PARALLAX

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Comments on "Parallax": While this article does not present any new theory, it is an effective review of the correct methods of obtaining elevations from stereoscopic photographs by means of the differential parallax. The article stresses the fact that "the parallax of one of the two points, whose height difference is being measured, must be known." That is to say, the correct photo base must be used in applying the parallax formula. This requirement was pointed out by Louis Desjardins (P.E., April, 1944) and reemphasized in our article "A Corrected Master Parallax Table" (P.E., Sept., 1947).

It is believed that Mr. Robbins' article should help to eliminate the confusion which has existed regarding the use of the parallax formula and the attendant nomenclature. *B. J. Lane, Jr. and C. T. MacNeil, for the Publications Committee.*

THE writer has read the many articles that have been printed in PHOTOGRAMMETRIC ENGINEERING on the subject of finding heights from parallax measurements by means of formulae, tables or graphs, and has noticed that almost every one of them uses the same symbols but allots different meanings to them. For instance, B or Bm or Bx means (1) "photo-base" of either photograph, or (2) "photo-base at sea level," or (3) "photo-base" at some other datum at which H is or is not zero. The result of this in most cases is that approximations are made which, with very little or no extra work, may be avoided. In flat country, these approximations amount to little and they are perhaps of little importance, in view of the fact that heights obtained by simple parallax measurements are essentially approximate, their accuracy depending on the (unknown) amount of tilt present and on the (unknown) inclination of the air base. But in hilly or mountainous country, they very definitely mean the difference between a reasonable answer and a false one. Furthermore, if several control elevations per overlap are available, and the effects of tilt minimized by the use of correction graphs (as suggested in the article "*Practical Applications of the Stereo-Comparagraph Plotting Machine*" by Albert L. Nowicki, PHOTOGRAMMETRIC ENGINEERING, IX, 2,107), they again decrease the accuracy of the results, even though the country be flat.

This confusion of notation leads in many cases to (a) rather cumbersome formulae and (b) to their incorrect use.

The same confusion exists even in the definition of the word "parallax." It has been defined incorrectly by some, and another author states that the word is frequently (but wrongly) used to mean "parallax reading" (on any type of parallax measuring device). Consequently, he uses it with this false meaning, and coins another phrase—"variable photo-base"—to convey what the word "parallax" should convey.

The purpose of this article therefore is to put forward parallax formulae which are exact, and a notation which will avoid any confusion and will ensure that these formulae are correctly used. In view of the recent report of the Nomenclature Committee,[1] and as the author is concerned with education in photogrammetry, he suggests that a strict notation of some sort be standardized and employed in the various articles concerned in the new edition of the MANUAL OF PHOTOGRAMMETRY—not necessarily the exact notation of this article but one which leaves no room for doubt.

Figure 1 illustrates diagrammatically conditions under which the ideal stereo-pair are exposed. L_1 and L_2 are positions of the camera lens at the instants of successive exposures; p_1, the image on the left-hand photo of its

[1] PHOTOGRAMMETRIC ENGINEERING, XIV, 2, 318.

principal point, and p_1' the image of this point on the right-hand photo; p_2 the image on the right-hand photo of its principal point and p_2' the image of this point on the left-hand photo; a_1 the image on the left-hand photo of any ground point A, a_2 its image on the right-hand photo and a_3 a point on the left-hand photo such that L_1a_3 is parallel to L_2a_2. Then a_1a_3 is the absolute stereoscopic parallax of the ground point A on the stereo-pair, and this is the linear distance subtended on the photograph by the angle of parallax at the lens (vide Louis Desjardins, PHOTOGRAMMETRIC ENGINEERING, X,2,90).

If x_1 is the coordinate along the photo-base of a_1 and x_2 that of a_2, the principal points being the origins, and coordinates being $+\infty$ to the right on positive prints, then $a_1a_3 = p_1a_1 + p_1a_3 = p_1a_1 + p_2a_2 = x_1 - x_2$ (with strict attention to algebraic signs) which is the definition of "Absolute Stereoscopic Parallax" given in the present edition of the MANUAL OF PHOTOGRAMMETRY. It can be

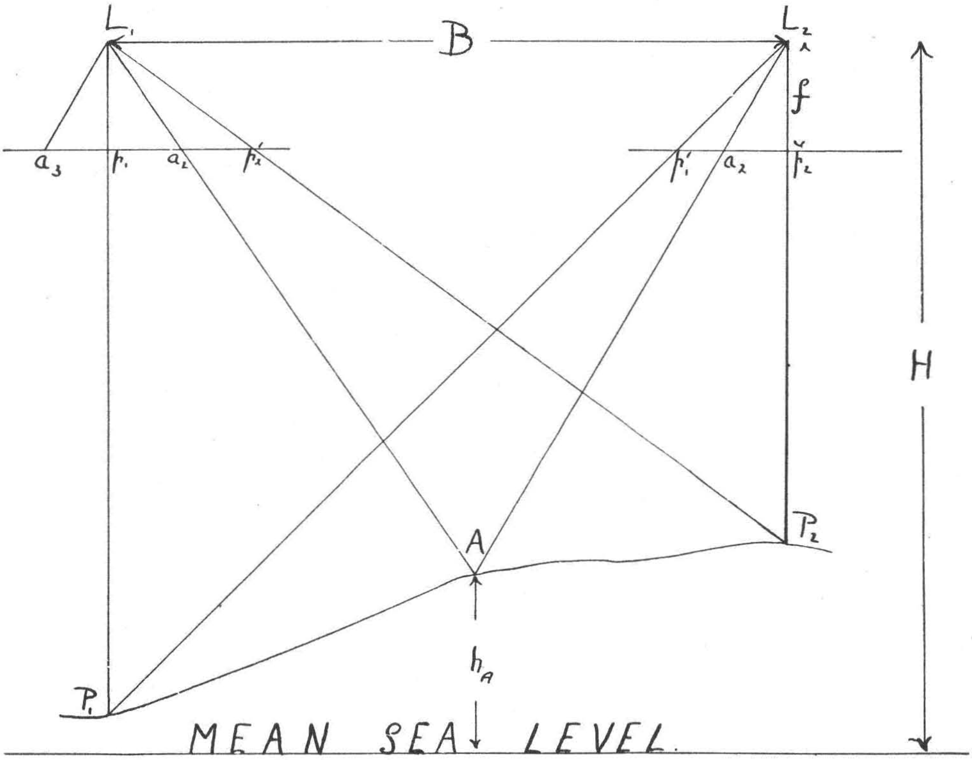

FIG. 1. A diagrammatic illustration of conditions under which the ideal stereo-pair is exposed.

seen at once that the absolute stereoscopic parallax (or simply parallax, as it shall be called for the remainder of this article) of a point cannot be measured directly by simple instruments, such as the stereocomparagraph or stereoscope and parallax bar. The separation of corresponding images—the distance a_1a_2 when the two photographs are placed together and viewed under a stereoscope—depends entirely on the physical characteristics of the stereoscope and has nothing to do with parallax. But the *difference* between the separations of the corresponding images of two ground points is the same as the difference of parallax between these two points. Therefore a single reading on the parallax bar is meaningless, but the difference between two such readings is a measure of the difference of parallax.[2] Consequently, in order to evaluate the height of any point on a stereo-pair, the parallax of one point at least must be known. It has been usual to measure the "photo-base" on both photographs—p_1p_2' and p_2p_1' on the figure—and to take their mean. In some of the articles previously published, it has then been assumed that this is the parallax of the control point C—this is an approximation. Rigidly, as can readily be deduced from the figure, p_1p_2' is the parallax of P_2 and p_2p_1' that of P_1. This is of the greatest importance, and it is the failure to recognize this that has contributed to the approximations in many of the existing formulae.

The notation to be used now is as follows:

f Focal length of camera.

B Air-base—i.e. the distance in space between the perspective centres (or positions of the camera lens at the instants of exposure) of two successive photographs.

H Mean height of aircraft above datum (usually M.S.L.) at the two instants of exposure.

h_A The height above the *same* datum of any ground point A.

Δh_{AB} $h_B - h_A$

b_2 p_1p_2' } See figure.

b_1 p_2p_1' }

p_A The absolute stereoscopic parallax of A.

Δp_{AB} $p_B - p_A$

C A ground control point.

The fundamental parallax formula, derived by simple geometric considerations from the figure, is:

$$\frac{p_A}{f} = \frac{B}{H - h_A} \quad \text{or} \quad p_A = \frac{fB}{H - h_A}. \tag{1}$$

Considering an increase in parallax Δp corresponding to an increase in height Δh,

$$p + \Delta p = \frac{fB}{H - (h + \Delta h)}$$

$$\therefore \Delta p = fB\left[\frac{1}{H - (h + \Delta h)} - \frac{1}{H - h}\right]$$

$$\therefore \Delta p_{AB} = \frac{fB\Delta h_{AB}}{(H - h_A - \Delta h_{AB})(H - h_A)}. \tag{2}$$

It is to be noted that equation 1 is differentiated, it follows that

$$\Delta p = \frac{fB\Delta h}{(H - h)^2}.$$

[2] Here it may be mentioned that an increase in ground height corresponds to an increase in parallax and to a decrease in the separation of corresponding images. The failure to recognize this has led some manufacturers of simple parallax measuring devices to graduate their micrometers so that an increase in reading corresponds to an increase in separation (and, therefore, to a decrease in parallax and in ground height), while others have graduated them in the opposite way. More confusion!

This is not true because Δh is not a small quantity and does not tend to zero, differentiation being therefore not permissible.

Using equation (1), (2) becomes:

$$\Delta p_{AB} = \frac{p_A \Delta h_{AB}}{H - h_A - \Delta h_{AB}}. \qquad (3)$$

Finally the equation may be put in the form:

$$\Delta h_{AB} = \frac{\Delta p_{AB}(H - h_A)}{p_A + \Delta p_{AB}}. \qquad (4)$$

These equations are by no means new, but they are the old ones in their simplest form, and the suffixes enable them to be used correctly. Their chief merits are that they are simple, general and mathematically exact. Finally, there is no question of "the variable photo-base at sea level" being introduced with a consequent second set of formulae for this special case.

The way to use these formulae, assuming one control elevation C on any part of the photograph, is as follows:

(i) $$p_c = b_1 + \Delta p_{p_1c} = b_2 + \Delta p_{p_2c}.$$

These two values will not be exactly the same due to the existence of tilt and inclination of the air-base, so p_c should be taken as their mean.

(ii) (a) If finding spot elevations, use equation (4) to find Δh between C and each point in turn.

(ii) (b) If contouring, use equation (3) to find Δp between the reading on the parallax bar when set on C and the reading to be set for each contour successively.

The form of these formulae illustrates the important fact, frequently overlooked in the past, that the parallax of one of the two points, whose height difference is being measured, must be known. The simplest way of using this fact is to take all measurements from the control point.

For those who like using tables, the old parallax tables have been shown by various authors to be incorrect. A new table, however, appeared in an excellent article in PHOTOGRAMMETRIC ENGINEERING, XIII,3,453—"*A Corrected Master Parallax Table*" by B. B. Lane, Jr. and G. T. McNeil. This table, if used correctly, gives a rigidly correct result. It is stressed in the article: " . . . that B_m and H . . . must correspond; that is, B_m must be the photo-base at the ground elevation which is H feet below the camera station." This introduction of an arbitrary datum from which all heights are measured—that of the as yet unknown height of one of the principal points—is confusing and the article makes no mention of how to obtain B_m at datum, assuming that the control elevation is not one of the principal points. However, if K were written as $[p_c(H-h_c)/2{,}500{,}000]$, the datum for all heights is automatically M.S.L. and there is no confusion with a "variable photo-base" at datum or at M.S.L. (It is worth noticing that $K=(fB/2{,}500{,}000)$, $p_A(H-h_A)$ being constant for all points of the stereo-pair and equal to fB.) Then the method of using the table, as illustrated in the article, ensures that all Δp's and Δh's are measured from the control point—as previously shown to be the easiest way of using the formulae.

For those, however, who prefer to use graphs, PHOTOGRAMMETRIC ENGINEERING, XIV,3,409—"Master Parallax Graphs" by H. R. Paterson may be used. This is constructed from the equation $dp=(B_m h/H-h)$, but, with the notation of the article which defines B_m as "the measured image base length in inches" (which presumably means the photo-base b), this is approximate and good only for flat country. Rigidly B_m should be the parallax of the lower point of the two, and the equation will then become equation (3) of this article. Therefore, in order to use the graph for photographs of mountainous areas, all measurements (of p or h) must be made from a control point and Δh abstracted from the graph for a "Base Length" which is the parallax of this control point, found as in (i) above. It must then be multiplied by the conversion factor of $(H-h_c/10,000)$ which is constant for all heights on any one over-lap. It is easier to set this constant factor on a computing machine or a slide rule, and multiply all Δh's by it, than to multiply by a variable $(H/10,000)$ where H is the height of the aeroplane above the lower point of the two whose elevation difference is being measured (the method suggested in this article). The Secondary Differential Parallax Graph is then unnecessary.

Having proceeded thus far, variations may be made according to taste although the writer would suggest that they are not necessary. One important aid to accuracy however may be introduced—a method of obtaining the true altitude of the aeroplane when the photographs were taken. It is well known that altimeters at their present stage of development give very inaccurate absolute heights, but reasonably accurate differential heights. If, therefore, an image of the altimeter appears on every photograph in a strip, and if the images of three or more control points appear on one stereo-pair, the mean height above sea-level of the aeroplane for this pair may be computed by using equation (2) which will give two equations with B and H as the only unknowns. H may then be found and the heights of the other photographs of the strip obtained by applying to this value of H the height differences as obtained from altimeter readings. Alternatively, H may be found from two control points if a minor control plot at a known scale is available which will give a value for B. Finally, if a minor control plot and one control point only are available, H can be computed from the relationships $b_1+\Delta p_{p1c}$ (or $b_2+\Delta p_{p2c}$)—preferably their mean $=p_c=(fB/H-h_c)$. It is doubtful however if this will give a value for H which is any better than that given by the altimeter as b and B can be measured only inaccurately (compared with measurements of Δp with the parallax bar) and the value of b includes errors due to tilt and inclination of the air-base. Furthermore, H, a large quantity, is being deduced from the measurement of b, a small quantity, and of B, a small length measured relatively inaccurately and multiplied by a large number (the scale of the plot); a small error in either will therefore produce a large error in H. The best method of course is to have as many control elevations as possible, in order that the effects of these errors on the measured Δp's may be ironed out.

The writer finishes this article by stressing once again the importance of the fact that these formulae are mathematically correct ONLY for a perfect stereo-pair—which have no tilt and which are both taken from the same altitude. Consequently, in practice, the results will be approximations and exact results may be obtained only by using more expensive instruments which allow the tilts and inclinations of the air-base to be set in them. What the equations will do, however, is to give the best result possible when using the simple instruments previously mentioned.