EXPLICIT THREE-POINT RESECTION IN SPACE

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I. INTRODUCTION

THERE are numerous iterative solutions to the problem of three-point space resection, but no efficient explicit solution to the problem. The complexity of an explicit solution is the main reason why iterative solutions are numerous and more generally known as the only practical solutions. Geometers have long recognized that given three face angles in space and the corresponding ground lengths, a direct solution to the problem of exterior orientation is possible by solving a fourth-degree equation whose coefficients are formed from the given data. The camera coordinates of the three images and the ground coordinates of the corresponding objects comprise the original raw data from which the three face angles and corresponding slope ground lengths are computed.

In general, iterative solutions have been employed because they are simpler and in most cases give results with fictitious photographs that exceed the precision of the uncalibrated lens and film. These facts, combined with the fact that the explicit solution has been only theoretically possible, serve to reduce the space problem almost wholly to one of differential mathematics. The impractical part of the explicit solution has not been so much solving the fourth-degree equation as it has been forming the coefficients of the degree equation. Then too, photogrammetry has never been considered a geodetic tool, but rather a compilation tool replacing primarily the plane table and alidade in compiling topographic detail on a geodetic framework established by ground control parties.

What value then does a rather complicated explicit solution have? To the analytical photogrammetrist and mathematician, it is an intellectual achievement in which the satisfaction of either solving or deriving is its own reward. There are other practical uses, however. We have no way of evaluating, for example, an iterative solution except by its application to a fictitious photograph. The explicit solution provides a set of data for evaluating iterative solutions on actual space exposures. If an iterative solution gives results comparable to those of an explicit, we are justified in saying the iterative solution is sufficiently accurate from at least theoretical considerations. Thirdly, no terrestrial geodesist would consider extending primary control by any but explicit formulas, even though all his observations are reciprocal and repeated. Then certainly we can not expect to extend geodetic control from the air, where the observations are neither reciprocal nor repeated, by any calculations less than explicit ones. Finally, there are certain research problems requiring a photographic record from which the geometry of phenomena is computed by analytical photogrammetry. The geometrical relations of the phenomena become a coordinate system by which all physical data are evaluated. Clearly the geometrical reference of the physical data must be explicitly exact.

The purpose of this paper is to describe an explicit solution to the problem of resecting the position of the camera in space from three control points—a solution that is no more lengthy than the differential analytical geometry solutions. The only conditions imposed are those of strong solid geometry; and they are no less than the rigorous specifications for "strength of figure" that the geodesist imposes on his ground quadrilateral triangulation schemes.

The method consists of first solving a fourth-degree equation for m by Ferrari's solution of the biquadratic. Knowing m, the perspective ray lengths are computed directly, after which the six elements of exterior orientation are computed with the perspective ray lengths as the additional required data beyond the ground and camera coordinates. m is simply the ratio of one perspective ray to another. The fourth-degree m-equation described in this report has relatively simple coefficients and consequently presents an explicit solution that is practical insofar as the equation is simply formed. The body of this report describes the explicit solution. A numerical example is woven into the description to assist the reader in understanding the solution. Derivation of the m-equation is given in the Appendix at the end of this report. The quartic in m reduces to a simple quadratic when four points are used. The development of the m-quadratic is also given.

II. COMPUTATION PROCEDURE

A. The computation procedure is readily broken down into five operations:

1. Computation of constants for coefficients of m-equation from camera and ground data.
2. Forming coefficients of m-equation, and solution of m-equation by Ferrari's method.
3. Computation of perspective ray lengths.
4. Computation of space coordinates.
5. Computation of space orientation.

B. Computation of constants for coefficients of m-equation from camera and ground data.

Fictitious photograph data[1]

1. Given photograph coordinates (millimeters)

	x	y	z
a.	−46.5384847	29.92755493	0
b.	46.3825116	17.69356712	0
c.	− 2.5773321	−42.57624638	0
p.p.	0	0	100

2. Given ground coordinates (feet)

	X	Y	Z
A.	14,158.3027	17,102.38904	500.0
B.	17,696.36364	8,870.49290	200.0
C.	10,000.	10,000.	0

3. Values to be determined

$LA = 10,598.9339$	$LB = 11,001.49989$	$LC = 11,093.49358$
$XL = 14,158.46096$	$YL = 12,402.66566$	$ZL = 10,000.$
$t = 3°0'0''$	$s = 330°0'0''$	$AZ = 30°0'0''$

Consider Figure 1. It is necessary to decide in the computation the pair of perspective ray lengths of which m shall be the ratio. Let $m = (LB'/LA')$. Solving for m where m is equal to LB/LA, the computer selects as a pivot point, ground

[1] The fictitious data are given to a rather ridiculous number of decimal places to illustrate the numerical accuracy of the solution.

point C or whatever point is not included in the ratio. Selecting C as the pivot point, the following values are required for the coefficients of the m equation.

1. $\cos A'LB' = .65605899$
2. $\sin \gamma_1 = .96540730$
3. $\sin \gamma_2 = .92688849$
4. $\cos \gamma_1 = .26074653$
5. $\cos \gamma_2 = .37533681$
6. $A'B'^2 = 80{,}372{,}000.0$
7. $D_1 = 11{,}804.34512$
8. $D_2 = 11{,}741.79875$

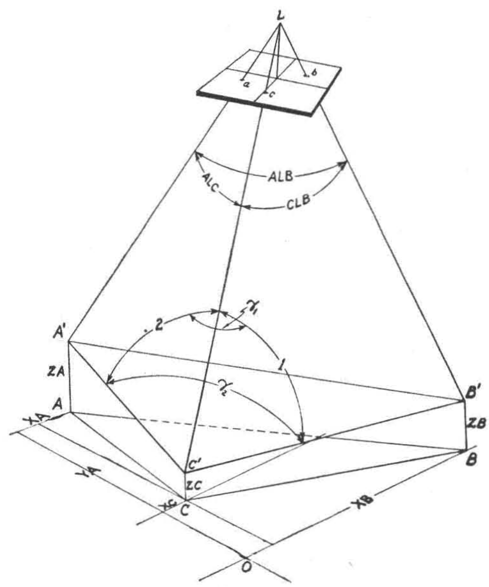

FIG. 1.

Where the $A'LB'$ is the angle at L between A' and B' or a and b; γ, the dihedral angle between planes $A'LC'$ and $B'LC'$; γ_2, the angle at C between slope lines $A'C'$ and $B'C'$; $A'B'^2$, the square of the slope line $A'B'$; and D_1, D_2, the diameters of circles $A'LC'$ and $B'LC'$. Insofar as the face angles are used in computing γ, and the diameters D_1 and D_2, angles $B'LC'$ and $C'LA'$ are also required. The three face angles at L are computed by the conventional angle-between-two-lines formula.

$$\cos A'LB' = \frac{xa \cdot xb + yb \cdot ya + f^2}{La \cdot Lb} = .65605899$$

$$\sin A'LB' = .75470958$$

$$\cos B'LC' = \frac{xb \cdot xc + yb \cdot yc + f^2}{Lb \cdot Lc} = .75197131$$

$$\sin B'LC' = .65919584$$

$$\cos C'LA' = \frac{xc \cdot xa + yc \cdot ya + f^2}{Lc \cdot Ld} = .71193541$$

$$\sin C'LA' = .70224495$$

where

$$La = (xa^2 + ya^2 + f^2)^{1/2} = 114.2868719$$

$$Lb = (xb^2 + yb^2 + f^2)^{1/2} = 111.6440759$$

$$Lc = (xc^2 + yc^2 + f^2)^{1/2} = 108.7169692$$

then

$$\cos \gamma_1 = \frac{\cos A'LB' - \cos B'LC' \cdot \cos C'LA'}{\sin B'LC' \cdot \sin C'LA'} = .26074653$$

and

$$\cos\gamma_2 = \frac{(XA-XC)(XB-XC)+(YA-YC)(YB-YC)+(ZA-ZC)(ZB-ZC)}{A'C'\cdot B'C'}$$

$$= .37533681$$

$$D_1 = \frac{B'C'}{\sin B'LC'} = 11{,}804.34512, \qquad D_2 = \frac{A'C'}{\sin A'LC'} = 11{,}741.38161$$

$$B'C' = [(XB - XC)^2 + (YB - YC)^2 + (ZB - ZC)^2]^{1/2} = 7781.37520$$

$$A'C' = [(XA - XC)^2 + (YA - YC)^2 + (ZA - ZC)^2]^{1/2} = 8245.325950$$

$$A'B' = [(XA - XB)^2 + (YA - YB)^2 + (ZA - ZB)^2]^{1/2} = 8965.04322$$

$$A'B'^2 = 80{,}372{,}000.0.$$

Thus we have computed the minimum data required to form the coefficients of the m-equation.

C. Forming the coefficients of the m-equation and solution of the m-equation by Ferrari's method. The quartic in m is written as follows:

$$A_1m^4 + B_1m^3 + C_1m^2 + E_1m + F_1 = 0.$$

Here the caps denote coefficients and not ground objects. The coefficients are obtained by substituting the constants computed above in the right members of the equalities shown below.

$$A_1 = \frac{D_2}{D_1}\cdot\left(1 - \frac{D_1^2 \sin^2\gamma_2}{A'B'^2}\right)$$

$$B_1 = -2\left[\cos\gamma_1\cos\gamma_2 + \cos A'LB'\cdot\frac{D_2}{D_1}\cdot\left(1 - \frac{2D^2}{A'B'^2}\sin^2\gamma_2\right)\right]$$

$$C_1 = \left[\frac{D_1}{D_2} + \frac{D_2}{D_1} + 4\cos\gamma_1\cos\gamma_2\cos A'LB' - \frac{A'B'^2}{D_1\cdot D_2}\sin^2\gamma_1 \right.$$
$$\left. - (4\cos^2 A'LB' + 2)\left(\frac{D_1D_2}{A'B'^2}\sin^2\gamma_2\right)\right]$$

$$E_1 = -2\left[\cos\gamma_1\cos\gamma_2 + \cos A'LB'\cdot\frac{D_1}{D_2}\cdot\left(1 - \frac{2D_2^2}{A'B'^2}\sin^2\gamma_2\right)\right]$$

$$F_1 = \frac{D_1}{D_2}\cdot\left(1 - \frac{D_2^2\sin^2\gamma_2}{A'B'^2}\right).$$

Numerical value of coefficients

$$A_1 = -\ .486866835$$
$$B_1 = 2.387037152$$
$$C_1 = -\ 3.797357154$$
$$E_1 = 2.373002132$$
$$F_1 = -\ .4761703759.$$

At first, forming the above coefficients may seem to be a very tedious task until one notices that the coefficient A, and the constant term F, are nearly alike, and also that B, and E, are nearly alike. We now write the fourth-degree polynomial with the numerical value of its coefficients:

$$.486866835m^4 - 2.387037152m^3 + 3.797357154m^2 - 2.373002132m + .47617037589 = 0.$$

It is desirable that the coefficient of m^4 be unity; therefore dividing $f(m)$ through by A, we obtain

$$m^4 - 4.90285429m^3 + 7.79958066m^2 - 4.87402706m + .97803001 = 0.$$

According to Descartes rule of signs, $f(m)$ shows four variations and therefore has four, two or no real positive roots. However, we are interested in only one of the four roots. The desired root is real and positive because $m = (LB/LA)$ and both LB and LA are real and positive. In solving the m-equation by Ferrari's solution we end up with a pair of quadratics, both of which must be solved to evaluate the desired root. A simple method of determining which of the roots is the desired root will be described when we have found the four roots. Continuing with the m-equation, let the coefficient of m^3 be a_1, m^2, b_1; m, c_1, and the constant term d_1. Then $b_1m^2+c_1m+d_1$ is transferred to the right side of the equality sign. This gives

$$m^4 - 4.90285429m^3 = - 7.79958066m^2 + 4.87402706m - .97803001$$

$\frac{a^2}{4}{}_1m^2$ is added to each side

$$\frac{a^2}{4}{}_1m^2 = 6.0094950 \cdot m^2$$

$$m^4 - 4.90285429m^3 + 6.009495m^2 = - 1.79008566m^2 + 4.87402706m - .97803001.$$

Now the left side is a perfect square: $(m^2-2.451427145m)^2$. If the right side were a perfect square, the solution would be immediate. Since m is probably rational, we do not expect this. In order to make the right side a perfect square, we add

$$y_1\left(m^2 + \frac{a_1}{2}m\right) + \frac{y_1{}^2}{4}$$

to both sides, which is equivalent to

$$\left(m^2 - 2.451427145m + \frac{y_1}{2}\right)^2$$

on the left side and

$$(y_1-1.79008566)m^2+\left[4.87402706-y(2.451427145)\right]m+\left(\frac{y^2}{4}-.97803001\right)$$

on the right side. We seek to determine y_1 so that the above becomes the square of a linear expression, say l_1m+n. In general, if

$$um^2 + vm + w = (l_1m + n)^2$$

the discriminant

$$v^2 - 4u \cdot w = 0.$$

That is,

$$u = l_1^2, \quad v = 2l_1 \cdot n, \text{ and } w = n^2$$

or

$$4l^2 \cdot n^2 - 4l^2 \cdot n^2 = 0.$$

Thus the right-hand side of the equation will be the square of a linear expression

$$l_1 m + n_1 \quad \text{if} \quad y_1$$

satisfies the equation.

$$[4.87402706 - y(2.451427145)]^2 = 4(y_1 - 1.79008566)\left(\frac{y^2}{4} - .97803001\right)$$

which expanded gives a cubic in y_1.

$$23.7561397816 - y(23.89664448) + y^2(6.009495047)$$
$$= y^3 - (1.79008566)y^2 - (3.91212004)y + 7.003029984.$$

Collecting,

$$y_1^3 - 7.799580707y_1^2 + 19.98452444y_1 - 16.753109798 = 0.$$

This equation is readily solved by Cardan's formulas insofar as any root of y_1 satisfying the cubic is suitable. In conformance with the cubic solution we let

$$P = b' - \frac{a'^2}{3}, \qquad Q = c' - \frac{a' \cdot b'}{3} + \frac{2a'^3}{27}$$

where a', b' are the coefficients of y^2 and y, and c' is the constant term. This gives a cubic of the form $y'^3 + py' + Q = 0$. Then by formula

$$b' = 19.98452444 \qquad c' = -16.753109798$$

$$-\frac{a'^2}{3} = -20.27781974 \qquad a'b'/3 = 51.95697042$$

$$P = -0.29329530 \qquad 35.203860622$$

$$\frac{2a'}{27} - \quad -35.14633146$$

$$Q = 0.057529162$$

or

$$y'^3 - .293295295y' + .059529160 = 0.$$

A root of this equation is most simply calculated trigonometrically

$$y_1 = y' - \frac{a}{3} \qquad y' = 2\sqrt{\frac{-P}{3}} \cdot \cos\frac{\phi}{3}$$

where

$$\cos\phi = \frac{+Q\sqrt{27}}{2P\sqrt{-P}} = .973697344$$

$$\phi = 160°13'02''.915 \qquad \frac{\phi}{3} = 53°24'20''.972$$

$$\cos\frac{\phi}{3} = .59614325 \qquad y' = .3727971219$$

and

$$y' - \frac{a'}{3} = 2.97265736 = y_1.$$

Substituting this value of y_1 back in the discriminant, $V^2 = 4\,uw$

$$u = 1.1825716969$$
$$v = -2.4132258777$$
$$w = 1.23114293$$
$$l = \sqrt{u} = 1.087461124$$
$$n = \sqrt{w} = 1.109568804$$

or

$$n = \frac{-V}{2l} = -1.109568804.$$

Now we may write the pair of quadratics which give the four roots and in particular the one desired root.

General equations:

$$m^2 + \frac{a_1}{2}\cdot m + \frac{y_1}{2} = lm + n$$

$$m^2 + \frac{a_1}{2}\cdot m + \frac{y_1}{2} = -lm - n.$$

Substituting the numerical values of l, η_1 and y,

(1) $m^2 - 2.451427145m + 1.4863286784 = 1.087461124m - 1.109568804$

(2) $m^2 - 2.451427145m + 1.4863286784 = -1.087461124m + 1.109568804.$

Collecting,

$$m^2 - 3.538888269m + 2.5958974824 = 0$$
$$m^2 - 1.363966021m + 0.3767598744 = 0.$$

Solving these two quadratics we obtain the following four roots.

$$m_1 = 2.500905049$$
$$m_2 = 1.037983224$$
$$m_3 = .979205069$$
$$m_4 = .384760952.$$

We note that all four roots are real and positive. Which of the four roots is the required root may be computed by an equation that gives the length of the omitted perspective ray LC' at the same time.

Let the first value be m_1, the second m_2, the third m_3, and the fourth m_4. Usually two of the roots will be sufficiently large or small to make evaluating absurd. Let us suppose that the choice lies between m_2 and m_3.

Since all values of m satisfy the equation

$$A'B'^2 = LA'^2 + LA'^2 m^2 - 2LA'^2 m \cos A'LB',$$

another relation must be selected. For this relation we express LC in terms of LA and LB. $LA \cdot \cos A'LC' + (D_2^2 - LA^2)^{1/2} \sin A'LC' = LB \cdot \cos B'LC' + (D_1^2 - LB^2)^{1/2} \sin B'LC' = LC$. The root m that gives a value of LA and LB that satisfies the above equality is the correct root, and in determining this value of m we automatically obtain LC. In this problem, the desired root was found to be 1.037983224 by the procedure outlined above.

D. Perspective ray lengths.

The lengths of LA and LB may be computed directly.

$$LA' = \frac{A'B'}{(1 + m_2^2 - 2m_2 \cos A'LB')^{1/2}} = \frac{8965.04322}{.845844384}$$

$$LA' = 10,598.9274$$

$$LB' = LA' \cdot m_2 = (10598.9274)(1.037983224) = 11,001.50883$$

LC' remains to be computed.

$$LA' \cos A'LC' + (D_2^2 - LA'^2)^{1/2} \sin A'LC'$$
$$= LB' \cos B'LC' + (D_1^2 - LB'^2)^{1/2} \sin B'LC' = LC$$
$$LC' = 11,093.48998$$

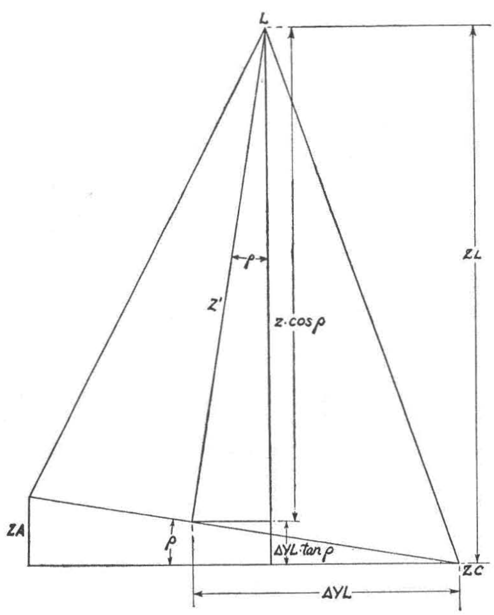

FIG. 2. Section through hillside principal plane.

Summarizing,

$$LA' = 10,598.9274$$
$$LB' = 11,001.50883$$
$$LC' = 11,093.48998.$$

E. Space Coordinates (Figure 2)

If the hillside plane is parallel to the datum plane, the determination of the Z ordinate of L is immediate. We assume this not to be the case. The normal Z' to the hillside plane is computed.

$$Z' = \frac{LA' \cdot LB' \cdot LC'}{D_1 \cdot D_2} \cdot \frac{\sin \gamma_1}{\sin \gamma_2}$$
$$= 9720.85245$$

Z' is inclined an angle ρ to the vertical. ρ is the dihedral angle between the hillside plane and the datum plane. The cosine of ρ is equal to the area of the datum triangle divided by the area of the hillside triangle.

$$\cos \rho = \frac{AC \cdot BC \cdot \sin \gamma_0}{AC' \cdot BC' \cdot \sin \gamma_2} = .99815463$$

where AC, BC are the datum ground lengths and γ_0 is the angle at C in the datum plane.

$$\cos \gamma_0 = \frac{AC^2 + BC^2 - AB^2}{2AC \cdot BC} = .3745910926$$

$$AC = [(XA - XC)^2 + (YA - YC)^2]^{1/2} = 8230.15188$$

$$BC = [(XB - XC)^2 + (YB - YC)^2]^{1/2} = 7778.80454$$

$$AB = [(XA - XB)^2 + (YA - YB)^2]^{1/2} = 8960.02232.$$

The vertical component of Z' is simply the cosine of ρ times Z'.

$$Z_N' = Z' \cos \rho = 9702.082505.$$

The foot of Z_N' is above the datum plane an amount equal to $\tan \rho \cdot \Delta Y'L$, where ΔYL is the horizontal distance from the lowest elevation to the foot of Z' along the hillside plane principal line.

$$\Delta YL = \frac{1}{2}\left[\frac{(A'C'^2 + LC'^2 - LA'^2)}{AC} \sin \phi_{BC} - \frac{(B'C'^2 + LC'^2 - LB'^2)}{BC} \sin \phi_{AC}\right] \frac{\cos^2 \rho}{\sin \gamma_0}$$

$$= 4883.364471$$

where ϕ_{BC} and ϕ_{AC} are the angles in the datum plane at C between the datum trace of the hillside principal line and the datum lines AC and BC.

$$\cos \phi_{AC} = \frac{\tan \rho_{AC}}{\tan \rho} = .998629095$$

$$\cos \phi_{BC} = \frac{\tan \rho_{BC}}{\tan \rho} = .422628905$$

and

$$\tan \rho_{AC} = \frac{ZA - ZC}{AC} = .0607522202$$

$$\tan \rho_{BC} = \frac{ZB - ZC}{BC} = .0257108915$$

finally

$$ZL = Z' \cdot \cos \rho + \Delta YL \tan \rho + ZC$$

$$\begin{aligned} Z' \cos \rho &= 9702.91381 \\ \Delta YL \cdot \tan \rho &= 297.08251 \\ ZC &= 0.00000 \\ \hline ZL &= 9999.99632 \end{aligned}$$

Now XL and YL are computed. Let

$$XL = X'L + XC$$
$$YL = Y'L + YC$$

then

$$X'L = \frac{1}{2 \sin \gamma_0} \left[\mu_1(YB - YC) - \mu_2(YA - YC)\right] = 2402.67931$$

$$Y'L = \frac{1}{2 \sin \gamma_0} \left[\mu_1(XB - XC) - \mu_2(XA - XC)\right] = 4158.44511$$

where

$$\mu_1 = \frac{AC^2 + RC^2 - RA^2}{AC \cdot BC} \qquad \mu_2 = \frac{BC^2 + RC^2 - RB^2}{AC \cdot BC}$$

and

$$RA^2 = LA'^2 - (ZL - ZA)^2 = 22{,}087{,}332.33$$
$$RB^2 = LB'^2 - (ZL - ZB)^2 = 24{,}993{,}269.06$$
$$RC^2 = LC'^2 - (ZL - ZC)^2 = 23{,}065{,}593.94$$

Then since $XL = X'L + YC$ and $YL = Y'L + YC$,

$$XL = 12{,}402.67931$$
$$YL = 14{,}158.44511$$
$$ZL = 9{,}999.9963$$

F. Space Orientation

At this point the determination of the angular elements of exterior orientation is extremely simple. The direction cosines of the photograph nadir point referred to the principal point are obtained from the simultaneous solution of three cosine-of-the-angle-between-two-lines equations.

$$La \cdot \cos MA = xa \cdot \cos \alpha_n + ya \cos \beta_n + f \cdot \cos t$$
$$Lb \cdot \cos MB = xb \cdot \cos \alpha_n + yb \cos \beta_n + f \cdot \cos t$$
$$Lc \cdot \cos MC = xc \cdot \cos \alpha_n + yc \cos \beta_n + f \cdot \cos t$$

The photograph pyramid edges have been previously determined.

$$La = 114.2868719$$
$$Lb = 111.6440759$$
$$Lc = 108.7169692$$

$$\cos MA = \frac{ZL - ZA}{LA'} = \frac{9499.9963}{10598.9274} = .8963167631$$

$$\cos MB = \frac{ZL - ZB}{LB'} = \frac{9799.9963}{11001.50883} = .8907865685$$

$$\cos MC = \frac{ZL - ZC}{LC'} = \frac{9999.9963}{11093.48998} = .901429245$$

Setting up the numerical values of the equations and solving by successive division and subtraction, we obtain the following set of data:

	$f\cdot\cos t$ +	$x\cdot\cos\alpha_n$ +	$y\cdot\cos\beta_n$ =	$L\cdot\cos M$
a.	100	−46.5384847	29.92755493 =	102.437239086
b.	100	46.3825116	17.69356712 =	99.451043264
c.	100	− 2.5773321	−42.57624638 =	98.000655486
	0	−92.92099633	12.23398781 =	2.986195822
	0	−43.96115259	72.50380131 =	4.436583600
		−1	.131660101 =	.0321369328
		−1	1.649269799 =	.1009205477
		0	1.517609698 =	.0687836149
			$\cos\beta_n$ =	.0453236527
		(by substitution of $\cos\beta_n$)	$\cos\alpha_n$ =	− .0261696166
			$\cos t$ =	.99862919

and

$$\tan s = \frac{\cos\alpha_n}{\cos\beta_n} = -\ .577394253$$

$$t = 3°0'01''.36$$

$$s = 329°59'53''.196.$$

The azimuth of the principal lines is computed by the tangent of an angle-between-two-lines-formula referred to the radials from the ground nadir

$$\tan Az = \frac{\tan\theta_A \pm \tan\theta_a}{1 \mp \tan\theta_A\cdot\tan\theta_a}$$

where

$$\tan\theta_A = \frac{XL - XA}{YL - YA}$$

$$\tan\theta_a = \frac{x'a}{y'\cdot\cos t}$$

and

$$x'a = xa\cos s - ya\sin s$$

$$y'a = xa\sin s + ya\cos s.$$

Thus each control point and image point provide a set of data for an azimuth computation. Rotating the photo coordinates through angle s and translating the origin along the principal line $f\cdot \tan t$ we obtain the following values for the new coordinates.

	a.	b.	c.
x'	−25.33809974	49.01496342	−23.52134664
y'	43.94667800	−13.11124895	−40.82410716

then

$\tan\theta_a = -.577356066$	$\tan\theta_b = 3.743521745$	$\tan\theta_c = .576954055$
$\tan\theta_A = -.000030606$	$\tan\theta_B = -1.001622824$	$\tan\theta_C = 1.730753286$
$\tan A_Z = .5773458637$	$\tan A_Z = .577290878$	$\tan A_Z = .577313801$
$A_Z = 29°59'59''.317$	$A_Z = 29°59'50''.811$	$A_Z = 29°59'54''.358$

mean

$$\text{Az.} = 29°59'54''.828.$$

Comparing the solved-for value of the unknowns with the correct value, the following differences or errors are noted:

Variable	Error	True Value
LA'	$-.0055$ (feet)	10,598.9339
LB'	.0089	11,001.49989
LC'	$-.0036$	11,093.49358
XL	$-.0158$	14,158.46096
YL	.0136	12,402.66566
ZL	$-.0036$	10,000.0
t	$1''.36$	3°
S	$6''.804$	330°
A_Z	$-5''.172$	30°

These errors are due entirely to the inability to preserve the absolute value of any set of numbers after compound arithmetical operations have been performed on them with a 10-bank calculating machine.

III. APPENDIX

A. Most of the analytical geometry formulas employed in the computation of space coordinates and space orientation are self-evident. Ferrari's solution of the biquadratic and Cardan's solution of the cubic may be found in any textbook on the theory of equations. The fourth degree equation in m, the reduction to a second-degree equation in m when four control points are used, and the equality in the omitted perspective ray length may not be so evident. The derivation of these equations is given to provide additional insight into the mathematics of the explicit solution.

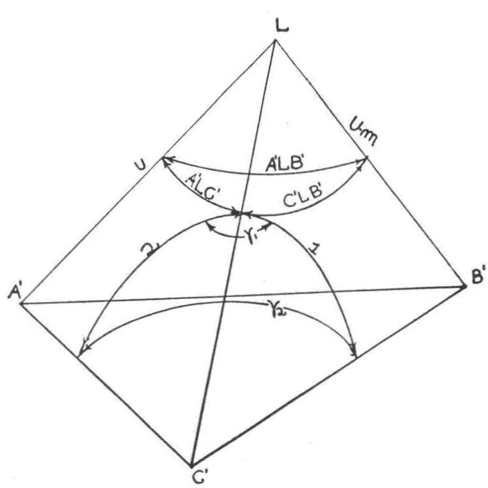

FIG. 3.

B. Derivation of m-equation. The elements of the derivation are illustrated in Figure 3. The picture plane and datum plane are omitted from the space pyramid to make more clear the relation of the basic parts of the pyramid in the development of the equation.

Let C' be the vertex of spherical triangle $A_SB_SC_S$ whose sides are 1, 2, and γ_2, and whose angle at C_S is γ_1. Now by the law of cosines in spherical trigonometry, $\cos\gamma_2 = \cos 1 \cos 2 + \sin 1 \sin 2 \cos\gamma_1$.

Let $LA' = U$ and $LB' = U\cdot m$

By the law of sines in plane trigonometry,

$$\frac{\sin 1}{U \cdot m} = \frac{\sin B'LC'}{B'C'} \quad \text{and} \quad \frac{\sin 2}{U} = \frac{\sin A'LC'}{A'C'}.$$

The chord of any circle divided by the sine of the inscribed angle is equal to the diameter of the circle. That is,

$$D_{AC} = \frac{A'C'}{\sin A'LC'} \qquad D_{BC} = \frac{B'C'}{\sin B'LC'}$$

therefore,

$$\frac{\sin 1}{U \cdot m} = \frac{1}{D_1}, \qquad \frac{\sin 2}{U} = \frac{1}{D_2}$$

or

$$\sin 1 = \frac{U \cdot m}{D_1}, \qquad \sin 2 = \frac{U}{D_2}$$

where

$$D_1 = D_{BC} \quad \text{and} \quad D_2 = D_{AC}.$$

Having an expression for the sine, we may write an expression for the cosines:

$$\cos 1 = \frac{(D_1^2 - U^2 m)^{1/2}}{D_1}, \qquad \cos 2 = \frac{(D_2^2 - U^2)^{1/2}}{D_2}$$

γ_1 is a dihedral angle at the intersection of planes $A'LC'$ and $C'LB'$. Since γ_1 is a dihedral angle, it may be computed with the apex angles.

$$\cos \gamma_1 = \frac{\cos A'LB' - \cos A'LC' \cdot \cos B'LC'}{\sin A'LC' \cdot \sin B'LC'}$$

and the face angles are computed from the image coordinates and camera constants.

γ_2 is the angle at C in the hillside plane between slope lengths $C'A'$ and $C'B'$

$$\cos \gamma_2 = \frac{(XA-XC)(XB-XC)+(YA-YC)(YB-YC)+(ZA-ZC)(ZB-ZC)}{A'C' \cdot B'C'}.$$

Recapitulating, we have an expression for *sin 1, cos 1, sin 2, cos 2,* and the values of γ_1 and γ_2 are known. Substituting in the original spherical cosine formula. we have

$$\cos \gamma_2 = \frac{(D_1^2 - U^2 m^2)^{1/2}(D_2^2 - U^2)^{1/2} + U^2 m \cos \gamma_1}{D_1 D_2}.$$

Transposing,

$$\cos \gamma_2 D_1 D_2 - U^2 m \cos \gamma_1 = (D_1^2 - U^2 m^2)^{1/2}(D_2^2 - U^2)^{1/2}$$

Squaring,

$$\cos^2 \gamma_2 D_1^2 \cdot D_2^2 - U^2 m 2 \cos \gamma_1 \cos \gamma_2 D_1 D_2 + U^4 m^2 \cos^2 \gamma_1$$
$$= D_1^2 \cdot D_2^2 - D_1^2 \cdot U^2 - D_2^2 U^2 m^2 + U^4 m^2.$$

Two trigonometric identities are substituted.

$$D_1^2D_2^2 - D_1^2D_2^2 \cos^2 \gamma_2 = D_1^2D_2^2 \sin^2 \gamma_2$$
$$U^4m^2 - U^4m^2 \cos^2 \gamma_2 = U^4m^2 \sin^2 \gamma_1.$$

Substituting these identities and factoring out the U^2 terms, we obtain

$$U^2(D_1^2 + D_2^2m^2 - 2 \cos \gamma_1 \cos \gamma_2 D_1D_2m) - U^4m^2 \sin^2 \gamma_1 = D_1^2D_2^2 \sin^2 \gamma_2.$$

Now

$$A'B'^2 = LA'^2 + LB^2 - 2LA' \cdot LB' \cos A'LB'.$$

Let $LA' = U$ and $LB' = U \cdot m$

$$\begin{aligned} A'B'^2 &= U^2 + U^2m^2 - 2U^2m \cos A'LB' \\ &= U^2(1 + m^2 - 2m \cos A'LB') \\ U^2 &= \frac{A'B'^2}{1 + m^2 - 2m \cos A'LB'}. \end{aligned}$$

Substituting the right-hand expression for U^2,

$$A'B'^2 \frac{(D_1^2 + D_2^2m^2 - 2 \cos \gamma_1 \cos \gamma_1 D_1D_2m)}{1 + m^2 - 2 \cos A'LB'} - \frac{A'B'^4 \sin^2 \gamma_1 m^2}{(1 + m^2 - 2m \cos A'LB')^2}$$
$$= D_1D_2 \sin^2 \gamma_2.$$

Multiplying both sides of the equation by

$$\frac{(1 + m^2 - 2m \cos A'LB')^2}{A'B'^2},$$

we obtain

$$\begin{aligned} &m^4(D_2^2) - m^3(2D_2^2 \cos A'LB' + 2 \cos \gamma_1 \cos \gamma_2 D_1D_2) \\ &\quad + m^2(D_1^2 + D_2^2 + 4 \cos \gamma_1 \cos \gamma_2 \cos A'LB' \cdot D_1D_2 - A'B'^2 \sin^2 \gamma_1) \\ &\quad - m(2D_1^2 \cos A'LB' + 2 \cos \gamma_1 \cos \gamma_2 D_1D_2) + D_1^2 \\ &= m^4\left(\frac{D_1^2D_2^2}{A'B'^2} \sin^2 \gamma_2\right) - m^3\left(4 \cos ALB \frac{D_1^2D_2^2}{A'B'^2} \sin^2 \gamma_2\right) \\ &\quad m^2(2 + 4 \cos^2 A'LB')\left(\frac{D_1^2D_2^2}{A'B'^2} \sin^2 \gamma_2\right) - m\left(4 \cos A'LB' \frac{D_1^2D_2^2}{A'B'^2} \sin^2 \gamma_2\right) \\ &\quad + \frac{D_1^2D_2^2}{A'B'^2} \sin^2 \gamma_2. \end{aligned}$$

Collecting the coefficients of like powers and the constant terms,

$$\begin{aligned} &m^4\left(D_2^2 - \frac{D_1^2D_2^2}{A'B'^2} \sin^2 \gamma_2\right) \\ &\quad - m^3 2\left(D_2^2 \cos A'LB' + \cos \gamma_1 \cos \gamma_2 D_1D_2 - 2 \cos A'LB' \frac{D_1^2D_2^2}{A'B'^2} \sin^2 \gamma_2\right) \\ &\quad + m^2[D_1^2 + D_2^2 + 4 \cos \gamma_1 \cos \gamma_2 \cos A'LB'D_1D_2 - A'B'^2 \sin^2 \gamma_1 \end{aligned}$$

$$- (2 + 4 \cos^2 A'LB') \frac{(D_1{}^2D_2{}^2)}{A'B'^2} \sin^2 \gamma_2 \Big)\Big]$$

$$- m2 \left(D_1{}^2 \cos ALB + \cos \gamma_1 \cos \gamma_2 D_1 D_2 - 2 \cos A'LB' \frac{D_1{}^2 D_2{}^2}{A'B'^2} \sin^2 \gamma_2 \right)$$

$$+ \left(D_1{}^2 - \frac{D_1{}^2 D_2{}^2}{A'B'^2} \sin^2 \gamma_2 \right) = 0.$$

Simplifying further and dividing through by D_1D_2, we have a fourth-degree equation in m.

$$A_1 m^4 + B_1 m^3 + C_1 m^2 + E_1 m + F_1 = 0$$

where

$$A_1 = \frac{D_2}{D_1} \left(1 - \frac{D_1{}^2}{A'B'^2} \sin^2 \gamma_2 \right)$$

$$B_1 = - 2 \left[\cos \gamma_1 \cos \gamma_2 + \cos A'LB' \cdot \frac{D_2}{D_1} \left(1 - 2 \frac{D_1{}^2}{A'B'^2} \sin^2 \gamma_2 \right) \right]$$

$$C_1 = \Big[\frac{D_1}{D_2} + \frac{D_2}{D_1} + 4 \cos \gamma_1 \cos \gamma_2 \cos A'LB' - \frac{A'B^2}{D_1 D_2} \sin^2 \gamma_1$$

$$- (2 + 4 \cos^2 A'LB') \left(\frac{D_1 D_2}{A'B'^2} \sin^2 \gamma_2 \right) \Big]$$

$$E_1 = - 2 \left[\cos \gamma_1 \cos \gamma_2 + \cos A'LB' \frac{D_1}{D_2} \left(1 - \frac{2D_2{}^2}{AB'^2} \sin^2 \gamma_2 \right) \right]$$

$$F_1 = \frac{D_1}{D_2} \left(1 - \frac{D_2{}^2}{A'B'^2} \sin^2 \gamma_2 \right).$$

C. Reduction to a second degree equation when four points are used. Let there be a fourth point G (Figure 4) on the opposite side of plane $A'LB'$ from C'. Let the angles at G' corresponding to γ_1 and γ_2 be δ_1 and δ_2 and the diameters corresponding to D_1 and D_2 be D_3 and D_4. Prior to the substitution of

$$U^2 = \frac{A'B'^2}{1 + m^2 - 2m \cos A'LB'}$$

in the previous derivation the following pair of equations are written:

$$U^2(D_1{}^2 + D_2{}^2 m^2 - 2 \cos \gamma_1 \cos \gamma_2 D_1 D_2 m) - U^4 m^2 \sin^2 \gamma_1 = D_1{}^2 D_2{}^2 \sin^2 \gamma_2$$

$$U^2(D_3{}^2 + D_4{}^2 m^2 - 2 \cos \delta_1 \cos \delta_2 D_3 D_4 m) - U^4 m^2 \sin^2 \delta_1 = D_3{}^2 D_4{}^2 \sin^2 \delta_2.$$

Eliminating the $U^4\, m^2$ term we obtain

$$U^2 \big[(D_1{}^2 + D_2{}^2 m^2 - 2 \cos \gamma_1 \cos \gamma_2 D_1 D_2 m) \sin^2 \delta_1$$

$$- (D_3{}^2 + D_4{}^2 m^2 - 2 \cos \delta_1 \cos \delta_2 D_3 D_4 m) \sin^2 \gamma_1 \big]$$

$$= D_1{}^2 D_2{}^2 \sin^2 \gamma_2 \sin^2 \delta_1 - D_3{}^2 D_4{}^2 \sin^2 \delta_2 \sin^2 \gamma_1.$$

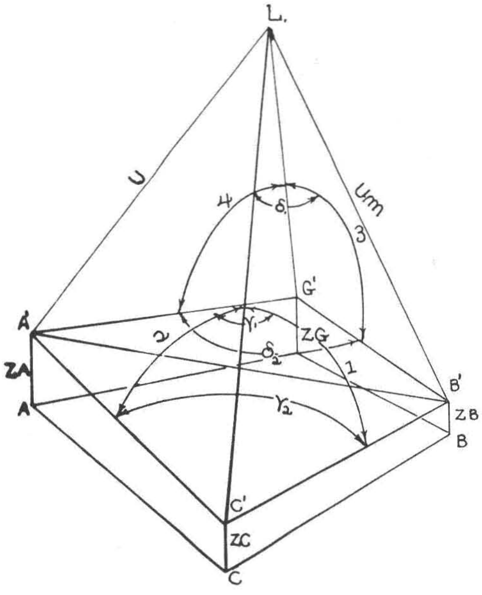

FIG. 4.

Now substituting

$$U^2 = \frac{A'B'^2}{1 + m^2 - 2m \cos A'LB'}$$

we have a quadratic in m

$$\eta m^2 + \xi m + w = 0$$

where

$$\eta = A_2 - K$$

$$\xi = B_2 + 2 \cos A'LB' \cdot K$$

$$w = C_2 - K$$

and

$$A_2 = (D_2^2 \sin^2 \delta_1 - D_4^2 \sin^2 \gamma_1)$$

$$B_2 = 2(\cos \delta_1 \cos \delta_2 \sin^2 \gamma_1 D_3 D_4 - \cos \gamma_1 \cos \gamma_2 \sin^2 \delta_1 D_1 D_2)$$

$$C_2 = (D_1^2 \sin^2 \delta_1 - D_3^2 \sin^2 \gamma_1)$$

$$K = \left(\frac{D_1^2 D_2^2}{AB^2} \sin^2 \gamma_2 \sin^2 \delta_1 - \frac{D_3^2 D_4^2}{AB^2} \sin^2 \gamma_1 \sin^2 \delta_2\right).$$

Again we note that the A_2 and C_2 terms are nearly alike while the K term is repeated. Solution of the quadratic in m here gives the additional required data for an ex)licit four-point solution that is completely general in that no special conditions are imposed on the geometry of the control points, such as defining a single plane. Just as often as not four control points will be available in which case the tedious solution of the quartic is avoided. The plot of the quartic and the equivalent pair of quadratics are shown in Figure 5 for general interest.

D. The equality in the omitted point. Reference is made to Figure 3.

$$\sin 1 = \frac{LB'}{D_1}, \qquad \sin 2 = \frac{LA'}{D_2}.$$

By the law of the sum and difference of sines of two angles.

$$\sin LB'C' = \sin (1 + A'LC')$$

$$\sin LA'C' = \sin (2 + B'LC')$$

or

$$\sin LB'C' = \frac{LB'}{D_1} \cos B'LC' + \frac{(D_1^2 - LB'^2)^{1/2}}{D_1} \sin B'LC'$$

$$\sin LA'C' = \frac{LA'}{D_2} \cos A'LC' + \frac{(D_2^2 - LA'^2)^{1/2}}{D_2} \sin A'LC'$$

and

$$LC' = D_2 \sin LA'C' = D_1 \sin LB'C'$$

or

$$\begin{aligned} LC' &= LB' \cos B'LC' + (D_1^2 - LB'^2)^{1/2} \sin B'LC' \\ &= LA' \cos A'LC' + (D_2^2 - LA'^2)^{1/2} \sin A'LC'. \end{aligned}$$

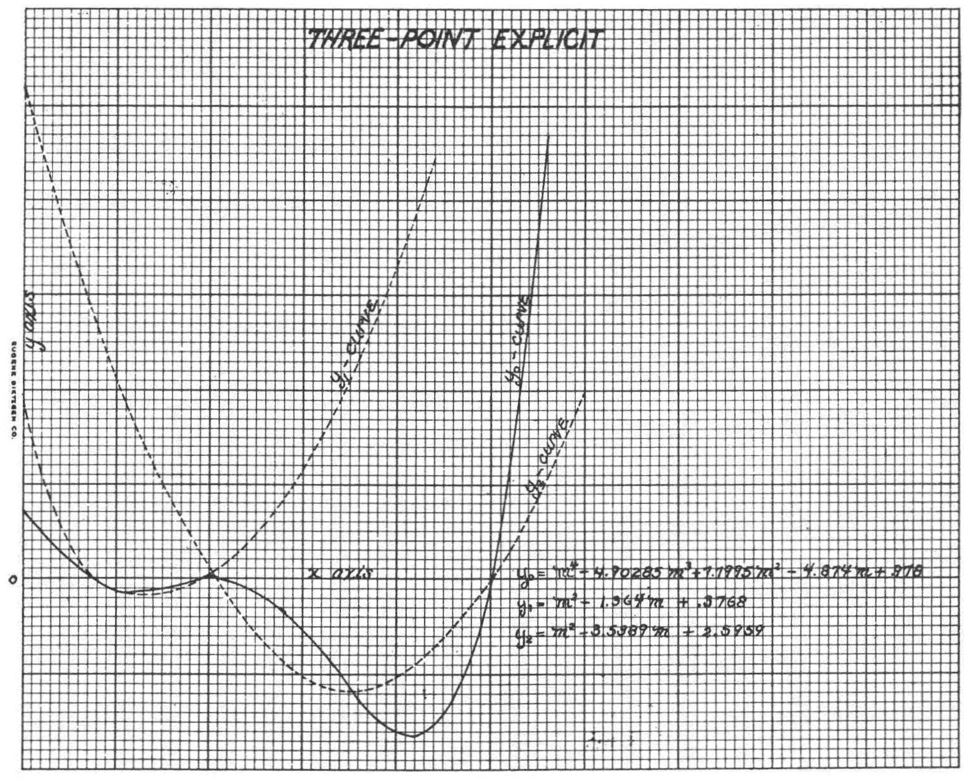

FIG. 5.

NEWS NOTE

AERIAL PHOTOS FOR ATOMIC ENERGY COMMISSION

Aerial photography of a several hundred square mile area for a new facility of the Atomic Energy Commission near Arco, Idaho was completed by Aero Service Corporation of Philadelphia within eight days after the recent contract award. Speed performance of the aerial phase of the survey was necessary to complete this before snowfall.

Compilation of the precise photo mosaics for the area is moving ahead rapidly. The AEC contract also calls for Aero Service to deliver topographic maps, compiled at a 10 foot contour interval and horizontal scale of 1 inch equals 1,000 feet.