
- N: the nadir point of the perspective center in the datum plane
- n: the nadir point of the perspective center in the photo plane
- O: the perspective center
- p: the principal point

Length symbols, other than coordinates

- f: the principal distance
- H: the elevation of the perspective center (O) above the datum plane
- h: the elevation of a ground point (G) above the datum plane

Note: h here is not the true elevation above sea level. The latter must be corrected for curvature and refraction by a function of the distance from air station to ground point, but this distance need only be known roughly.

Angle symbols

- A or A_z: the horizontal azimuth of the principal plane measured clockwise from an arbitrarily chosen zero direction
- s or s_z: the swing angle in the plane of the photograph measured clockwise from an arbitrarily chosen zero direction to the trace of the principal plane
- t or t_z: the tilt of the photograph perpendicular from the vertical, zero tilt being vertically down.

The alternatives A_z, s_z, and t_z will be used only at one stage in order that these angles be differentiated from A_x, A_y, s_x, s_y, t_x and t_y, and these latter will be defined only in order to demonstrate the symmetrical nature of the perspective relationships revealed.

Cartesian coordinate systems

In all systems used, the perspective center O will be the origin. In general, six systems will be employed, two in respect to measurements on the photograph, two in respect to measurements of ground points, and two intermediate systems connecting the photograph and ground systems.

$(\bar{x}, \bar{y}, f)$ *First photo-coordinate system:* This system defines the position of an image g on a tilted photograph. The orientations of the $\bar{x}$ and $\bar{y}$ axes are arbitrary.

(x, y, f) *Second photo-coordinate system:* This system also defines the position of g on the photograph, but here the y axis lies in the principal plane. The $+y$ direction corresponds to the direction of the horizon from p.

(*Note the use of the overline to differentiate between the above two systems*)

(u, v, z): *First intermediate system.* This system defines the position of g as before. The v and z axes lie in the principal plane, but z is vertical and its positive direction is down.

$(\bar{u}, \bar{v}, f)$: *Second intermediate system:* This system defines the position of an image of G on a non-tilted or truly vertical photograph taken from the same perspective center as the tilted photograph with a camera having the same principal distance f. The f and $\bar{v}$ axes lie in the principal plane of the tilted photograph. f is vertical with its positive direction downwards.

(X, Y, Z) *First ground coordinate system:* This system defines the position of G. Here the Y and Z axes lie in the principal plane of the tilted photograph. Z is vertical with its positive direction downwards.

$(\overline{X}, \overline{Y}, Z)$ *Second ground coordinate system:* This system also defines the position of G and the Z axis is vertical as before, with its positive direction downwards, but here the orientation of the $\overline{X}$ and $\overline{Y}$ axes is arbitrary. Usually, but not necessarily, either the $+\overline{Y}$ axis is North or the $+\overline{X}$ axis lies in the general direction of the line of flight.

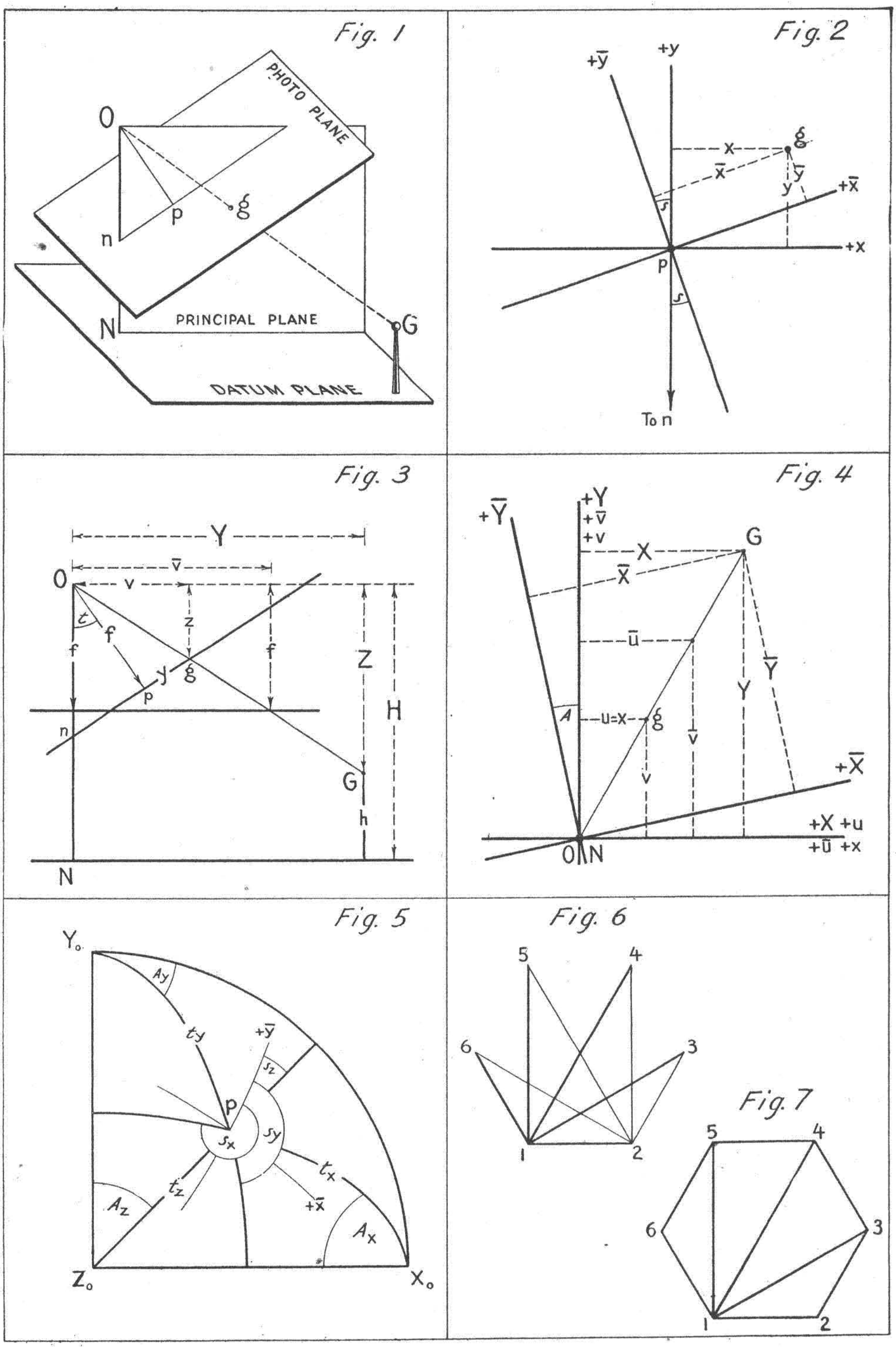
Observe that the two intermediate systems and the first ground coordinate system are in reality one system, as the angular orientation of the coordinate axes are the same. They have been differentiated from each other so that one refers to images in the oblique photo-plane only, another to corresponding image points in a horizontal photo-plane and the third to ground points.

Convention for subscripts

Numeral subscripts refer to individual G points, G_1, G_2, etc. For example, X_1 is the X coordinate of G_1 from O. X_{12} is the X coordinate of G_2 from G_1; in other words, $X_{12}=X_2-X_1$.

The Perspective Transformation in Single Steps

The complete transformation from the first photo-coordinate system to the second ground-coordinate system may be analyzed simply and clearly by stating the single-step transformations from one system of coordinates to the next. These will now be given without proof, as the proofs are either well known or else can be easily deduced from Figures 1, 2, 3, and 4.



FIGS. 2, 3 and 4 are orthogonal projections in the photo plane, principal plane and the datum plane respectively.

FIG. 5 is an orthogonal projection of an octant of a sphere centered at *O*, the point of tangency of the projection being at *Z*.

The swing transformation. Figure 2.

$$x = \bar{x}\cos s - \bar{y}\sin s, \qquad y = \bar{x}\sin s + \bar{y}\cos s$$

$$\bar{x} = y\sin s + x\cos s, \qquad \bar{y} = y\cos s - x\sin s.$$

The tilt transformation. Figures 1 and 3.

$$u = x, \qquad v = (f\sin t + y\cos t), \qquad z = (f\cos t - y\sin t)$$

$$x = u, \qquad y = (v\cos t - z\sin t), \qquad f = (v\sin t + z\cos t).$$

The tilt projection (by similar triangles, noting that $u/\bar{u} = v/\bar{v}$) Figures 1 and 3.

$$\bar{u} = fu/z, \qquad \bar{v} = fv/z$$

$$u = \bar{u}z/f, \qquad v = \bar{v}z/f.$$

The scale change. Figure 3.

$$X = Z\bar{u}/f, \qquad Y = Z\bar{v}/f$$

$$\bar{u} = Xf/Z, \qquad \bar{v} = Yf/Z.$$

At this point in order to get signs correct, note that

$$Z = H - h$$

$$Z_{12} = Z_2 - Z_1 = h_1 - h_2.$$

The azimuth transformation. Figure 4.

$$\overline{X} = [Y\sin A + X\cos A], \qquad \overline{Y} = [Y\cos A - X\sin A]$$

$$X = [\overline{X}\cos A - \overline{Y}\sin A], \qquad Y = [\overline{X}\sin A + \overline{Y}\cos A].$$

THE FUNDAMENTAL RELATIONSHIPS

Two differences of height formulas and a fundamental check

If X and Y are put through the scale change, tilt projection, and tilt transformation

$$X = Zx/(f\cos t - y\sin t)$$

$$Y = Z(f\sin t + y\cos t)/(f\cos t - y\sin t).$$

Solving in each case for Z

$$Z = X(f\cos t - y\sin t)/x \tag{1}$$

$$Z = Y(f\cos t - y\sin t)/(f\sin t + y\cos t). \tag{2}$$

Making the azimuth transformations for X and Y and the swing transformations for x and y

$$Z = \frac{(\overline{X}\cos A - \overline{Y}\sin A)(f\cos t - \bar{x}\sin s\sin t - \bar{y}\cos s\sin t)}{(\bar{x}\cos s - \bar{y}\sin s)} \tag{3}$$

$$Z = \frac{(\overline{X}\sin A + \overline{Y}\cos A)(f\cos t - \bar{x}\sin s\sin t - \bar{y}\cos s\sin t)}{(f\sin t + \bar{x}\sin s\cos t + \bar{y}\cos s\cos t)}. \tag{4}$$

Ratioing expressions (3) and (4) and setting up three such expressions, one for each of three control points, makes it obvious theoretically that solutions for A, s, and t can be found, provided the $\overline{X}$ and $\overline{Y}$ of each control point is known.

But this necessitates determining the horizontal position of the air station as a preliminary. As one of the principal purposes of this study is to avoid doing just this, a somewhat different approach is required. Nevertheless, expressions (1) and (2) are of importance here as they provide two independent means of determining the difference of height between the air station and a point on the ground. Furthermore, as they will only give the same result when the exterior orientation is correctly determined, a fundamental and independent check is provided for whatever procedure is actually used in determining A, s, and t. The simplest form of this check is obtained by ratioing (1) and (2) resulting in

$$X(f \sin t + y \cos t) - Yx = 0. \tag{5}$$

A General Equation for A, s, and t

For A, s, and t a general formula which is independent of the position of the air station is now developed by considering the perspective transformation of differences of coordinates between two ground points. Using the specified subscript notation and combining the azimuth transformation and the scale change

$$\begin{aligned}
\overline{X}_1 &= Z_1(\bar{v}_1 \sin A + \bar{u}_1 \cos A)/f \\
\overline{X}_2 &= Z_2(\bar{v}_2 \sin A + \bar{u}_2 \cos A)/f \\
\overline{X}_{12} &= \overline{X}_2 - \overline{X}_1 = [(Z_2\bar{v}_2 - Z_1\bar{v}_1) \sin A + (Z_2\bar{u}_2 - Z_1\bar{u}_1) \cos A]/f.
\end{aligned}$$

By substituting $(H-h_1)$ for Z_1 and $(H-h_2)$ for Z_2 and solving for H

$$H = \frac{\overline{X}_{12}f + (h_2\bar{v}_2 - h_1\bar{v}_1) \sin A + (h_2\bar{u}_2 - h_1\bar{u}_1) \cos A}{(\bar{v}_2 - \bar{v}_1) \sin A + (\bar{u}_2 - \bar{u}_1) \cos A}. \tag{6}$$

Similarly for Y_{12}

$$H = \frac{Y_{12}f + (h_2\bar{v}_2 - h_1\bar{v}_1) \cos A - (h_2\bar{u}_2 - h_1\bar{u}_1) \sin A}{(\bar{v}_2 - \bar{v}_1) \cos A - (\bar{u}_2 - \bar{u}_1) \sin A}. \tag{7}$$

By ratioing (6) and (7) and making the tilt projection and transformation and the swing transformation on the $\bar{u}$s and $\bar{v}$s there is eventually obtained

$$\begin{aligned}
&+ \overline{X}_{12}\bar{x}_{12}f(\cos A \sin s - \sin A \cos s \cos t) \\
&+ \overline{X}_{12}\bar{y}_{12}f(\cos A \cos s + \sin A \sin s \cos t) \\
&- \overline{Y}_{12}\bar{x}_{12}f(\sin A \sin s + \cos A \cos s \cos t) \\
&- \overline{Y}_{12}\bar{y}_{12}f(\sin A \cos s - \cos A \sin s \cos t) \\
&+ Z_{12}\bar{x}_{12}f \cos s \sin t - Z_{12}\bar{y}_{12}f \sin s \sin t \\
&- \overline{X}_{12}k_{12} \sin A \sin t - \overline{Y}_{12}k_{12} \cos A \sin t - Z_{12}k_{12} \cos t = 0
\end{aligned} \tag{8}$$

where

$$k_{12} = (\bar{x}_1\bar{y}_2 - \bar{x}_2\bar{y}_1).$$

The beautiful symmetry of this equation is not apparent until there are introduced the additional symbols $A_z, A_y, A_x, s_z, s_y, s_x, t_z, t_y$ and t_x. If there is assumed a sphere centered at O with radius f, as shown in Figure 5, X_0, Y_0, and Z_0 represent the directions of the $\overline{X}$, $\overline{Y}$ and Z coordinate axes respectively, forming a quadrantal triangle. p is the principal point through which pass great circles at right angles to one another representing the directions of the $\bar{y}$ and $\bar{x}$ axes on the photograph. The tilts of the principal distance f from the Z, $\overline{Y}$ and $\overline{X}$ directions are Z_0p, Y_0p and X_0p respectively or t_z, t_y, and t_x.

Reference to the figure now makes clear the meanings of A_z, A_y, A_x, s_z, s_y and s_x, and equation (8) can easily be transformed by spherical trigonometry to the symmetrical form

$$\begin{aligned} &\overline{X}_{12}f \sin t_x(\bar{x}_{12} \cos s_x - \bar{y}_{12} \sin s_x) \\ &\qquad + \overline{Y}_{12}f \sin t_y(\bar{x}_{12} \cos s_y - \bar{y}_{12} \sin s_y) \\ &\qquad + Z_{12}f \sin t_z(\bar{x}_{12} \cos s_z - \bar{y}_{12} \sin s_z) \\ &\quad = k_{12}(\overline{X}_{12} \cos t_x + \overline{Y}_{12} \cos t_y + Z_{12} \cos t_z). \end{aligned} \tag{9}$$

Here, to digress for a moment from the main theme, it should be pointed out that in the symbols recommended by the Nomenclature Committee of the American Society of Photogrammetry t_x and t_y are given definitions which make them the complements of these angles as shown in Figure 5. For the sake of consistency and in order to make the precise geometry of photogrammetry clearer it would seem to be very desirable to give the angles called x-tilt and y-tilt in the committee's report the symbols T_x and T_y.

The Explicit Method of Computation

For practical working purposes, a compromise between the forms of (8) and (9) is desirable, as the chief concern is the determination of A_z, S_z, and t_z. Dividing throughout by cos t_z and dropping the z subscript, we obtain the working formula

$$\begin{aligned} &+ \overline{X}_{12}\bar{x}_{12}f \cos s_x \sin t_x/\cos t \\ &- \overline{X}_{12}\bar{y}_{12}f \sin s_x \sin t_x/\cos t \\ &+ \overline{Y}_{12}\bar{x}_{12}f \cos s_y \sin t_y/\cos t \\ &- \overline{Y}_{12}\bar{y}_{12}f \sin s_y \sin t_y/\cos t \\ &+ Z_{12}\bar{x}_{12}f \cos s \tan t \\ &- Z_{12}\bar{y}_{12}f \sin s \tan t \\ &- \overline{X}_{12}k_{12} \sin A \tan t \\ &- \overline{Y}_{12}k_{12} \cos A \tan t \\ &\qquad = Z_{12}k_{12}. \end{aligned} \tag{10}$$

The problem now is to solve for (f cos s tan t), (f sin s tan t), (sin A tan t) and (cos A tan t).

For, when this is done

$$\tan s = (f \sin s \tan t)/(f \cos s \tan t) \tag{11}$$

$$\tan A = (\sin A \tan t)/(\cos A \tan t) \tag{12}$$

$$\tan t = (\sin A \tan t)/\sin A = (\cos A \tan t)/\cos A \tag{13}$$

$$(f \tan t) = (f \cos s \tan t)/\cos s = (f \sin s \tan t)/\sin s \tag{14}$$

$$f = (f \tan t)/\tan t. \tag{15}$$

Note also that $-(f \cos s \tan t)$ and $-(f \sin s \tan t)$ are equal to $\bar{y}_n$ and $\bar{x}_n$ respectively. In other words, they are the coordinates of n from p on the photograph.

Though theoretically three control points are all that are necessary to solve for A, S, and t from equation (10), an explicit general solution with only three points does not seem to be possible. An exhausting, if not exhaustive, literal analysis has been made of equation (10), therefore, and it has been found that if six control points are used, provided at least two of which have elevations differing considerably from the others, a comparatively simple linear solution is possible. Before proceeding further, however, it should be pointed out that equation (10) is not the only one that can be developed from equations (6) and (7). It has been chosen as it appears at present to lead to the simplest solutions. If (6) alone is considered, an equation can be developed in which the Y coordinate does not appear. Similarly it is possible to develop an equation from the fundamental check equation (5) for the solution of A, s, and t which is entirely independent of the elevations of the ground control points and the position of the air station. Full investigation of this last equation in respect to its practical numerical solution will, however, have to be deferred.

To return to the consideration of the numerical solution of equation (10), the literal analysis has demonstrated that, if five control lines radiating from a single point are used (the heavy lines in figure 6) and the first four unknowns in equation (10) are eliminated by any rigorous procedure, then the coefficients of ($\sin A \tan t$) and ($\cos A \tan t$) are reduced to zero.

Provided then, that this distribution of control lines is adhered to, the terms in equation (10) containing ($\sin A \tan t$) and ($\cos A \tan t$) can be ignored and the equation becomes

$$
\begin{aligned}
&+ \overline{X}_{12}\bar{x}_{12}(f \cos s_x \sin t_x)/\cos t - \overline{X}_{12}\bar{y}_{12}(f \sin s_x \sin t_x)/\cos t \\
&+ \overline{Y}_{12}\bar{x}_{12}(f \cos s_y \sin t_y)/\cos t - \overline{Y}_{12}\bar{y}_{12}(f \sin s_y \sin t_y)/\cos t \\
&+ Z_{12}\bar{x}_{12}(f \cos s \tan t) - Z_{12}\bar{y}_{12}(f \sin s \tan t) = Z_{12}k_{12}. \qquad (16)
\end{aligned}
$$

Thus the five radial line equations from any one point in a hexagon of control points may be reduced to a single equation in which ($f \cos s \tan t$) and ($f \sin s \tan t$) are the only unknowns. The analysis also shows that any two points of the hexagon used as centers will give two independent equations. Consequently, individual values of ($f \cos s \tan t$) and ($f \sin s \tan t$) may be easily determined from a six point control.

A different procedure is necessary to solve for ($\sin A \tan t$) and ($\cos A \tan t$). Here the analysis has shown that, if the three lines of a triangle of control are used to eliminate, for example, ($f \cos s_x \sin t_x)/\cos t$ and ($f \cos s_y \sin t_y)/\cos t$, the coefficient of ($f \cos s \tan t$) is reduced to zero. Thus the equations of four independent triangles in a hexagon of control (as in figure 7) can be used to obtain a single equation in which ($\cos A \tan t$) and ($\sin A \tan t$) are the only unknowns. A second independent equation in ($\cos A \tan t$) and ($\sin A \tan t$) can be obtained by eliminating ($f \sin s_x \sin t_x)/\cos t$, ($f \sin s_y \sin t_y)/\cos t$ and ($f \sin s \tan t$) in the triangular equations, thus making it possible to solve for ($\cos A \tan t$) and ($\sin A \tan t$), with six control points.

Finally the literal analysis has made it possible to devise a practical procedure for computing the coefficient of the unknowns in the two pairs of simultaneous equations directly from the original data. This is explained in the appendix.

Two Iterative Solutions of the Basic Formula

To solve equation (8) for A, s and t by iterative methods it is necessary to make preliminary estimates of their numerical values. The closer these estimates are to their true values, the shorter becomes the computation. Each of the two methods given here requires only three control points. The first is generally useful when the tilt is considerable, as in the case of high obliques. The second is suitable for near vertical photography.

The 1st iterative method

An orientation for the $\overline{X}$, $\overline{Y}$, and $\bar{x}\bar{y}$ coordinate systems is adopted such that, if primes indicate first estimates or approximations of the unknowns

$$\overline{X} = X', \quad \overline{Y} = Y', \quad \bar{x} = x' \quad \text{and} \quad \bar{y} = y'.$$

Also if the Δ prefix indicates the difference between the approximate and true values, then

$$A = A' + \Delta A, \quad s = s' + \Delta s, \quad \text{and} \quad t = t' + \Delta t.$$

Now, if W' is the numerical value of equation (8) when A', s', and t' have been used instead of the true values, then, by Taylor's theorem and with the powers of ΔA, Δs, and Δt above the first neglected,

$$\frac{\partial W'}{\partial A'}\Delta A + \frac{\partial W'}{\partial s'}\Delta s + \frac{\partial W'}{\partial t'}\Delta t = -W'. \tag{17}$$

If $W'=0$ the assumptions for A, s, and t have been correct. If it does not, then the procedure is as follows.

The partial differential coefficients in full from equation (8) without simplification are expressions almost as lengthy as that of the equation itself. But, as A' and s' have both been assumed as equal to zero, expression (17) without any simplifying departure from theory becomes

$$\begin{aligned}
&+ \text{arc } 1'[-\overline{X}_{12}k_{12} \sin t' - f\overline{X}_{12}\bar{x}_{12} \cos t' - f\overline{Y}_{12}\bar{y}_{12}]\Delta A \\
&+ \text{arc } 1'[-fZ_{12}\bar{y}_{12} \sin t' + f\overline{Y}_{12}\bar{y}_{12} \cos t' + f\overline{X}_{12}\bar{x}_{12}]\Delta s \\
&+ \text{arc } 1'[(f\overline{Y}_{12}\bar{x}_{12} + Z_{12}k_{12}) \sin t' + (fZ_{12}\bar{x}_{12} - \overline{Y}_{12}k_{12}) \cos t']\Delta t \\
= &-[(fZ_{12}\bar{x}_{12} - \overline{Y}_{12}k_{12}) \sin t' - (f\overline{Y}_{12}\bar{x}_{12} + Z_{12}k_{12}) \cos t' + f\overline{X}_{12}\bar{y}_{12}].
\end{aligned} \tag{18}$$

Arc $1'=0.0002909$ and is used as a constant multiplier of the coefficients merely in order that the computed values of ΔA, Δs, and Δt will be in minutes of arc.

Similar expressions to (18) for any two other lines of control are set up and three simultaneous equations obtained. Values of the unknowns are then computed. The precision of the result can now be checked by entering the derived values of A, s, and t in the full formula (8) for each line. If the resulting W'' values are not zero or sufficiently close to zero for practical purposes a closer approximation can be made by solving the equations (18) again, using the same coefficients but with the new W'' values. The resulting values of ΔA, Δs and Δt are increments to be added algebraically to those of the first approximation in order to get improved values of A, s and t from the original assumptions.

In the first approximation if any of the computed values of ΔA and Δs and Δt are large, say over two or three degrees, a quicker result in the long run will be obtained by measuring new coordinate values and recomputing the coefficients before making a second approximation. Finally, if more than 3 control points are available, equations (18) may be set up for each long line available and the final solution for A, s, and t made by the method of least squares. A much shorter alternative to the least square method which gives nearly as good results, if handled with discrimination, is to make a selection of the long control lines available, and then to form three groups of these such that the average lines representing each group are long and have widely varying azimuths from one another. Within each group equations (18) are developed for each line and are then treated as normal equations.

The 2nd iterative method

In the case of an oblique photograph, even when no apparent horizon line is visible on the photograph, there can generally be obtained estimates of A, s, and t which will be within a degree or two of the best values. Quite another situation exists in near vertical photography.

In these cases, though it is known that t is close to zero, there is little to provide even such a rough estimate of A and s.

However, by a suitable choice of a first system of coordinate axes on the photograph, s can be made to be very nearly equal to A. This is done by placing the photograph under a transparent plot of the ground control and orienting it so that image lines of object lines are on the average parallel to each other. Then on the photograph is traced a line through the principal point p parallel to the $+\overline{Y}$ direction on the ground plot and this line is considered as the zero or $+\bar{y}$ direction on the photograph. Obviously s, whatever actual value it may have, will be nearly equal to A.

Assuming this to be the case so that

$$A' = s' = \tfrac{1}{2}(A' + s'), \quad t' = 0 \quad \text{and} \quad \Delta t = t$$

we obtain a difference formula (19) from (8) in the same way that (18) was obtained in the 1st iterative method but with this result.

$$\begin{aligned} &- \text{arc } 1'(\overline{X}_{12}k_{12} + fZ_{12}\bar{y}_{12})t \sin \tfrac{1}{2}(A' + s') \\ &- \text{arc } 1'(\overline{Y}_{12}k_{12} - fZ_{12}\bar{y}_{12})t \cos \tfrac{1}{2}(A' + s') \\ &- \text{arc } 1'f(\overline{X}_{12}\bar{x}_{12} + \overline{Y}_{12}\bar{y}_{12})(A' - s') \\ &= -(\overline{X}_{12}\bar{y}_{12}f - \overline{Y}_{12}\bar{x}_{12}f - Z_{12}k_{12}). \end{aligned} \tag{19}$$

Similar expressions for two other lines are set up as before and values of

$$t \sin \tfrac{1}{2}(A + s), \quad t \cos \tfrac{1}{2}(A + s) \quad \text{and} \quad (A - s)$$

are obtained by solving three simultaneous equations. From them A, s and t are easily derived.

A second approximation can be made in the same manner as in the first iterative method, but here it should be particularly noted that the values of $t \sin\frac{1}{2}(A+s)$, $t \cos\frac{1}{2}(A+s)$ and $(A-s)$ obtained are not the total values but are increments to be applied to those of the first approximation before solving for A, s and t. Least square methods or alternatively grouping methods can also be applied as a final procedure if more than the minimum control is available.

Computing instructions with synthetic examples of each iterative method are given in the Appendix.

Conclusions with Reference to the Worked Out Examples

The merits of the six-point explicit method are: (1) no preconceived ideas concerning the orientation are necessary; (2) the method is independent of any knowledge of the principal distance, which in turn can be derived; and (3) no successive approximations are required. Its disadvantages are: (1) the computations, though well adapted to machine computing, are long; (2) no adequate check is provided for computing the N factors (see Appendix); and (3) at least two of the control points must have elevations in respect to the datum plane considerably different from those of the others. Also, seven significant figures are desirable throughout in developing the coefficients. This, however, is a minor disadvantage on a computing machine.

Enough experience with this method has not been accumulated to permit a positive assertion that the direct computation of the coefficients is superior to the usual numerical methods of eliminating unknowns in groups of simultaneous equations. It would appear, however, that the direct method is superior to the method of determinants, but it is doubtful whether it is superior to the Gaussian method of elimination.

In practice the method should be useful in analyzing or testing the orientation of photographs of mountainous country, especially low obliques, where preliminary estimates of tilt are difficult to make. It should prove especially valuable as a preliminary computation in cases where the common principal distance of a series of photographs is not known precisely—a condition which is likely to occur in wartime photography.

In the writer's opinion, the two iterative methods are generally to be preferred to the explicit method. In the vast majority of cases found in practice, it is possible to make the necessary preliminary estimates to within a degree or two, and when this is so, provided the initial data are reliable, a single approximation should suffice. Certainly not more than two will be necessary for practical purposes except on the rarest of occasions. In example 2 in the Appendix excessively large errors in the preliminary estimates of A, s and t have been purposely introduced. After the first approximation, the errors have been reduced to well within the 1° limit. The example has been continued through the second approximation with the original differential coefficients, in order to demonstrate the method, and a third approximation and perhaps even a fourth would probably be necessary to get a sufficiently close result. If, however, after the first approximation new coefficients had been computed, the second approximation would have sufficed. The third example, in which the errors of estimation are more realistic, has been carried beyond the first approximation, again to demonstrate the method. It is apparent, however, that for most purposes the first approximation has provided a sufficiently accurate result. Certainly errors in A and s of the order of $30'$ are insignificant when t is only $1°30'$. Considering the accuracy achieved, these two iterative methods are not too laborious. They are also self checking.

The computing forms used in the Appendix have been designed primarily for the purpose of demonstration. An experienced computer would undoubtedly wish to reorganize and condense them considerably. It should be noted that the two iterative methods are worked out on the same forms. This is possible because the two methods are based on similar logical developments from the same fundamental expression and are therefore similar to one another in form though not in detail.

In conclusion, grateful acknowledgment is due Mr. Everett L. Merritt, Topographic Engineer, U. S. Naval Photographic Interpretation Center for checking by independent analysis the validity of the methods and the computations and for several valuable suggestions concerning the presentation of this paper.

Appendix

Instructions for Obtaining the Exterior Orientation of a Photograph Independently of the Position of the Air Station

Instructions are given for one explicit and two iterative methods. The explicit method requires only a single solution of each of two pairs of simultaneous equations. The iterative methods each require the repeated solution of three simultaneous equations, but the coefficients of the unknowns remain unchanged throughout and third approximations are generally not necessary.

Furthermore the developments of the coefficients for the iterative methods, especially for the second method, are short in comparison with that of the explicit method.

Forms 1, 2, 3 and 4 apply to the explicit method and forms 5, 6 and 7 apply to both of the iterative methods. Form 7 can also be used to check the accuracy of the explicit computation.

Selection of Control Points

The images of the control points chosen must be well separated from one another. If less than six are available, one of the iterative methods must be employed. In any case, if the iterative methods are used, the best selection of control points will be that which gives three long control lines separated in azimuth from each other as nearly as possible by 60°.

In the case of a high oblique photograph when only three control points are available it is better that there be available one foreground point and two distant points rather than two foreground points and one distant point.

Corrections for Curvature and Refraction

From the flight index map or, if this is inadequate, from a rough horizontal resection of the air station by the tracing paper method, obtain approximate values of the horizontal distance of each selected control point from the air station.

In order to obtain h, the elevation of a control point from the datum *plane* of the air station, subtract E times the square of the horizontal distance from its elevation above sea level, where E is a number which varies slightly under different conditions. [A good average value for E when the distance is given in feet is $2/10^8$. The correction is obviously less than 2 feet for distances less than 10,000 feet.]

Example 1——Explicit Method

Assembling of Data (Form 1)

From the survey records with any common origin extract $\overline{X}$ and $\overline{Y}$ coordinates of the selected control points. Enter these with the h values, corrected for curvature and refraction, on form 1. With any convenient orientation of the $\bar{x}$ and $\bar{y}$ axis and with the principal point as origin, measure the $\bar{x}$ and $\bar{y}$ coordinates as precisely as means permit and enter these also.

Preliminary Computations

1st step. Obtain the coordinate differences $\overline{X}_{12}$, $\overline{Y}_{12}$, Z_{12}, etc. and the k factors and record these in table 1 of form 2.

2nd step. Obtain the Q and q factors of the form as shown in table 2 of form 2 and record these.

3rd step. Obtain the M factors of the form as shown on the top of form 3 and enter these in the places indicated.

4th step. Obtain the N factors of the form as shown on the top of form 4 and enter these in order in column 1 of form 4.

Before proceeding with the computation it is desirable to check the preliminary computations to safeguard against errors.

RIGOROUS SIX POINT DETERMINATION OF EXTERIOR ORIENTATION
— FORM I —

Example No. 1

DATA Note: The $\overline{X}$ and $\overline{Y}$ coordinates can be recorded as from any origin.

Point	$\overline{X}$	$\overline{Y}$	h	$\overline{x}$	$\overline{y}$
G_1	+10,384.7	+6,779.9	3000	+100	+100
G_2	+10,039.1	+1,121.1	0	+110	+ 10
G_3	+ 2,508.5	−3,252.5	1500	+ 60	−100
G_4	− 1,720.6	− 927.3	2500	− 40	− 90
G_5	− 3,291.2	+3,589.0	2000	− 90	− 10
G_6	− 852.8	+6,288.3	3500	− 65	+ 60

EQUATIONS FOR $f \cos s \tan t$ AND $f \sin s \tan t$

(G_1 as pivot) $+1.0164506\, f \cos s \tan t - 1.4276131\, f \sin s \tan t = +41.116246$

(G_2 as pivot) $+0.8680717\, f \cos s \tan t - 0.4632849\, f \sin s \tan t = +42.280620$

EQUATIONS FOR $\sin A \tan t$ AND $\cos A \tan t$

(G_1 as pivot) $+2.308621 \sin A \tan t + 3.825617 \cos A \tan t = 1.626001$

(G_1 as pivot) $+0.319738 \sin A \tan t + 1.292433 \cos A \tan t = 0.465634$

SOLUTIONS

$f \cos s \tan t = +53.7659429$	$\sin A \tan t = +0.1818530$	$t = 20° 00' 00.5''$
$f \sin s \tan t = +9.4802842$	$\cos A \tan t = +0.3152880$	$f = 149.9982$
$\tan s = +0.1763250$	$\tan A = 0.5767837$	In this synthetic example the known values are
$s = 9° 59' 59.6''$	$A = 29° 58' 33''$	$A = 30°$
$\sin s = +0.1736463$	$\sin A = +0.49963468$	$s = 10°$
$\cos s = +0.9848090$	$\cos A = +0.86623622$	$t = 20°$
Mean $f \tan t = 54.5953356$	Mean $\tan t = 0.3639732$	$f = 150$

Note that:

$$\overline{X}_{12} + \overline{X}_{23} + \overline{X}_{34} + \overline{X}_{45} + \overline{X}_{56} + \overline{X}_{16} = 0$$

$$\overline{X}_{13} + \overline{X}_{24} + \overline{X}_{35} + \overline{X}_{46} - \overline{X}_{15} - \overline{X}_{26} = 0$$

$$\overline{X}_{12} - \overline{X}_{14} + \overline{X}_{25} + \overline{X}_{34} - \overline{X}_{36} + \overline{X}_{56} = 0.$$

Similar checks can be made on the Y, Z, x and Y coordinate differences.

RIGOROUS SIX POINT DETERMINATION OF EXTERIOR ORIENTATION
—FORM 2—

Example Nº 1

TABLE 1 DIFFERENCES OF COORDINATES AND THE k FACTORS, e.g. $k_{12} = (\bar{x}_1\bar{y}_2 - \bar{x}_2\bar{y}_1)$

LINE	$\bar{X}$	$\bar{Y}$	Z	$\bar{x}$	$\bar{y}$	k
2-1 = 12	− 345.6	−5658.8	+3000	+ 10	− 90	−10000
3-1 = 13	−7876.2	−10032.4	+1500	− 40	−200	−16000
4-1 = 14	−12105.3	− 7707.3	+ 500	−140	−190	− 5000
5-1 = 15	−13675.9	− 3190.9	+1000	−190	−110	+ 8000
6-1 = 16	−11237.5	− 491.7	− 500	−165	− 40	+12500
3-2 = 23	− 7530.6	− 4373.7	−1500	− 50	−110	−11600
4-2 = 24	−11759.7	− 2048.4	−2500	−150	−100	− 9500
5-2 = 25	−13330.3	+ 2467.9	−2000	−200	− 20	− 200
6-2 = 26	−10891.9	+ 5167.2	−3500	−175	+ 50	+ 7250
4-3 = 34	−4229.1	+ 2325.2	− 1000	−100	+ 10	− 9400
5-3 = 35	− 5799.7	+6841.6	− 500	−150	+ 90	− 9600
6-3 = 36	− 3361.3	+9540.8	−2000	−125	+160	− 2900
5-4 = 45	− 1570.6	+4516.4	+ 500	− 50	+ 80	− 8500
6-4 = 46	+ 867.8	+ 7215.6	− 1000	− 25	+150	− 8250
6-5 = 56	+2438.4	+ 2699.2	− 1500	+ 25	+ 70	− 6050

TABLE 2 THE Q AND q FACTORS

$Q/10^8$	$q/10^4$
$(\bar{X}_{12}\bar{Y}_{13} - \bar{X}_{13}\bar{Y}_{12}) = Q_{123} = -0.4110271$	$(\bar{x}_{12}\bar{y}_{13} - \bar{x}_{13}\bar{y}_{12}) = q_{123} = -0.5600$
$(\bar{X}_{12}\bar{Y}_{14} - \bar{X}_{14}\bar{Y}_{12}) = Q_{124} = -0.6583794$	$(\bar{x}_{12}\bar{y}_{14} - \bar{x}_{14}\bar{y}_{12}) = q_{124} = -1.4500$
$(\bar{X}_{12}\bar{Y}_{15} - \bar{X}_{15}\bar{Y}_{12}) = Q_{125} = -0.7628657$	$(\bar{x}_{12}\bar{y}_{15} - \bar{x}_{15}\bar{y}_{12}) = q_{125} = -1.8200$
$(\bar{X}_{12}\bar{Y}_{16} - \bar{X}_{16}\bar{Y}_{12}) = Q_{126} = -0.6342126$	$(\bar{x}_{12}\bar{y}_{16} - \bar{x}_{16}\bar{y}_{12}) = q_{126} = -1.5250$
$(\bar{X}_{13}\bar{Y}_{14} - \bar{X}_{14}\bar{Y}_{13}) = Q_{134} = -0.6074212$	$(\bar{x}_{13}\bar{y}_{14} - \bar{x}_{14}\bar{y}_{13}) = q_{134} = -2.0400$
$(\bar{X}_{13}\bar{Y}_{15} - \bar{X}_{15}\bar{Y}_{13}) = Q_{135} = -1.1207092$	$(\bar{x}_{13}\bar{y}_{15} - \bar{x}_{15}\bar{y}_{13}) = q_{135} = -3.3600$
$(\bar{X}_{13}\bar{Y}_{16} - \bar{X}_{16}\bar{Y}_{13}) = Q_{136} = -1.0886795$	$(\bar{x}_{13}\bar{y}_{16} - \bar{x}_{16}\bar{y}_{13}) = q_{136} = -3.1400$
$(\bar{X}_{14}\bar{Y}_{15} - \bar{X}_{15}\bar{Y}_{14}) = Q_{145} = -0.6677691$	$(\bar{x}_{14}\bar{y}_{15} - \bar{x}_{15}\bar{y}_{14}) = q_{145} = -2.0700$
$(\bar{X}_{14}\bar{Y}_{16} - \bar{X}_{16}\bar{Y}_{14}) = Q_{146} = -0.8065893$	$(\bar{x}_{14}\bar{y}_{16} - \bar{x}_{16}\bar{y}_{14}) = q_{146} = -2.5750$
$(\bar{X}_{15}\bar{Y}_{16} - \bar{X}_{16}\bar{Y}_{15}) = Q_{156} = -0.2913392$	$(\bar{x}_{15}\bar{y}_{16} - \bar{x}_{16}\bar{y}_{15}) = q_{156} = -1.0550$
$(\bar{X}_{23}\bar{Y}_{24} - \bar{X}_{24}\bar{Y}_{23}) = Q_{234} = -0.3600688$	$(\bar{x}_{23}\bar{y}_{24} - \bar{x}_{24}\bar{y}_{23}) = q_{234} = -1.1500$
$(\bar{X}_{23}\bar{Y}_{25} - \bar{X}_{25}\bar{Y}_{23}) = Q_{235} = -0.7688707$	$(\bar{x}_{23}\bar{y}_{25} - \bar{x}_{25}\bar{y}_{23}) = q_{235} = -2.1000$
$(\bar{X}_{23}\bar{Y}_{26} - \bar{X}_{26}\bar{Y}_{23}) = Q_{236} = -0.8654939$	$(\bar{x}_{23}\bar{y}_{26} - \bar{x}_{26}\bar{y}_{23}) = q_{236} = -2.1750$
$(\bar{X}_{24}\bar{Y}_{25} - \bar{X}_{25}\bar{Y}_{24}) = Q_{245} = -0.5632829$	$(\bar{x}_{24}\bar{y}_{25} - \bar{x}_{25}\bar{y}_{24}) = q_{245} = -1.7000$
$(\bar{X}_{24}\bar{Y}_{26} - \bar{X}_{26}\bar{Y}_{24}) = Q_{246} = -0.8307562$	$(\bar{x}_{24}\bar{y}_{26} - \bar{x}_{26}\bar{y}_{24}) = q_{246} = -2.5000$
$(\bar{X}_{25}\bar{Y}_{26} - \bar{X}_{26}\bar{Y}_{25}) = Q_{256} = -0.4199923$	$(\bar{x}_{25}\bar{y}_{26} - \bar{x}_{26}\bar{y}_{25}) = q_{256} = -1.3500$

To check the Q, q, and k factors note that:

$$Q_{123} + Q_{134} = Q_{124} + Q_{234}$$

$$Q_{123} + Q_{135} = Q_{125} + Q_{235}$$

$$Q_{123} + Q_{136} = Q_{126} + Q_{236}$$

$$Q_{124} + Q_{145} = Q_{125} + Q_{245}$$

RIGOROUS SIX POINT DETERMINATION OF EXTERIOR ORIENTATION
-FORM 3-

Example №1

THE M FACTORS AND THE COEFFICIENTS OF $f \cos s \tan t$ AND $f \sin s \tan t$

POINT 1 AS PIVOT	POINT 2 AS PIVOT
$M_{12} = [q_{135} q_{146} Q_{134} Q_{156} - q_{134} q_{156} Q_{135} Q_{146}]$	$M_{23} = [q_{125} q_{246} Q_{126} Q_{245} - q_{126} q_{245} Q_{125} Q_{246}]$
$M_{13} = [q_{124} q_{156} Q_{125} Q_{146} - q_{125} q_{146} Q_{124} Q_{156}]$	$M_{24} = [q_{126} q_{235} Q_{125} Q_{236} - q_{125} q_{236} Q_{126} Q_{235}]$
$M_{14} = [q_{126} q_{135} Q_{123} Q_{156} - q_{123} q_{156} Q_{126} Q_{135}]$	$M_{25} = [q_{123} q_{246} Q_{126} Q_{234} - q_{126} q_{234} Q_{123} Q_{246}]$
$M_{15} = [q_{123} q_{146} Q_{126} Q_{134} - q_{126} q_{134} Q_{123} Q_{146}]$	$M_{26} = [q_{125} q_{234} Q_{123} Q_{245} - q_{123} q_{245} Q_{125} Q_{234}]$
$M_{16} = [q_{125} q_{134} Q_{124} Q_{135} - q_{124} q_{135} Q_{125} Q_{134}]$	$M_{21} = [q_{236} q_{245} Q_{235} Q_{246} - q_{235} q_{246} Q_{236} Q_{245}]$

Point 1 as pivot LINE	M	ZM	$ZM\bar{x}/10^5$	$ZM\bar{y}/10^5$	$ZMk/10^5$
12	-0.4143790	-1243.13790	-0.1243138	+1.1188241	+124.313790
13	+0.0423591	+ 63.53865	-0.0254155	-0.1276773	-10.166184
14	+0.1936686	+ 96.83430	-0.1355680	-0.1839852	- 4.841715
15	-0.4758822	-475.88220	+0.9041762	+0.5234704	-38.070576
16	+0.4819051	-240.95255	+0.3975717	+0.0963810	-30.119069
		SUMS →	+1.0164506	+1.4276131	+41.116246

Point 2 as pivot LINE	M	ZM	$ZM\bar{x}/10^5$	$ZM\bar{y}/10^5$	$ZMk/10^5$
23	-0.0175638	+ 26.34570	-0.0131729	-0.0289803	-3.056101
24	+0.1841952	-460.48800	+0.6907320	+0.4604880	+43.746360
25	-0.2791371	+558.27420	-1.1165484	-0.1116548	- 1.116548
26	+0.2230816	-780.78560	+1.3663748	-0.3903928	-56.606956
21	-0.1977129	-593.13870	-0.0593139	+0.5338248	+59.313870
		SUMS →	+0.8680717	+0.4632849	+42.280620

$$Q_{124} + Q_{146} = Q_{126} + Q_{246}$$

$$Q_{125} + Q_{156} = Q_{126} + Q_{256}.$$

Similar checks can be made on the q factors.

Note in addition that

$$q_{123} = k_{12} - k_{13} + k_{23}$$

$$q_{124} = k_{12} - k_{14} + k_{24} \quad \text{etc.}$$

RIGOROUS SIX POINT DETERMINATION OF EXTERIOR ORIENTATION
-FORM 4-

Example Nº 1

THE N FACTORS AND THE COEFFICIENTS OF $\sin A \tan t$ AND $\cos A \tan t$

$N_{123} = [Z_{14} Q_{156} + Z_{16} Q_{145} - Z_{15} Q_{146}]$	$N_{135} = [Z_{14} Q_{126} - Z_{12} Q_{146} - Z_{16} Q_{124}]$
$N_{124} = [Z_{15} Q_{136} - Z_{13} Q_{156} - Z_{16} Q_{135}]$	$N_{136} = -[Z_{12} Q_{145} + Z_{15} Q_{124} - Z_{14} Q_{125}]$
$N_{125} = [Z_{13} Q_{146} + Z_{16} Q_{134} - Z_{14} Q_{136}]$	$N_{145} = [Z_{12} Q_{136} + Z_{16} Q_{123} - Z_{13} Q_{126}]$
$N_{126} = [Z_{14} Q_{135} - Z_{13} Q_{145} - Z_{15} Q_{134}]$	$N_{146} = [Z_{13} Q_{125} - Z_{12} Q_{135} - Z_{15} Q_{123}]$
$N_{134} = [Z_{12} Q_{156} + Z_{16} Q_{125} - Z_{15} Q_{126}]$	$N_{156} = [Z_{12} Q_{134} + Z_{14} Q_{123} - Z_{13} Q_{124}]$

1	2	3	4	5	6
N_{abc}	$N_{abc}\bar{X}_{bc}/10^4$	$N_{abc}\bar{Y}_{bc}/10^4$	$N_{abc}Z_{bc}/10^4$	$\bar{x}_{ab}\bar{x}_{ac}/10^4$	$\bar{y}_{ab}\bar{y}_{ac}/10^4$
+0.9948043	-0.7491473	-0.4350976	-0.1492206	-0.0400	+1.8000
-1.2120253	+1.4253054	+0.2482713	+0.3030063	-0.1400	+1.7100
-0.3618336	+0.4823350	-0.0892969	+0.0723667	-0.1900	+0.9900
+1.0487203	-1.1422557	+0.5418947	-0.3670521	-0.1650	+0.3600
+0.1416279	-0.0598959	+0.0329313	-0.0141628	+0.5600	+3.8000
+1.7734719	-1.0285605	+1.2133385	-0.0886736	+0.7600	+2.2000
-2.2802533	+0.7664617	-2.1755445	+0.4560508	+0.6600	+0.8000
-2.1092060	+0.3312719	-0.9526018	-0.1054663	+2.6600	+2.0900
+2.6288561	+0.2281321	+1.8968774	-0.2628856	+2.3100	+0.7600
-1.0402081	-0.2536443	-0.2807730	+0.1560312	+3.1350	+0.4400

7	8	9	10	11	12
2 5	3 5	4 5	2 6	3 6	4 6
+0.0299659	+0.0174039	+0.0059688	-1.3484651	-0.7831757	-0.2685971
-0.1995428	-0.0347580	-0.0424209	+2.4372722	+0.4245439	+0.5181408
-0.0916437	+0.0169664	-0.0137497	+0.4775117	-0.0884039	+0.0716430
+0.1884722	-0.0894126	+0.0605636	-0.4112121	+0.1950821	-0.1321388
-0.0335417	+0.0184415	-0.0079312	-0.2276044	+0.1251389	-0.0538186
-0.7817060	+0.9221373	-0.0673919	-2.2628331	+2.6693447	-0.1950819
+0.5058647	-1.4358594	+0.3009935	+0.6131694	-1.7404356	+0.3648406
+0.8811832	-2.5339208	-0.2805244	+0.6923583	-1.9909378	-0.2204120
+0.5269852	+4.3817868	-0.6072657	+0.1733804	+1.4416268	-0.1997931
-0.7951749	-0.8802234	+0.4891578	-0.1116035	-0.1235401	+0.0686537
SUMS (multiplied by 10)					
+2.308621	+3.825617	-1.626001	+0.319738	+1.292433	-0.465634

No simple checks are apparent for the M and N factors, but it should be observed that the Ms can be developed in three ways.

$$
\begin{aligned}
M_{12} &= q_{135}q_{146}Q_{134}Q_{156} - q_{134}q_{156}Q_{135}Q_{146} \\
&= q_{136}q_{145}Q_{134}Q_{156} - q_{134}q_{156}Q_{136}Q_{145} \\
&= q_{136}q_{145}Q_{135}Q_{146} - q_{135}q_{146}Q_{136}Q_{145}.
\end{aligned}
$$

DETERMINATION OF EXTERIOR ORIENTATION BY ITERATION

f = 150 — FORM 5 — *Example No. 2*

DATA Note: The $\bar{X}$ and $\bar{Y}$ coordinates can be recorded as from any origin.

POINT	$\bar{X}$	$\bar{Y}$	h	$\bar{x}$	$\bar{y}$
G_1	+ 6073.1	+52333.4	+6000.0	+ 10.0	+ 70.0
G_2	−10468.8	+15551.5	+1500.0	−100.0	0.0
G_3	− 1426.0	+ 5876.3	0.0	− 30.0	− 90.0
G_4					
G_5					
G_6					

FINAL VALUES		
	A	
	s	
	t	

ASSUMED VALUES		
	A	0
	s	.0
	t	57°
	$\sin t$	0.8386706
	$\cos t$	0.5446390

APPROXIMATE VALUES

	1st Approx.	2nd Approx.
A	4° 22'	3° 55'
s	2° 27'	1° 53'
t	59° 49' 20"	60° 05'
$\sin A$	0.07614	
$\cos A$	0.99710	
$f \sin s$	6.41213	
$f \cos s$	149.86289	
$\sin t$	0.86447	
$\cos t$	0.50268	

VERTICAL SOLUTION ONLY

	1st Approx.	Sum of 1st and 2nd
$\tan\frac{1}{2}(A+s)$		
$\frac{1}{2}(A+s)$		
$\frac{1}{2}(A-s)$		
Sum = A		
Diff. = s		
$\sin\frac{1}{2}(A+s)$		
$\cos\frac{1}{2}(A+s)$		
Mean t		

COEFFICIENTS OF SIMULTANEOUS EQUATIONS

ΔA	Δs	Δt	−w'	−w"	−w'''
$t\,\cancel{\sin\frac{1}{2}(A+s)}$	$t\,\cancel{\cos\frac{1}{2}(A+s)}$	$\cancel{(A-s)}$			
−12.734	+15.211	+18.477	= +2034.05	+114.68	
−32.928	+22.487	+ 7.289	= −4080.32	+238.16	
− 7.289	+ 5.326	− 0.520	= −1213.17	+ 6.97	

ELIMINATION

+21.452	+51.559	= +11893.60	+74.355
+ 4.305	+14.130	= −3027.47	+74.718
	8.116	= +1374.33	+128.275

SOLUTIONS OF EQUATIONS

OBLIQUE	VERTICAL	1st Approx.	2nd Approx.	SUM
ΔA	$t\,\cancel{\sin\frac{1}{2}(A+s)}$	+262'.10	−27'.28	+234'.82
Δs	$t\,\cancel{\cos\frac{1}{2}(A+s)}$	+147'.45	−34'.50	+112'.95
Δt	$\cancel{(A-s)}$	+169'.33	+15'.81	+185'.14

The work on this synthetic example is not complete, as at least one more approximation is necessary. The known values are $A = 4°$ $s = 2°$ $t = 60°$

Development of Coefficients (Forms 3 and 4)

The coefficient of ($f \cos s \tan t$) with point 1 as pivot is

$$Z_{12}M_{12}\bar{x}_{12} + Z_{13}M_{13}\bar{x}_{13} + Z_{14}M_{14}\bar{x}_{14} + Z_{15}M_{15}\bar{x}_{15} + Z_{16}M_{16}\bar{x}_{16}$$

and with point 2 as pivot is

$$Z_{23}M_{23}\bar{x}_{23} + Z_{24}M_{24}\bar{x}_{24} + Z_{25}M_{25}\bar{x}_{25} + Z_{26}M_{26}\bar{x}_{26} + Z_{12}M_{21}\bar{x}_{12}.$$

The corresponding coefficients of ($f \sin s \tan t$) and the right hand members of

DETERMINATION OF EXTERIOR ORIENTATION BY ITERATION
—FORM 6—

Example No. 2

DIFFERENCES OF COORDINATES AND THE k FACTORS, e.g. $k_{12} = (\bar{x}_1\bar{y}_2 - \bar{x}_2\bar{y}_1)$

LINE	$\bar{X}/10^4$	$\bar{Y}/10^4$	$Z/10^4$	$\bar{x}$	$\bar{y}$	k
2-1 = 12	−1.6542	−3.6781	+0.4500	−110	−70	+7000
3-1 = 13	−0.7499	−4.6457	+0.6000	−40	−160	+1200
3-2 = 23	+0.9043	−0.9675	+0.1500	+70	−90	+9000

	LINE	12	13	23
PRODUCTS OF COORDINATE DIFFERENCES	$\bar{X}\bar{x}$	+181.962	+29.996	+63.301
	$\bar{X}\bar{y}$	+115.794	+119.984	−81.387
	$\bar{X}k$	−11579.400	−899.880	+8138.700
	$\bar{Y}\bar{x}$	+404.591	+185.828	−67.725
	$\bar{Y}\bar{y}$	+257.467	+743.312	+87.075
	$\bar{Y}k$	−25746.700	−5547.840	−8707.500
	$Z\bar{x}$	−49.500	−24.000	+10.500
	$Z\bar{y}$	−31.500	−96.000	−13.500
	Zk	+3150.000	+720.000	+1350.000
COEFFICIENTS FOR OBLIQUE METHOD	$-\bar{X}\bar{x}f\cos t$	−14865.	−2450.	−5171.
	$-\bar{X}k\sin t$	+9711.	+755.	−6826.
	$-\bar{Y}\bar{y}f$	−38620.	−111497.	−13061.
	Coefficient of ΔA = arc 1′ sum	−12.734	−32.928	−7.289
	$+\bar{X}\bar{x}f$	+27294.	+4499.	+9495.
	$+\bar{Y}\bar{y}f\cos t$	+21034.	+60726.	+7114.
	$-Z\bar{y}f\sin t$	+3963.	+12077.	+1698.
	Coefficient of Δs = arc 1′ sum	+15.211	+22.487	+5.326
	$+\bar{Y}\bar{x}f\sin t$	+50897.	+23377.	−8520.
	$-\bar{Y}k\cos t$	+14023.	+3036.	+4742.
	$+Z\bar{x}f\cos t$	−4044.	−1961.	+858.
	$+Zk\sin t$	+2642.	+604.	+1132.
	Coefficient of Δt = arc 1′ sum	+18.477	+7.289	−0.520
COEFFICIENTS NEAR VERTICAL METHOD	$Z\bar{x}f$			
	$Z\bar{y}f$			
	Coeff't. of $t\sin\frac{1}{2}(A+s) = -\text{arc }1'(\bar{X}k+Z\bar{y}f)$			
	Coeff't. of $t\cos\frac{1}{2}(A+s) = -\text{arc }1'(\bar{Y}k-Z\bar{x}f)$			
	Coefficient of $(A-s) = -\text{arc }1'(\bar{X}\bar{x}+\bar{Y}\bar{y})$			

the equations are obtained by substituting the corresponding $\bar{y}$ and k factors for the $\bar{x}$ factors.

On form 3, as indicated, obtain first ZM products and then the $ZM\bar{x}$, $ZM\bar{y}$ and ZMk products. Obtain the required coefficients and right hand members by summing each of the last three columns.

The coefficient of ($\sin A\ \tan t$) in the first of the second group of equations is

DETERMINATION OF EXTERIOR ORIENTATION BY ITERATION
- FORM 7-

Example No. 2

VALUES OF W WHEN A, s AND t ARE APPROXIMATE

THE C FACTORS (See form 5 for Trig. values etc.)		
APPROXIMATION	2	3
(1) f cos A sin s	+ 6.3935	
(2) f cos A cos s	+149.4279	
(3) f sin A sin s	+ 0.4882	
(4) f sin A cos s	+ 11.4104	
(5) cos t	+ 0.5027	
C_1 = (1)−(4)(5)	+ 0.6577	
C_2 = (2)+(3)(5)	+149.6733	
C_3 = (3)+(2)(5)	+ 75.6033	
C_4 = (4)−(1)(5)	+ 8.1965	
C_5 = f cos s sin t	+129.5519	
C_6 = f sin s sin t	+ 5.5431	
C_7 = sin A sin t	+ 0.0658	
C_8 = cos A sin t	+ 0.8620	
C_9 = cos t	+ 0.5027	

W WHEN $A' = s' = 0$ (1st approximation for obliques)			
LINE	12	13	23
$+\bar{X}\bar{y}f$	+17369.1	+17997.6	−12208.1
$-\bar{Y}\bar{x}f\cos t$	−33053.4	−15181.4	+ 5532.9
$+Z\bar{x}f\sin t$	− 6227.1	− 3019.2	+ 1320.9
$-\bar{Y}k\sin t$	+21593.0	+ 4675.5	+ 7302.7
$-Zk\cos t$	− 1715.6	− 392.2	− 735.2
W = SUM	− 2034.0	+ 4080.3	+ 1213.2

W WHEN $t' = 0$ AND $A' = s'$ (1st approximation for near-verticals)			
LINE	12	13	23
$+\bar{X}\bar{y}f$			
$-\bar{Y}\bar{x}f$			
$-Zk$			
W = SUM			

W IN FULL

	2ND APPROXIMATION			3RD APPROXIMATION		
LINE	12	13	23	12	13	23
$+\bar{X}\bar{x}C_1$	+ 119.672	+ 19.728	+ 41.631			
$+\bar{X}\bar{y}C_2$	+17331.267	+17958.398	−12181.458			
$-\bar{Y}\bar{x}C_3$	−30588.415	−14049.210	+ 5120.233			
$-\bar{Y}\bar{y}C_4$	− 2110.327	− 6092.553	− 713.710			
$+Z\bar{x}C_5$	− 6412.821	− 3109.247	+ 1360.295			
$-Z\bar{y}C_6$	+ 174.607	+ 532.136	+ 74.832			
$-\bar{X}kC_7$	+ 762.155	+ 59.230	− 535.688			
$-\bar{Y}kC_8$	+22192.636	+ 4805.291	+ 7505.520			
$-ZkC_9$	− 1583.457	− 361.933	− 678.624			
W = SUM	− 114.683	− 238.160	− 6.969			

$$N_{123}\overline{X}_{23}\bar{x}_{12}\bar{x}_{13} + N_{124}\overline{X}_{24}\bar{x}_{12}\bar{x}_{14}$$

$$N_{125}\overline{X}_{25}\bar{x}_{12}\bar{x}_{15} + N_{126}\overline{X}_{26}\bar{x}_{12}\bar{x}_{16}$$

$$N_{134}\overline{X}_{34}\bar{x}_{13}\bar{x}_{14} + N_{135}\overline{X}_{35}\bar{x}_{13}\bar{x}_{15}$$

$$N_{136}\overline{X}_{36}\bar{x}_{13}\bar{x}_{16} + N_{145}\overline{X}_{45}\bar{x}_{14}\bar{x}_{15}$$

$$N_{146}\overline{X}_{46}\bar{x}_{14}\bar{x}_{16} + N_{156}\overline{X}_{56}\bar{x}_{15}\bar{x}_{16}.$$

The coefficient of (cos A tan t) and the right hand member of this equation are obtained by substituting corresponding $\overline{Y}$ and Z factors respectively for the X factors.

DETERMINATION OF EXTERIOR ORIENTATION BY ITERATION

$f = 150$ — FORM 5 — *Example No. 3*

DATA Note: The $\bar{X}$ and $\bar{Y}$ coordinates can be recorded as from any origin.

POINT	$\bar{X}$	$\bar{Y}$	h	$\bar{x}$	$\bar{y}$
G_1	+ 416.7	−6377.3	+250.0	+ 1.0	−100.0
G_2	+6957.7	+ 206.6	+100.0	+100.0	0.0
G_3	+ 801.4	+6818.8	0.0	+ 10.0	+100.0
G_4					
G_5					
G_6					

FINAL VALUES	A	70° 00′
	s	71° 00′
	t	1° 30′
ASSUMED VALUES	A	—
	s	—
	t	0
	$\sin t$	0
	$\cos t$	1

APPROXIMATE VALUES

	1st Approx.	2nd Approx.
A	70° 29′	70° 00′
s	71° 29′	71° 00′
t	1° 31′	1° 30′
$\sin A$	0.94254	
$\cos A$	0.33408	
$f \sin s$	142.23470	
$f \cos s$	47.63708	
$\sin t$	0.02647	
$\cos t$	0.99965	

VERTICAL SOLUTION ONLY

	1st Approx.	Sum of 1st and 2nd
$\tan\frac{1}{2}(A+s)$	2.902045	2.8242216
$\frac{1}{2}(A+s)$	70° 59.′2	70° 30′ 10″
$\frac{1}{2}(A-s)$	− 29.′8	− 29′ 50″
Sum = A	70° 29.′4	70° 00′ 20″
Diff. = s	71° 29.′0	71° 00′ 00″
$\sin\frac{1}{2}(A+s)$	0.9454396	0.9426577
$\cos\frac{1}{2}(A+s)$	0.3257973	0.3337612
Mean t	90.′9	89.′9

COEFFICIENTS OF SIMULTANEOUS EQUATIONS

$\cancel{\Delta A}$ $t\sin\frac{1}{2}(A+s)$	$\cancel{\Delta s}$ $t\cos\frac{1}{2}(A+s)$	$\cancel{\Delta t}$ $(A-s)$	−W′	−W″	−W‴
−1.968	−1.850	−5.699	= +115.6	+1.825	
−0.230	−0.412	−11.531	= +655.0	+0.204	
+1.747	−1.963	−5.303	= +408.0	−2.883	

ELIMINATION

+0.385	+21.380	= −1262.5	−0.018
+7.095	+20.393	= −1004.9	+2.485
	−143.840	= +8570.55	−0.008

SOLUTIONS OF EQUATIONS

OBLIQUE	VERTICAL	1st Approx.	2nd Approx.	SUM
$\cancel{\Delta A}$	$t\sin\frac{1}{2}(A+s)$	+85.976	−1.255	+84.721
$\cancel{\Delta s}$	$t\cos\frac{1}{2}(A+s)$	+29.626	+0.372	+29.998
$\cancel{\Delta t}$	$(A-s)$	−59.584	−0.008	+59.592

In this synthetic example the known values are $A = 70°$ $s = 71°$ $t = 1° 30'$

The second equation in ($\sin A \tan t$) and ($\cos A \tan t$) is derived directly from the first by substituting corresponding $\bar{y}$ factors for the $\bar{x}$ factors.

On form 4 first obtain and enter in columns 2, 3 and 4 the $N\bar{X}$, $N\bar{Y}$ and NZ products respectively. Then in columns 5 and 6, the $\bar{x}\bar{x}$ and $\bar{y}\bar{y}$ factors.

Make and record as indicated in columns 7, 8, 9, 10, 11 and 12, the products $N\bar{X}\bar{x}\bar{x}$, etc. The sums of columns 7, 8 and 9 are the coefficients and right hand member of the first equation in ($\sin A \tan t$) and ($\cos A \tan t$), and the sums of columns 10, 11, 12 are those for the second equation.

Solution of Equations

On form 1 enter the coefficients and right hand members of the equations

DETERMINATION OF EXTERIOR ORIENTATION BY ITERATION
—FORM 6—

Example No. 3

DIFFERENCES OF COORDINATES AND THE k FACTORS, e.g. $k_{12}=(\bar{x}_1\bar{y}_2-\bar{x}_2\bar{y}_1)$

LINE	$\bar{X}/10^4$	$\bar{Y}/10^4$	$Z/10^4$	$\bar{x}$	$\bar{y}$	k
2-1=12	+0.65410	+0.65839	+0.01500	+99	+100	+10.000
3-1=13	+0.03847	+1.31961	+0.02500	+ 9	+200	+11.000
3-2=23	−0.61563	+0.66122	+0.01000	−90	+100	+10.000

	LINE	12	13	23
PRODUCTS OF COORDINATE DIFFERENCES	$\bar{X}\bar{x}$	+ 64.756	+ 0.346	+ 55.407
	$\bar{X}\bar{y}$	+ 65.410	+ 7.694	− 61.563
	$\bar{X}k$	+6541.000	+ 42.317	−6156.300
	$\bar{Y}\bar{x}$	+ 65.181	+ 11.877	− 59.510
	$\bar{Y}\bar{y}$	+ 65.839	+ 263.922	+ 66.122
	$\bar{Y}k$	+6583.900	+1451.571	+6612.200
	$Z\bar{x}$	+ 1.435	+ 0.225	− 0.900
	$Z\bar{y}$	+ 1.500	+ 5.000	+ 1.000
	Zk	+ 150.000	+ 27.500	+ 100.000
COEFFICIENTS FOR OBLIQUE METHOD	$-\bar{X}\bar{x}f\cos t$			
	$-\bar{X}k\sin t$			
	$-\bar{Y}\bar{y}f$			
	Coefficient of ΔA = arc 1′ sum			
	$+\bar{X}\bar{x}f$			
	$+\bar{Y}\bar{y}f\cos t$			
	$-Z\bar{y}f\sin t$			
	Coefficient of Δs = arc 1′ sum			
	$+\bar{Y}\bar{x}f\sin t$			
	$-\bar{Y}k\cos t$			
	$+Z\bar{x}f\cos t$			
	$+Zk\sin t$			
	Coefficient of Δt = arc 1′ sum			
COEFFICIENTS NEAR VERTICAL METHOD	$Z\bar{x}f$	+222.750	+ 33.750	−135.000
	$Z\bar{y}f$	+225.000	+750.000	+150.000
	Coeff't. of $t\sin\frac{1}{2}(A+s)=-\text{arc }1'(\bar{X}k+Z\bar{y}f)$	− 1.968	− 0.230	+ 1.747
	Coeff't. of $t\cos\frac{1}{2}(A+s)=-\text{arc }1'(\bar{Y}k-Z\bar{x}f)$	− 1.850	− 0.412	− 1.963
	Coefficient of $(A-s)=-\text{arc }1'(\bar{X}\bar{x}+\bar{Y}\bar{y})f$	− 5.699	− 11.531	− 5.303

to be solved as found on forms 3 and 4 in the places indicated. Solve for ($f\cos s\tan t$), ($f\sin s\tan t$), ($\sin A\tan t$) and ($\cos A\tan t$). Enter the values obtained under *solution* and compute values of A, s, t and f as shown by the example.

To check the accuracy of the whole work, W may be computed on form 7 for a single line. If no mistake has been made, the value should be zero.

Example 2——1st Iterative Method

Assembling of Data (Form 5)

From the survey records, make an accurate plot of the ground control on any convenient scale. Make an estimate of t, the tilt, and the direction of the principal line. Mark on the photograph a line through p, the principal point, to indicate the principal line. Estimate the horizontal direction of t and mark this direction on the ground control plot as the $+\overline{Y}$ direction. Measure the $\overline{X}$ and $\overline{Y}$ coordinates of the selected control points from any common origin and enter these with the h values, corrected for curvature and refraction, on form 5. Assume the $+\bar{y}$ direction as parallel to the estimated direction of the principal line on the photograph with the +direction towards the horizon from p. With p as origin, measure the $\bar{x}$ and $\bar{y}$ coordinates as precisely as means permit and enter these on form 5.

Note that no elaborate procedure is necessary in making these first estimates. The first approximate computation will not only reveal a bad estimate but will in itself be the means of obtaining a good estimate with which to start the process anew.

Record in the indicated box on form 5 the assumed values of A, the azimuth of the principal plane, s, the swing, t, sin t and cos t. Note in this case that A and s have both been assumed as zero. At the top left corner of form 5 record the value of f, the principal distance, as precisely as it is known.

Preliminary Computations (Form 6)

1st step. Obtain the coordinate differences $\overline{X}_{12}$, $\overline{Y}_{12}$, Z_{12}, etc. and the k factors and record these at the top of form 6.

2nd step. Obtain the products of coordinate differences such as $\overline{X}_{12}\,\bar{x}_{12}$, $\overline{X}_{12}\,\bar{y}_{12}$ and $Z_{12}\,k_{12}$ and record these in the places indicated on Form 6.

Development of Coefficients (Forms 6 and 7)

The coefficients of ΔA, Δs and Δt are determined on form 6, which is self explanatory.

The right hand member of each equation is denoted by W. As shown on form 7, the calculation of W is considerably simplified for first approximations. For 2nd approximations the full value of W with values of A, s and t as found in the first approximation must be used. The full equation for W for line 12 for example is

$$+\overline{X}_{12}\bar{x}_{12}C_1+\overline{X}_{12}\bar{y}_{12}C_2-\overline{Y}_{12}\bar{x}_{12}C_3-\overline{Y}_{12}\bar{y}_{12}C_4$$
$$+Z_{12}\bar{x}_{12}C_5-Z_{12}\bar{y}_{12}C_6-\overline{X}_{12}k_{12}C_7-\overline{Y}_{12}k_{12}C_8-Z_{12}k_{12}C_9.$$

Note that the C factors, as shown on form 7 are functions of A', s', t' and f and are common to each line. Values of W for third approximations are obtained similarly using the values of A, s and t as determined by the second approximation.

Solution of Equations

On form 5 in the places indicated, enter the computed coefficients and $-W$ values for the first approximation. Solve the equations and enter the values, which will be in minutes of arc, under solution of equations. Add these values to the assumed A, s, and t obtaining improved values of the latter.

Now on form 7 compute W for each line from the derived values of A, s, and t.

If the values of W so obtained are not all zero or approximately so, a second approximation is necessary. To do this enter the new $-W$ values under $-W''$

DETERMINATION OF EXTERIOR ORIENTATION BY ITERATION
—FORM 7—

Example No. 3

VALUES OF W WHEN A, s AND t ARE APPROXIMATE

THE C FACTORS (See form 5 for Trig. values etc.)

APPROXIMATION	2	3
① $f \cos A \sin s$	47.51791	
② $f \cos A \cos s$	15.91464	
③ $f \sin A \sin s$	134.06252	
④ $f \sin A \cos s$	44.90058	
⑤ $\cos t$	0.99965	
C_1 = ①−④⑤	+ 2.63306	
C_2 = ②+③⑤	+149.93020	
C_3 = ③+②⑤	+149.97159	
C_4 = ④−①⑤	− 2.60068	
$C_5 = f \cos s \sin t$	1.26084	
$C_6 = f \sin s \sin t$	3.76463	
$C_7 = \sin A \sin t$	0.02497	
$C_8 = \cos A \sin t$	0.00884	
$C_9 = \cos t$	0.99965	

W WHEN $A' = s' = 0$ (1st approximation for obliques)

LINE	12	13	23
$+\bar{X}\bar{y}f$			
$-\bar{Y}\bar{x}f \cos t$			
$+Z\bar{x}f \sin t$			
$-\bar{Y}k \sin t$			
$-Zk \cos t$			
W = SUM			

W WHEN $t' = 0$ AND $A' = s'$ (1st approximation for near-verticals)

LINE	12	13	23
$+\bar{X}\bar{y}f$	+9811.5	+1154.1	−9234.5
$-\bar{Y}\bar{x}f$	−9777.1	−1781.6	+8926.5
$-Zk$	− 150.0	− 27.5	− 100.0
W = SUM	− 115.6	− 655.0	− 408.0

W IN FULL

	2ND APPROXIMATION			3RD APPROXIMATION		
LINE	12	13	23	12	13	23
$+\bar{X}\bar{x}C_1$	+ 170.506	+ 0.912	+ 145.889			
$+\bar{X}\bar{y}C_2$	+9806.934	+1153.563	−9230.153			
$-\bar{Y}\bar{x}C_3$	−9775.238	−1781.138	+8924.779			
$-\bar{Y}\bar{y}C_4$	+ 171.226	+ 686.377	+ 171.962			
$+Z\bar{x}C_5$	+ 1.872	+ 0.284	− 1.135			
$-Z\bar{y}C_6$	− 5.647	− 18.823	− 3.765			
$-\bar{X}kC_7$	− 163.329	− 1.057	+ 153.723			
$-\bar{Y}kC_8$	− 58.202	− 12.832	− 58.452			
$-ZkC_9$	− 149.947	− 27.490	− 99.965			
W = SUM	− 1.825	− 0.204	+ 2.883			

on form 5 and solve the equations again using the same coefficients but with $-W''$ values instead of $-W'$ values as the right hand members of the equations.

The algebraic sum of the solutions obtained in the first and second approximates are now applied to the assumed values and a further improvement in A, s and t is obtained. A third and further approximation may be made in the same way, but in most cases found in practice, these will be unnecessary.

Example 3——2nd Iterative Method

Assembling of data (Form 5)

From the survey records with any common origin, extract the $\bar{X}$ and $\bar{Y}$ coordinates of the selected control points. Enter these with the h values, corrected for curvature and refraction if necessary, on form 5.

On a transparent sheet make an accurate plot of the ground control on any convenient scale.

Place the photograph under the plot and orient it so that image lines of object lines are on the average parallel to each other. Trace a line onto the photograph passing through p parallel to the $+\overline{Y}$ direction on the plot and adopt this line as the $+\bar{y}$ direction.

With p as origin, measure the $\bar{x}$ and $\bar{y}$ coordinates as precisely as means permit and enter these on form 5. Record in the indicated box on form 5 that t is assumed as zero and that $\sin t = 0$ and $\cos t = 1$. Record the known value of f as precisely as it is known at the top left corner of the form.

Preliminary Computations (Form 6)

These are the same as in the 1st Iterative Method.

Development of Coefficients (Forms 6 and 7)

The coefficients of $t \sin\frac{1}{2}(A+s)$, $t \cos\frac{1}{2}(A+s)$ and $(A-s)$ are determined on form 6, which is self explanatory. W is determined on form 7 in the same way as in the 1st iterative method with a still greater simplification for first approximations.

Solution of Equations

The solution of the simultaneous equations and further approximations are carried out in the same way as for the 1st iterative method with this difference. After the 1st approximate solution, values of $t \sin\frac{1}{2}(A+s)$, $t \cos\frac{1}{2}\ (A+s)$ and $(A-s)$ are obtained and values of A, s and t are then derived as shown in the box in the lower right hand corner of form 5. Especially note that, if further approximations are required, A, s and t must be derived from the sum of the solutions of $t \sin\frac{1}{2}(A+s)$, $t \cos\frac{1}{2}(A+s)$ and $(A-s)$ obtained by each approximation.