THE GEOMETRY OF PHOTORECTIFICATION

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PART I—INTRODUCTION

NO device has yet been perfected for maintaining the aerial camera in an absolutely vertical position at the instant of exposure in flight. In consequence, even the so-called vertical photographs, despite every effort and precaution to the contrary, are almost invariably characterized by slight distortions and scalar differences introduced by small angular deviations between the camera axis and a truly vertical line.

Optical rectification is the process of projecting the image of a tilted photograph into a horizontal reference plane. The photographic print produced by such a projection will possess all the characteristics of a vertical photograph taken at the same position in space.

Numerous machines have been devised in the course of the last half-century to accomplish this result, and there is no dearth of literature descriptive of the operating techniques applicable to any specific apparatus. Singularly lacking, however, is any generalized exposition of those relationships which must be maintained in any rectifying device, irrespective of its design, if a theoretically correct horizontalized projection is to be obtained.

It is the purpose of this writing, therefore, to develop a single consistent and easily comprehensible theorem of such universal application that it may serve both to explain the principles of presently known methods of rectification and to establish simple criteria by which the validity of suggested procedures may be examined. To this end the geometrical approach has been considered most appropriate, since every student or practitioner of photogrammetry may be presumed to possess an adequate knowledge of the elements of plane geometry and trigonometry, and is therefore adhered to throughout.

PART II—THE FUNDAMENTALS OF PERSPECTIVE PROJECTION AS APPLIED TO AERIAL PHOTOGRAPHS

Accepting the hypothesis that any graphical representation of the relative positions of ground points constitutes a map, a single aerial photograph is indeed a very complete map of a small section of the earth's surface. On the other hand, from the viewpoint that a map must necessarily be an orthographic projection of a portion of the earth's surface, or at least some projection approximating that ideal as nearly as may be, an aerial photograph can never be regarded as a true map unless the terrain depicted should happen to be a perfect plane, for an aerial photograph is always a perspective projection.

The presence of relief in the ground therefore introduces into the aerial photograph image displacements quite independent of the relative horizontality of the photographic plate. To illustrate, Figure 1 depicts a horizontal plane containing four objects, A, B, C and D, the elevations of which vary as do those of all points in accidented terrain. Point D lies in the datum plane. The map positions for all these points must lie in the datum plane, with the result that true distances between them are measured by the distances DA', $A'B'$, $B'C'$ and $C'D$, which form a square. Upon this horizontal plane is superimposed a truly vertical photograph having the same scale as the square. If each of the objects forming this square were of the same elevation, their images would also form a square on the photograph. But the variation in elevation of the four objects causes their images to appear at A'', B'', C'' and D on the photograph, and the square formed by the objects on the horizontal ground or datum plane is thus distorted into the figure $A''B''C''D$ on the photograph.

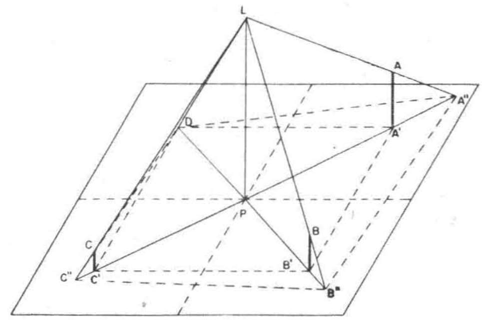


FIG. 1. A horizontal plane containing four objects, A, B, C, and D, the elevation of which vary as do those of all points in accidented terrain.

It is apparent from Figure 1 that the image displacement due to relief is radial from the plumb or nadir point P: an object appearing at the nadir point will not be displaced on the photograph regardless of the relief of the ground or the datum plane which is used. Points higher than the datum plane are evidently displaced radially outward from the nadir point, and points lower are displaced radially inward. In mapping, inward displacement rarely occurs since sea level is ordinarily selected as the datum plane and ground positions are generally higher than sea level.

Investigation of the image displacements resulting from tilt introduces the problem of perspective projection of features from a horizontal to an inclined plane. Figure 2 shows a bundle of rays cut by two planes, I and II, which intersect along a line SI' at an angle t. Plane I may be considered as the plane of

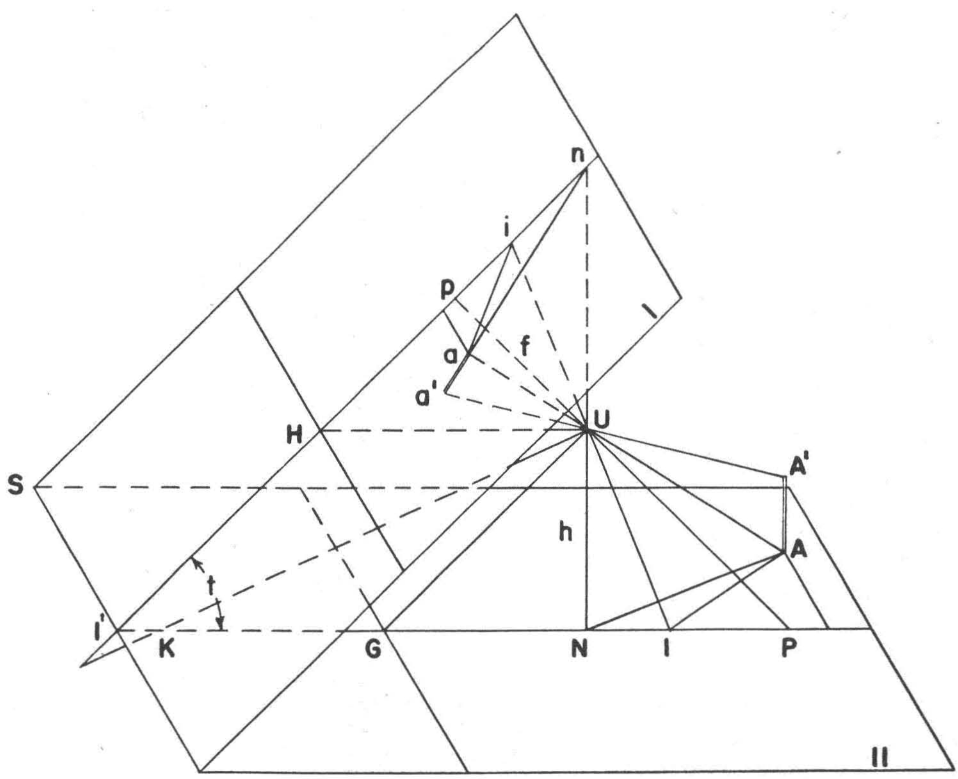


FIG. 2. A bundle of rays cut by two planes, I and II, which intersect along a line SI' at an angle t.

the oblique photograph, and plane II as the object plane, or plane of the ground photographed. Point U, the perspective center, is analogous to the lens of the aerial camera. The foot of the perpendicular from U to plane I falls at the point p, which is the principal point of the photograph. The foot of the perpendicular from U to plane II falls at point N, the nadir point of the exposure station. The lengths of the two perpendiculars are f and h, the perspective distance (or, for all practical purposes, the focal length) of the aerial camera and the flight height, respectively.

The principal plane of the system is the vertical plane containing the camera axis Up. This plane, UNI', is perpendicular to the planes I and II, and therefore also to their intersection, line SI'. The intersection of the principal plane with the plane of the photograph establishes the principal line of the photograph. A line drawn through point U in the principal plane parallel to plane I will strike plane II at G, and another line similarly drawn parallel to plane II will intersect plane I at H. Then lines through either H or G parallel to SI' contain all the infinitely distant points of the other plane, and hence the line through H parallel to the intersection of planes I and II and perpendicular to the principal plane is the true horizon on plane I and contains the images of all infinitely distant points on plane II.

The bisector of the interior angle, NUP or nUp, between the perpendiculars to the planes I and II, intersects the planes at points i and I, respectively. The bisector of the exterior angle, NUp or PUn, between these perpendiculars intersects planes I and II at k and K respectively. These four points, i, I, k and K, are known as the isocenters. In the plane of the photograph, then, there are two isocenters, but usually only one will appear within the area of the photograph. Any point such as A in plane II has a corresponding projected image position, a, in plane I. Such pairs of points as A and a are termed homologous points.

The isocenter of the aerial photograph possesses the unique property that angles between rays therefrom to any images on the photograph are precisely equal to the corresponding angles on the object plane, regardless of the magnitude of the tilt. Proof of this relationship may be demonstrated by reference to Figure 2. wherein it will be seen that, since Ii is the bisector of the equal opposite angles NUP and nUp, angles NUI and pUi are equal, making the right triangles NUI and pUi similar. Angles NIU and piU are therefore equal, and line Ii intersects planes I and II at the same angle. Then angles aip and AIP are necessarily equal by symmetry about the line Ii. This angle-true relationship is, however, valid only when the ground points involved lie in the datum plane, such as A and P. If a point, such as A', lies in a different plane, this angle-true relationship no longer holds. Consequently, in a tilted photograph of ground with relief, there is no point having angle-true properties.

The point A' in Figure 2 may be assumed to be any point on the ground higher than the datum plane. Its orthographic projection on the datum plane is A, whose image on the photograph will appear at point a. The image of A' itself will fall at a', on the line radial from the nadir point n through a. That this is true is obvious, as the plane containing the line Nn and the point A will be vertical, and will also contain the point A'. The intersection of this vertical plane with plane I is in the line na, and therefore the image of A' must likewise fall on this line of intersection, or at a'. However, in a tilted photograph, the position of the nadir point and the elevation of any point above or below the datum are indeterminate from the photograph alone, and therefore only the image of that point is in evidence. In Figure 2 that image is a'; if a ray is drawn from i to a', the resulting angle between lines ia' and ip would obviously not be equal to the corresponding angle measured at point I on the ground.

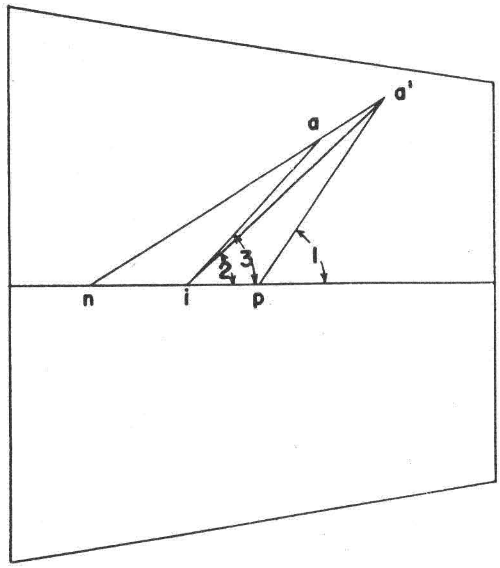
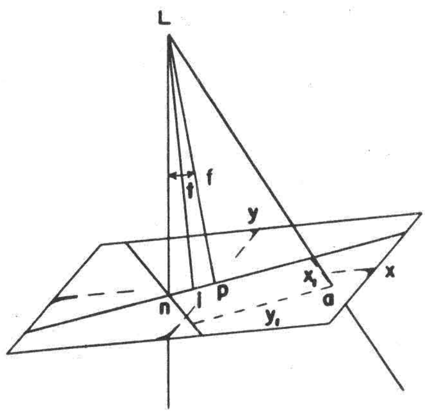


Fig. 3

Fig. 4



L = Perspective center
p = Principal point
n = Nadir point
i = Isocenter
a = An image
nip = Principal line
x, y = Fiducial marks
x_1, y_1 = Coordinates of point a referred to the principal line as X-axis and to a perpendicular thereto through n as Y-axis
Lp = Perspective distance = f
t = Tilt

Figure 3 is a plan view of this condition. If a tilted aerial photograph is used as though it were truly vertical, angles would be measured from the principal point, as angle 1. Theoretically, angle 3 is the correct one, but practically, since there is no evidence of position other than that of a' on the photograph, there is no point on the photograph from which angular measurement is true when the effects of relief and tilt are combined.

It is possible, as an academic exercise, to determine both the position in plan and the elevation of any point imaged on the tilted aerial photograph by purely analytical methods. Such methods are, however, much too tedious and complex to be of utility in photogrammetric mapping, for which relatively simple and expeditious procedures are primary requisites. This consideration makes evident the necessity for rectification of the aerial photograph, since thereby the aberrations, both angular and scalar, introduced by tilt may be entirely eliminated, and that perspection regained which would have been obtained had the axis of the aerial camera been truly vertical at the instant of exposure. Relief distortions of course cannot be so removed, but the image displacements occasioned by relief are easily calculable and, in fact, are employed in all stereoscopic mapping operations as the means of determining relative differences in elevation.

It should be pointed out that Figure 2 is in one sense misleading, since it conveys the impression that the principal line is parallel to the edges of the tilted photograph and hence also to the standard rectangular coordinate axes of the photograph as established by the fiducial marks and having the principal point as origin. This condition seldom exists in practice; a more typical appearance of the tilted photograph is presented in Figure 4. In what follows, the coordinate axes of reference in the plane of the photograph will be the principal line as the axis of abscissae and a perpendicular thereto as the axis of ordinates. The origin may be chosen quite arbitrarily, but the nadir point is most convenient and will be so employed

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PART III—THE GEOMETRIC PRINCIPLES OF RECTIFICATION

Refer now to Figure 5a, wherein the plane of the paper represents the principal plane of an aerial photograph whose principal line appears as AB. A perpendicular to AB at p, the principal point of the photograph, of length f equal to the principal distance of the aerial camera, locates the perspective center at point U. The lens height at a given map scale may be represented by h, and the line IR a distance h below point U is then the trace of the map plane in the principal plane of the photograph. The perpendicular to IR through U

locates n and n_0, the nadir points of the photograph and map plane respectively, and the bisector of angle pUn locates i and $i_,$, the respective isocenters.

The tilt angle t of the photograph is the angle between the photograph and map planes measured in the principal plane, and the angle BIR between the map plane trace IR and the principal line of the photograph AB, extended, is therefore equal to t.

Through the point U pass a plane parallel to the map plane intersecting the plane of the photograph at H. Now, as was indicated in the preceding section concerning the elements of perspective projection, the point H will be the vanishing point of all lines imaged on the photograph parallel to the principal line on the map, and a line through H perpendicular to the principal plane will be the horizon of the photograph. Observe that angle BHU = angle $BIR = t$.

The map plane IR is of course a representation at some greatly reduced scale of the ground plane of which AB is a photograph, and the reprojection of AB through the perspective center U onto the map plane is then a rectification of the tilted photograph which may be produced at any scale desired simply by varying the distance h.

A rectifying camera which operated in this manner would however be capable at best of only approximate solutions at small angles of tilt since, as will be demonstrated later herein, conjugate focus relationships between object and image planes would not be maintained.

The problem, therefore, is to discover the locus of position of the perspective center, and the location of the projection plane corresponding to any position taken by the perspective center, which will produce a true rectification while permitting the use of the actual aerial negative and a projection lens of whatever focal length may be most convenient. The method of determination which will be employed in this investigation will be to find what scale relationship must obtain between the negative and a known correct horizontal projection and then to find the possible positions of the perspective point and the projection plane which will maintain this scale relationship.

Referring to Figure 5a, triangle HUn is similar to triangle In_0n, whence

$$\frac{In_0}{HU} = \frac{In}{Hn}$$

$$In_0 = HU \times \frac{In}{Hn} \cdot \qquad (1)$$

Also, triangle HUp is similar to triangle Ip_0p, and

$$\frac{Ip_0}{HU} = \frac{Ip}{Hp}$$

$$Ip_0 = HU \times \frac{Ip}{Hp} \cdot \qquad (2)$$

Subtracting (1) from (2),

$$Ip_0 - In_0 = HU\left(\frac{Ip}{Hp} - \frac{In}{Hn}\right).$$

But $Ip_0 - In_0 = n_0p_0$; therefore

$$n_0p_0 = HU\left(\frac{Ip}{Hp} - \frac{In}{Hn}\right). \qquad (3)$$

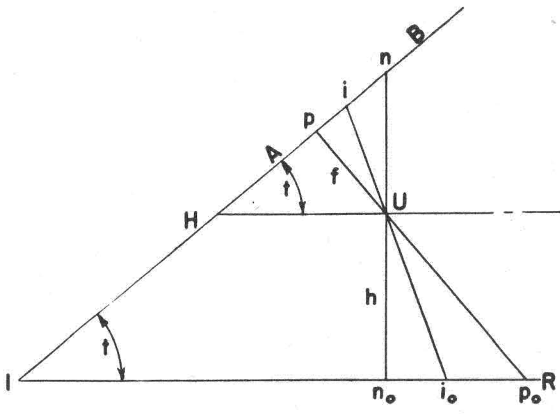

Fig. 5a

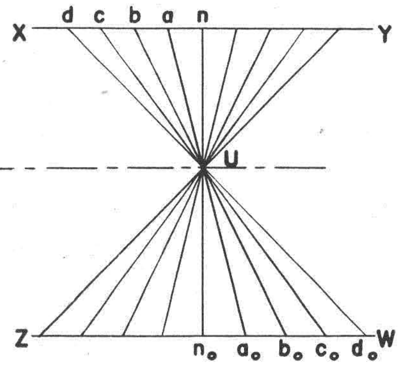

Fig. 5b

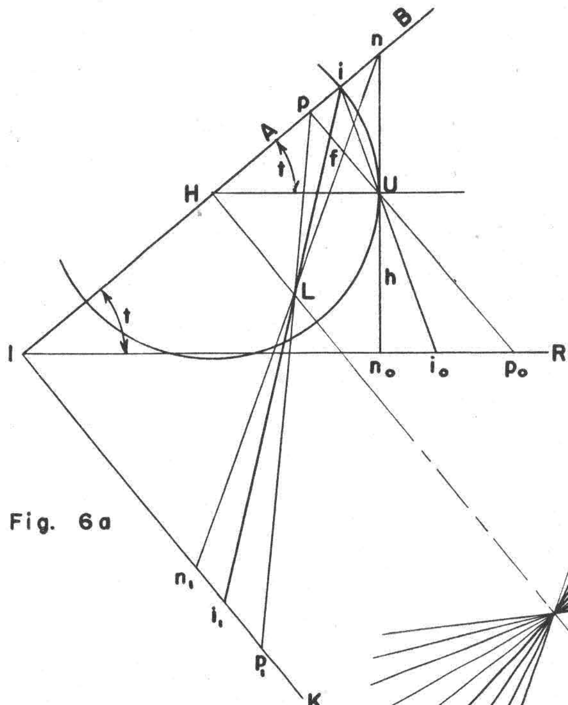

Fig. 6a

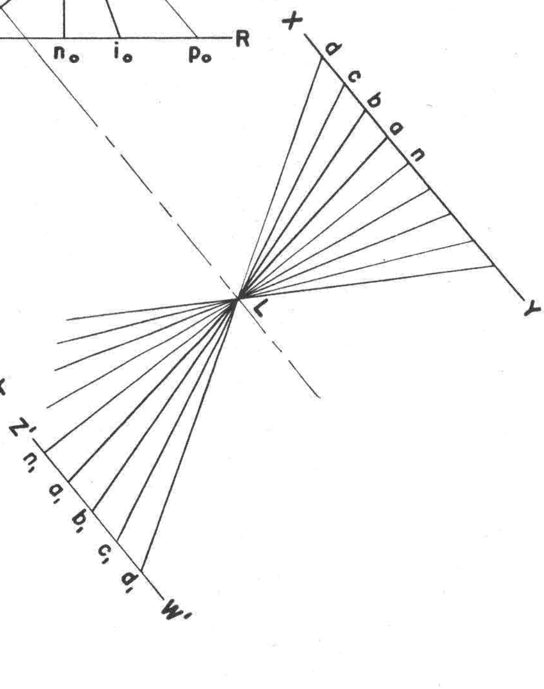

Fig. 6b

Through line nn_0 pass a plane perpendicular to the principal plane. The plane so constructed will contain U, and the intersections of this plane with the negative and map planes, respectively, will be lines XY and ZW, each perpendicular to nn_0 and consequently parallel to each other. This plane is shown in Figure 5b. Let XY be divided, in both directions from n, into equal lengths indicated by a, b, c, d, etc., and from these points of equal division draw rays through U to cut ZW at a_0, b_0, c_0, d_0, etc.

Any triangle such as triangle cUb will be similar to its projection c_0Ub_0, whence

$$\frac{c_0b_0}{cb} = \frac{Un_0}{Un}.$$

From the similar triangles HUn and In_0n in Figure 5a

$$\frac{nn_0}{Un} = \frac{In}{Hn}$$

and, by proportion

$$\frac{nn_0 - Un}{Un} = \frac{In - Hn}{Hn}$$

but $nn_0 - Un = Un_0$, and $In - Hn = IH$, or

$$\frac{Un_0}{Un} = \frac{IH}{Hn}.$$

Substituting equalities,

$$\frac{c_0b_0}{cb} = \frac{IH}{Hn}$$

and

$$c_0b_0 = cb \times \frac{IH}{Hn}. \tag{4}$$

Since, on XY, $cb = ba = an$, etc., and, on ZW, $c_0b_0 = b_0a_0 = a_0n_0$, etc., equation (4) establishes the scale relationship which must obtain in the direction perpendicular to the principal plane between any corresponding points in a correct rectification and in the photograph, since it is obvious that an identical relationship could similarly have been established for any two homologous points other than n and n_0, which have been used to avoid inclusion in the figure of any more elements than are necessary.

Equation (3) has similarly established the scale relationship which must obtain in the principal plane for any distance n_0p_0 on the principal line of the rectified projection. To relate this directly to the negative distance np, observe that, in Figure 5a, triangle pUn is similar to triangle p_0Un_0, whence

$$\frac{n_0p_0}{np} = \frac{h}{f}. \tag{5}$$

In Figure 6a, let the perspective point take any position L in the principal plane. Draw HL, and construct IK parallel to HL. Through L draw rays from

p, i and n to intersect IK at p_1, i_1, and n_1. IK represents the trace in the principal plane of the plane upon which the photograph will be projected through the new perspective center L.

Triangle nIn_1 is similar to triangle nHL, whence

$$\frac{In_1}{HL} = \frac{In}{Hn}$$

$$In_1 = HL \times \frac{In}{Hn} \cdot \tag{6}$$

Also, triangle pIp_1 is similar to triangle pHL, so that

$$\frac{Ip_1}{HL} = \frac{Ip}{Hp}$$

$$Ip_1 = HL \times \frac{Ip}{Hp} \cdot \tag{7}$$

Subtracting (6) from (7),

$$Ip_1 - In_1 = HL\left(\frac{Ip}{Hp} - \frac{In}{Hn}\right).$$

But $Ip_1 - In_1 = n_1p_1$; therefore

$$n_1p_1 = HL\left(\frac{Ip}{Hp} - \frac{In}{Hn}\right). \tag{8}$$

Dividing (8) by (3),

$$\frac{n_1p_1}{n_0p_0} = \frac{HL}{HU}$$

and

$$n_1p_1 = n_0p_0 \times \frac{HL}{HU} = \frac{h}{f} \times np \times \frac{HL}{HU} \cdot \tag{9}$$

Through line nn_1 pass a plane perpendicular to the principal plane intersecting the negative and map planes in XY and $Z'W'$, parallel perpendiculars to the principal plane. As before, divide XY into equal divisions at a, b, c, d, etc., and from these points draw rays through L intersecting $Z'W'$ at a_1, b_1, c_1, d_1, etc., as illustrated by Figure 6b.

Any triangle such as cLb will be similar to its projection c_1Lb_1, so that

$$\frac{c_1b_1}{cb} = \frac{Ln_1}{In} \cdot$$

From the similar triangles HLn and In_1n in Figure 6a,

$$\frac{nn_1}{Ln} = \frac{In}{Hn}$$

and hence

$$\frac{nn_1 - Ln}{Ln} = \frac{In - Hn}{Hn}$$

but $nn_1 - Ln = Ln_1$, and $In - Hn = IH$, or

$$\frac{Ln_1}{Ln} = \frac{IH}{Hn} \cdot$$

Substituting equalities,

$$\frac{c_1b_1}{cb} = \frac{IH}{Hn}$$

and

$$c_1b_1 = cb \times \frac{IH}{Hn} \cdot \qquad (10)$$

From equations (9) and (10) we have now the scale relationships which exist in the principal plane and perpendicular thereto between the photograph and its projection onto IK. Comparing equations (4) and (10), it is seen that $c_1b_1 = c_0b_0 = cb \times (IH/Hn)$, which indicates that the scale ratio perpendicular to the principal plane between the projection and map planes, $(c_1b_1/c_0b_0$, remains unity for any position of the perspective point within the principal plane.

If, however, a correct rectification is to be achieved, the scale ratio in the principal plane between distances on the projection and in the map plane must also remain unity. Since, from equation (9), for any position of the perspective point

$$n_1p_1 = \frac{h}{f} \times np \times \frac{HL}{HU},$$

and, from equation (5), $n_0p_0 = (h/f) \times np$, this ratio, (n_1p_1/n_0p_0), can remain unity only if $HL = HU$. Consequently, the perspective point must always be placed a distance HU from H, or its locus is a circle with center at H and radius $HU = f \csc t$, where f is the perspective distance of the aerial camera.

The projection plane IK must be parallel to the plane through LH perpendicular to the principal plane for any position of L. Furthermore, H, the horizon of the photograph, must be focused at infinity on the rectified print, which requires that H lie in the focal plane of whatever projection lens is used. This condition will be met if the focal length of the projection lens is equal to the perpendicular distance F from the point H to the line IL which represents the trace of the nodal plane of the lens (see Figure 7). To find the unique position of a given lens, draw a circle with center at H and radius F, and construct IQ tangent to this circle. The intersection of IQ with the perspective center locus at L is then the proper lens position for projection upon the plane IK which is placed parallel to a plane through HL perpendicular to the principal plane.

It is important to note that there is no necessary fixed position of the axis I; if it remain in the position defined as the intersection of the negative and map planes, the result will be the maintenance on any projection produced the same scale as that which would have been obtained by projection on the map plane itself. If I be placed anywhere along AB produced, the scale ratios in the

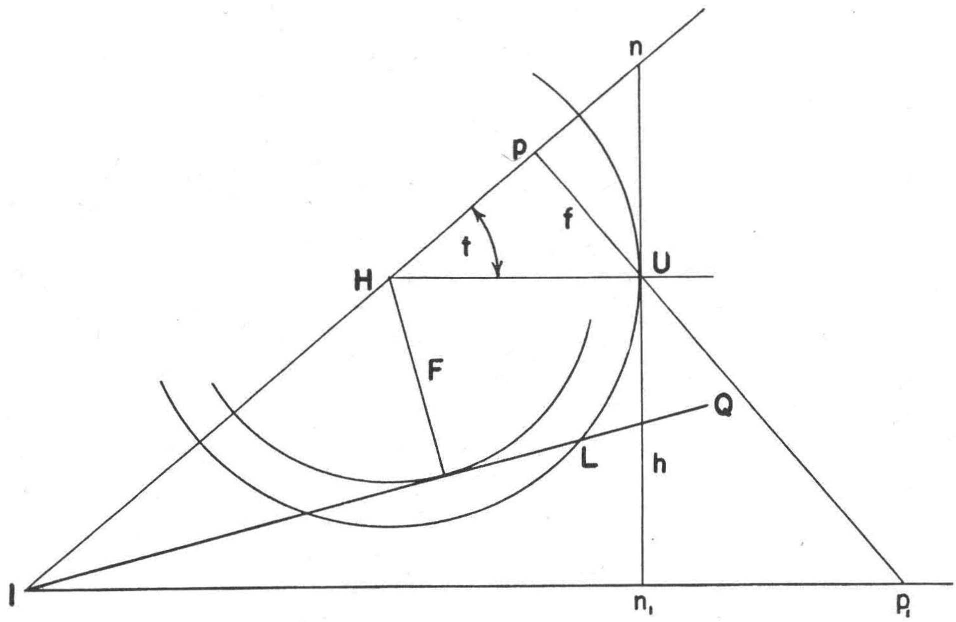

Fig. 7

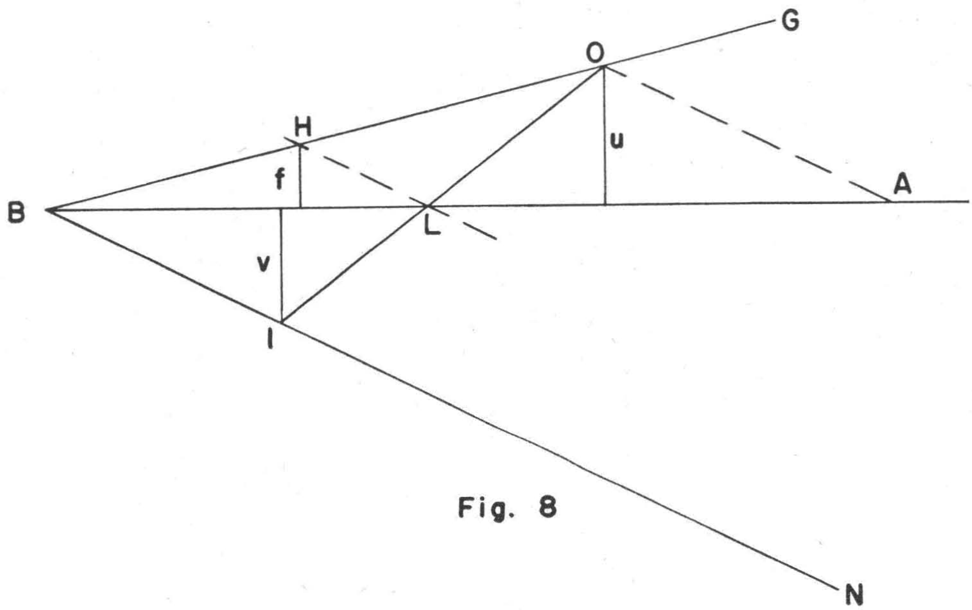

Fig. 8

principal plane and perpendicular thereto will both be changed by the same proportional amount, and the result will be a correct rectification at some different scale.

There remains to be proven that, if the lens and projection planes be located as just described, all object points on the photograph will be correctly focused on the projection. To provide this in a general case, we refer to Figure 8, where BG and BN are, respectively, an object plane and an image plane intersecting at

B. L is a lens located between BG and BN, the nodal plane of the lens passing through B in coincident intersection. The focal length f of the lens is equal to the perpendicular distance to the nodal plane of the lens from a point H where a plane through L parallel to the image plane intersects the object plane. Let O be any object point in the object plane and I its image, projected through L, in the image plane. Through O pass a plane parallel to the image plane intersecting the nodal plane of the lens at A. Let u and v be, respectively, the perpendiculars from O and I to the lens plane.

Triangle OAL is similar to triangle IBL, so that

$$\frac{u}{v} = \frac{AL}{LB}$$

and triangle BOA is similar to triangle BHL, so that

$$\frac{u}{f} = \frac{AB}{LB} = \frac{AL + LB}{LB} = \frac{AL}{LB} + 1.$$

Substituting equalities,

$$\frac{u}{f} = \frac{u}{v} + 1$$

$$uv = uf + fv$$

or

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u} \cdot \qquad (11)$$

Equation (11) will at once be recognized as the familiar conjugate foci formula, indicating that all points in the object plane BG are correctly focused on the image plane BN. Comparison will show that the optical system of Figure 7 is exactly the same as that in Figure 8, proving that the photograph will be correctly imaged on the projection.

We may now state what has just been demonstrated in the form of a proposition which includes all the basic requirements for rectification:

Given an aerial photograph of known tilt angle t and perspective distance f, from its perspective point (U) pass a plane parallel to the object plane photographed to intersect the plane of the photograph in a line which will be the horizon of the photograph (H). With the horizon line as center and radius $f \csc t$ construct a circle in the principal plane. Taking any point (L) on this circle as a center of perspective, a correct horizontalized perspection, or rectification, of the photograph will be obtained by projection through the point selected upon any plane placed perpendicular to the principal plane and parallel to a line from the perspective point to the photograph horizon.

Two convenient and useful corollaries follow directly from the basic principles just enunciated, and these may now be demonstrated. Figure 9 represents the principal plane of a tilted aerial photograph whose principal line is AB, principal point p and isocenter i, which has been rectified by projection through the perspective center, or lens point, L, upon the plane IK, as indicated. The principal distance of the tilted photograph is pU, and point U is its original perspective center.

In the right triangle pUi angle $pUi = \frac{1}{2}t$, and consequently angle $piU = 90° - \frac{1}{2}t$. In the right triangle pHU, angle $pHU = t$, and consequently

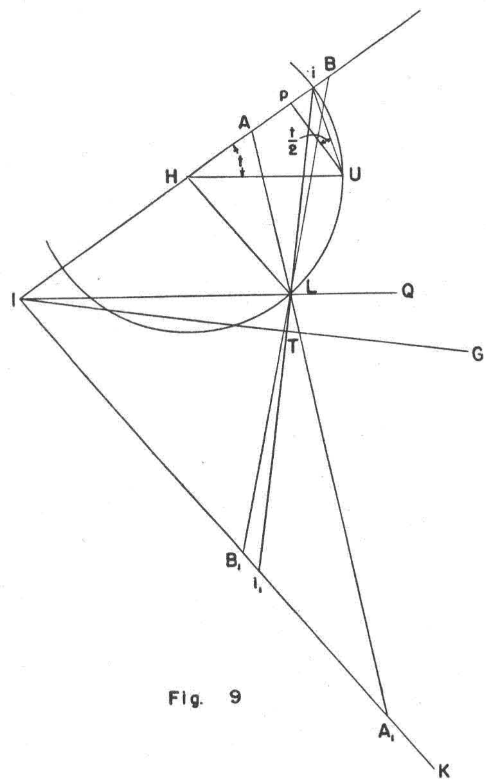

Fig. 9

angle $HUp = 90° - t$. By addition, angle $HUi = 90° - t + \frac{1}{2}t = 90° - \frac{1}{2}t$. Therefore, in triangle HUi angle HiU = angle HUi, and side Hi = side HU. Hence the isocenter of the tilted photograph lies on the locus of the perspective center of rectification.

Referring again to Figure 9, IQ represents the plane of the lens L, and IG represents the bisector of angle BIK formed in the principal plane by the object and image planes. Draw line iL, and extend it to intersect IG at T and IK at i_1. Now, since IK is parallel to HL, angle HLi = angle Ii_1L. In triangle HLi sides HL and Hi are equal, making angle HLi = angle LiH. By substitution of equalities, angle TiI (= angle LiH) = angle Ti_1I, so that in triangle iIi_1 side Ii = side Ii_1. Angle $TIi = TIi_1$, IG being the bisector of angle BIK, and triangle TIi is congruent to triangle TIi_1. Therefore angle ITi = angle ITi_1, and, since iTi_1 is by construction a straight line, each must be a right angle. Consequently the perpendicular through the perspective center to the bisector of the angle in the principal plane between object and image planes passes through the isocenters of the tilted photograph and of its horizontalized projection.

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PART IV—PRACTICAL APPLICATIONS OF THE PRINCIPLES OF RECTIFICATION

The application of these principles to the design of a practical rectifying camera may be accomplished adequately in various ways, and the basic characteristics of a number of different representative instruments will now be investigated in some detail. In the general case where every variable element is provided for independently, a rectifier would possess a total of seven different settings. A simple machine of this kind is illustrated in Figure 10, where A is the photograph holder, L the lens, B the image plane upon which the photograph is projected, and S is a source of illumination. The movements provided are: A, L and B each may revolve about a horizontal axis in order that coincident intersection of their planes, extended, may be obtained; the distances between A and L and between L and B may be varied to maintain sharp imagery; the photograph may be displaced in the plane of its holder so that its isocenter will fall on the locus of the perspective center of rectification; and the photograph may be rotated in its holder so that its principal line will lie in the plane of tilt.

Although seven independent settings are thus provided, at least one of them must be held constant to serve as a datum, or reference, upon which to establish the other settings. Perhaps the most convenient setting to hold constant is the lens, whose axis would usually be held horizontal, i.e., parallel to the guides

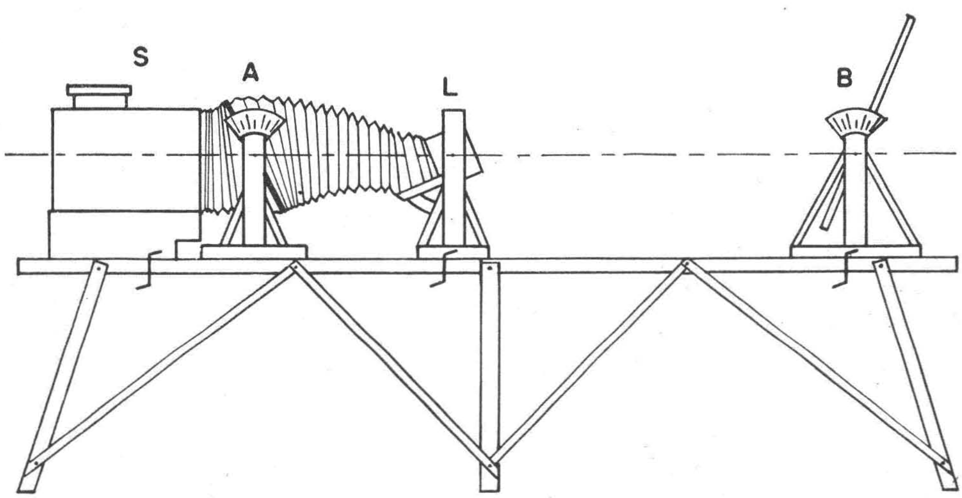


Fig. 10. A Simple Form of Rectifier

which determine the path of lateral motion of the negative carrier, lens mount and image plane, in an apparatus of the kind illustrated in Figure 10. The adoption of this expedient requires the intersection of the object and image planes in the fixed nodal plane of the lens, and thereby greatly facilitates the determination of the other settings. Alternatively, the principal point of the negative may be held fast on, say, the axis of rotation of the negative holder, in which case the lens axis must necessarily be either rotated or translated vertically.

One or the other of these two settings is therefore always eliminated, or, to say the same thing differently, held constant, in all modern rectifiers. An example of the type of machine in which the lens is permanently fixed in position is the Zeiss Automatic Rectifier, whose fundamental geometry is shown in Figure 11, wherein all elements of the drawing are identical to those previously used with the addition of the lens axis, represented by line AB intersecting the object, or negative, plane at z and the image plane at z_1, and line HS, a perpendicular dropped from horizon point H to map plane IR. The angle between the object and lens planes is a, and that between image and lens planes is b.

Five settings are required for this machine, or for others of the same type, comprising angles a and b as defined in the preceding paragraph, the distances, measured along the lens axis, from lens to negative plane (zL) and from lens to image plane (Lz_1), and the displacement of the negative in its carrier (pz), which is the distance, measured along the principal line, from the principal point of the negative to that point, z, at which the lens axis must intersect the plane of the negative. Formulae for computation of these settings may be derived from Figure 11. Referring thereto, in triangle MLH, HL being parallel to IK,

$$\sin b = \frac{F}{HL} = \frac{F}{f \csc t} = \frac{F}{f} \sin t \tag{12}$$

and from triangle IMH

$$\sin a = \frac{F}{IH}$$

but, from triangle ISH

$$IH = HS \csc t = h \csc t$$

whence, by substitution

$$\sin a = \frac{F}{h \csc t} = \frac{F}{h} \sin t. \tag{13}$$

Now in triangle zHL, angle $HLz = 90° - b$, angle zHL = angle $HIK = a + b$, and angle $HzL = 90° - a$. By the law of sines

$$\frac{zL}{HL} \equiv \frac{\sin (a + b)}{\sin (90° - a)},$$

$$zL = HL \times \frac{\sin (a + b)}{\cos a} = f \csc t \times \frac{\sin (a + b)}{\cos a},$$

$$zL = F \frac{\sin (a + b)}{\cos a \sin b} \cdot \tag{14}$$

In right triangle ILz_1,

$$Lz_1 = Iz_1 \sin b$$

but triangle HLz is similar to triangle Iz_1z, so that

$$\frac{Iz_1}{HL} = \frac{zz_1}{zL} = \frac{zL + Lz_1}{zL} = 1 + \frac{Lz_1}{zL},$$

$$Iz_1 = HL + HL \frac{Lz_1}{zL} \cdot$$

Substituting equalities,

$$Iz_1 = f \csc t + \frac{Lz_1 \cos a}{\sin (a + b)} = \frac{F}{\sin b} + \frac{Lz_1 \cos a}{\sin (a + b)}$$

and

$$Lz_1 = \sin b \left[\frac{F}{\sin b} + \frac{Lz_1 \cos a}{\sin (a + b)} \right] = F + \frac{Lz_1 \cos a \sin b}{\sin (a + b)},$$

$$Lz_1\left[1 - \frac{\cos a \sin b}{\sin (a + b)}\right] = F$$

$$Lz_1[\sin (a + b) - \cos a \sin b] = F \sin (a + b)$$

$$Lz_1(\sin a \cos b + \cos a \sin b - \cos a \sin b) = F \sin (a + b)$$

$$Lz_1 = F \frac{\sin (a + b)}{\sin a \cos b} \cdot \qquad (15)$$

As a convenient convention, consider the distance pz to be negative in sign when p is downward, or toward I, from AB and positive when upward, or away from I, so that $pz = Hp = Hz$ in Figure 11. From triangle HzL, by the law of sines,

$$\frac{Hz}{HL} = \frac{\sin (90° - b)}{\sin (90° - a)} = \frac{\cos b}{\cos a},$$

$$Hz = HL \times \frac{\cos b}{\cos a} = \frac{F}{\sin b} \times \frac{\cos b}{\cos a} = \frac{F}{\cos a \tan b}$$

and, from triangle HUp,

$$Hp = \frac{f}{\tan t}$$

therefore

$$pz = \frac{f}{\tan t} - \frac{F}{\cos a \tan b} \cdot \qquad (16)$$

If the calculation of pz produces a positive result, then the photograph must be displaced upward in the negative carrier by that amount, assuming the negative first to have been placed in the carrier with its principal point on the lens axis, and vice versa. As has previously been stated, this displacement is measured along the principal line of the photograph, which will have been discovered by the tilt analysis preceding rectification, and the photograph is positioned in the negative carrier so that its principal line coincides with the permanently marked principal line of the rectifier.

The separation of the nodal points of the rectifier lens has not been considered in deriving these equations for the several settings which must be calculated, nor is it necessary to do so. The object plane is simply placed at the computed distance (zL) from the incident node of the lens and the image plane at its corresponding distance (Lz_1) from the emergent node. The geometry of the figure is in no wise altered thereby, since L may be regarded as representing the *two* perspective centers of the rectifying lens separated by the intra-nodal distance, and I may serve similarly to represent two points separated by an equal and parallel amount. An identical consideration, it will be understood, applies equally to the optical systems of the other devices to be described.

The automatic feature of the Zeiss machine which justifies the inclusion of that description in its nomenclature is a mechanical coupling, called an inverted, which automatically controls the image and object distances to preserve the conditions of optical image definition throughout the full range of settings. In other models of the Zeiss Rectifier, the lens itself may be inclined, and a second inverter is then included which maintains linear coincidence of object, image and nodal planes.

In the special case where unit magnification is always maintained, that is, where the ratio f/h is unity, angles a and b will be equal, and

$$pz = \frac{f}{\tan t} - \frac{F}{\cos a \tan b} = \frac{f}{\tan t} - \frac{F}{\sin b}$$

$$pz = f \csc t(\cos t - 1)$$

$$pz = -f \tan \tfrac{1}{2}t$$

or

$$pz = -pi$$

which means that the lens axis passes through the isocenter and displacement of the aerial negative in the plane of its holder becomes unnecessary if the axis of rotation of the negative is fixed at the isocenter.

Figure 12 illustrates this condition, wherein it will be observed that since HL remains parallel to IK, angle HLI = angle $LIK = a$. In triangle HIL, therefore, $HI = HL = Hi$, so that point I falls on the locus of the perspective point. Further triangle HMI is similar to triangle ILi, whence

$$\frac{iL}{HM} = \frac{Ii}{HI} = 2$$

$$iL = 2HM = 2F$$

and, triangle ILi being congruent to triangle ILi_1, $i_1L = iL = 2F$.

This is the system employed in the oldest known fully automatic rectifier, the Scheimpflug-Kammerer Universal Transformer. In this machine the focal length of the rectifier lens is made equal to that of the aerial camera, and consequently the rectifier tilt a is identical to the camera tilt t. The negative must be capable of displacement in the plane of its holder, whose trace is represented by line IH in Figure 12, by the amount $f \tan \frac{1}{2} t$, so that as any tilt t is set into the machine the isocenter will fall on the optical axis. The negative holder is also rotatable in its plane in order to position the principal line of the photograph in the plane of tilt.

The automatic control incorporated into this machine consists of a pair of levers placed normal to the object and image planes, whose respective traces in the principal plane are lines IH and IK in Figure 12, and linked on a common geared sleeve which lies in the nodal plane of the lens, whose trace is line IQ, causing the simultaneous tilt rotation of both object and image planes by the same angular amount but in opposite directions. Linear concurrence of the three planes is always maintained except in the rare instance of zero tilt, when all three of course become parallel.

It will be observed that displacement of the image screen in its own plane is not necessary to receive the correct rectified image. Actually, however, it is advantageous to do so since if, prior to tilting, the two planes have been aligned parallel to each other and with the principal point of the negative and the center of the image screen on the optical axis; then after tilting, the marked center point of the image screen will coincide with the correct nadir point of the image. Thus, in Figure 12, it will be seen that n_1i_1 is equal to pi. This is useful because, if the nadir point is identified on the photograph, then the correct setting of the rectifier may be obtained simply by manipulating the three movements, tilt, rotation and displacement, of the planes until the image of the marked nadir point falls on the center of the image screen.

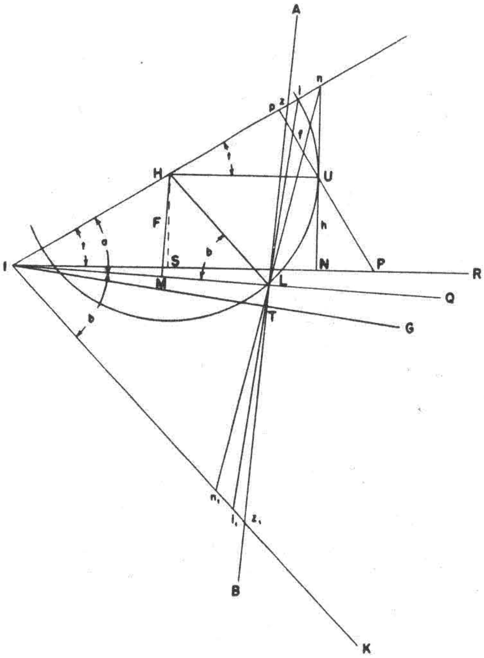
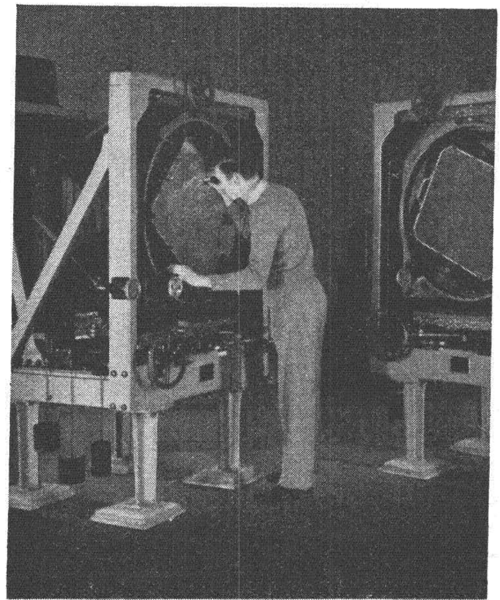


FIG. 11



The Brock Correcting Projector

The Zeiss Automatic Rectifier in Use

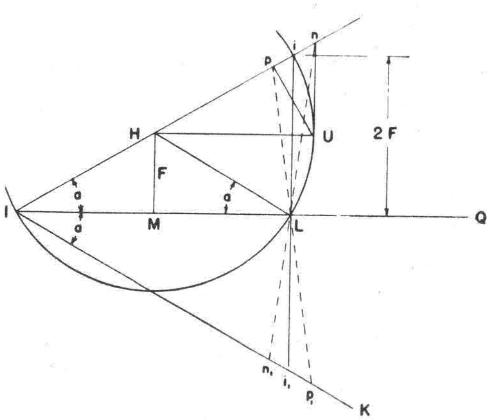
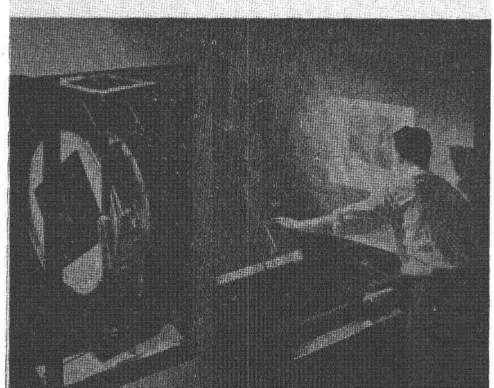


FIG. 12



A Universal Rectifier of the International Aero Service Corporation

The Brock Correcting Projector, as the rectifier used in the Brock process of mapping from aerial photographs is called, represents a further development of the older Scheimpflug machine. It provides, in addition to the levers coupling the equal and opposite tilt rotation of the object and image planes, a mechanical linkage which automatically decenters the two planes by the amount $f \tan \frac{1}{2} t$, thereby eliminating the necessity for manual displacement of negative and image screen. Equality of focal length of aerial camera and rectifier objectives is not, however, maintained, and the rectifier tilt angle is calculated, as in the universal machines, by the formula $\sin a = F/f \sin t$.

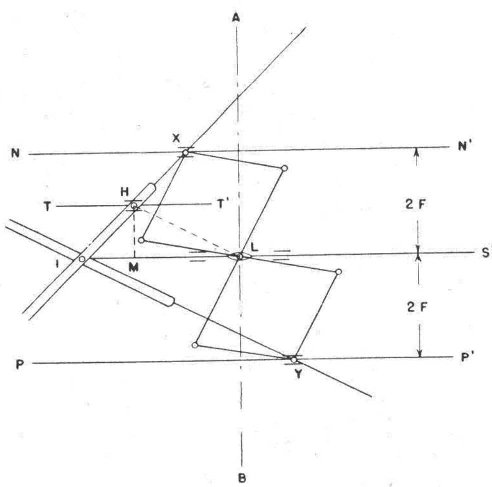
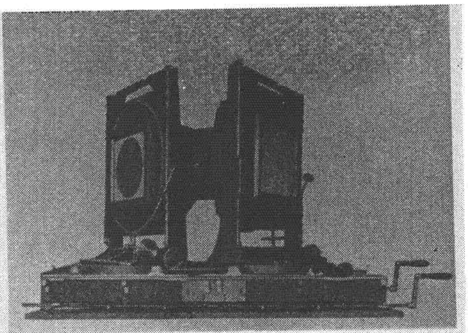
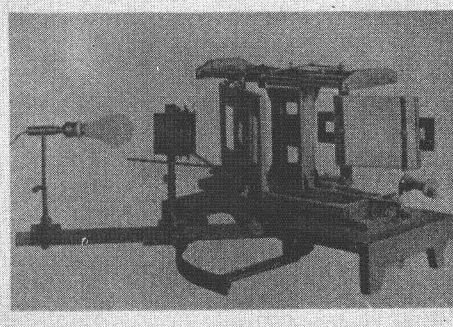


FIG. 13



The Scheimpflug - Kammerer Universal Transformer



S. Finsterwalder's Photocartograph

Incidentally, in the Brock process the value of f remains arbitrarily fixed insofar as rectification is concerned, for the negatives from any aerial camera, of whatever focal length of lens, are enlarged by direct ratio projection prior to rectification to that size which would have been obtained had a given lens been used.

Another interesting outgrowth of the Scheimpflug Rectifier is the Photocartograph, invented by S. Finsterwalder, which differs from its predecessor essentially in that the axes of tilt rotation of the object and image planes are not fixed on the lens axis. Instead, as shown schematically in plan in Figure 13, the two axes, X of the negative or object plane and Y of the image plane, may be shifted in two planes NN' and PP', respectively, each perpendicular to the lens axis AB and parallel to the lens plane IS' and equidistant $2F$ therefrom. The point I is a tie pin which is free to move in the lens plane and maintains continuous linear coincidence of the three planes. The horizon line of the photograph lies in the line H, perpendicular to the plane of the drawing, which travels on the slide TT' parallel to the lens plane and $HM=F$ distant from the lens plane. $HL=f \csc t$. A set of expansion links pivoting about lens point L connects X and Y so as to keep image plane IY parallel to HL for any position of object plane IX. Since I is free to move toward or away from L while H remains fixed for a given value of $f \csc t$, it is seen that IH may vary and, accordingly, the scale of rectification, since $IH=h \csc t$ as before. The photograph tilts accommodated are necessarily limited, but it is possible to rectify to any scale desired within the limits of the machine. The negative may be rotated and displaced in its holder for proper positioning.

The procedure of enlargement, or of reduction if appropriate, of all camera plates to the dimensions of a standard equivalent in the Brock process has been mentioned primarily because it is immediately suggestive of the possibility of adapting the photograph to the rectifier, rather than the converse, and thereby eliminating completely all rectifier settings and calculations therefor. The essential elements of the rectifying apparatus might be as illustrated in Figure 9, but having its object plane, image plane and lens all permanently fixed in any convenient position. Obviously, then, any photograph exposed at that particular tilt, t, in a camera having that principal distance, f, for which the rectifier was arranged could be horizontalized therein since its own unique geometric requirements for correct rectification would correspond exactly to those of the rectifier. In the general case, this can be interpreted as meaning that the radius of the locus of the perspective center of rectification of the photograph, $f \csc t$, equals that of the machine, and the photograph will be completely oriented in position for rectification if it be placed so that its isocenter falls on the isocenter point of the rectifier and its horizon point coincides also with that of the machine. Now any other photograph, produced by a camera of different principal distance, f', at some other tilt, t', will have a different locus radius $f' \csc t'$. However, if the product $f' \csc t'$ is finite, as it will be for all but perfectly vertical photographs, it can be made equal to the product $f \csc t$ of the machine simply by direct enlargement or reduction of the photograph. If this be done, and the enlarged or reduced photograph be placed in the rectifier with its isocenter at the isocenter point of the rectifier, the photograph horizon must fall on the rectifier horizon since the distance $Hi=f \csc t$ and the photograph may be horizontalized by projection in the rectifier without altering the permanent settings of the latter. The rectified print will not, of course, be at the desired map scale unless by unusual coincidence, and must therefore be enlarged or reduced by direct ratio projection to bring it to scale. Assuming unit magnification in the rectifier and the following notation,

f = focal length of aerial camera lens.
t = actual tilt of photograph.
F = focal length of rectifier lens.
t' = fixed tilt accommodation of rectifier.
H = lens height of exposure station.
m = desired map scale.
$h=mH$ = lens height at scale of map,

the enlargement or reduction factor, E, to be applied to the rectified print may be derived as follows:

$$\text{Scale of rectified print} = \frac{f}{H} \times \frac{F \csc t'}{f \csc t} = \frac{F \csc t'}{H \csc t}$$

$$E \times \frac{F \csc t'}{H \csc t} = m = \frac{h}{H}$$

$$E = \frac{h}{F \csc t' \sin t}. \tag{17}$$

If the magnification obtained in the rectifier itself is not unity, but rather some factor k, then:

$$\text{Scale of rectified print} = \frac{Fk \csc t'}{H \csc t}$$

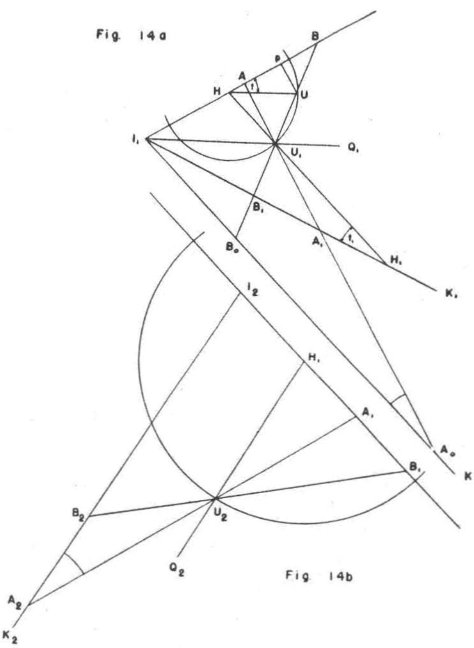
and the enlargement factor becomes

$$E = \frac{h}{Fk \csc t' \sin t}. \tag{18}$$

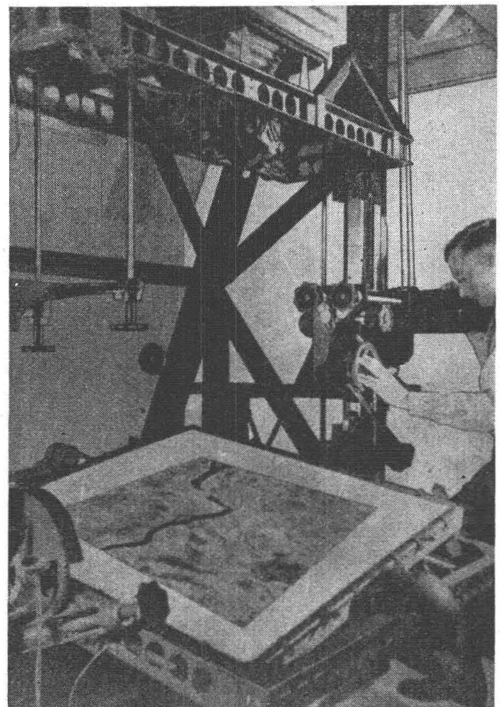
From the mass production point of view, where the ultimate desideratum is the establishment of a simple repetitive operation, the process just described possesses manifest advantages. It is, however, limited practically to the rectification of photographs of large tilt since the rapid rate of change of the sine function of small angles would require prohibitive enlargements of the aerial negative.

One deficiency common to all of the preceding types of rectifier is that an undesirably weak intersection of the projected rays with the image plane may be produced even though the horizontalization be entirely correct. To ameliorate this condition, there has recently been proposed a procedure of "stage rectification," understood to have been conceived originally by Mr. O. N. Miller of the American Geographical Society and further developed by Mr. Robert Singleton of the International Aero Service Corporation, which, as the name implies, accomplishes horizontalization by more than one projection. Thus, in Figure 14a, AB is the principal line of a tilted photograph which may be rectified in the usual manner by projection through the perspective center U_1 upon the plane IK, producing a rectification A_0B_0. It will be seen that the ray from the point A on the tilted photograph intersects the projection plane in a comparatively flat angle at A_0. To avoid possible uncertainty of imagery, therefore, the projection plane is rotated about I_1, the trace of the line of coincident intersection of object, lens and image planes, to take some position such as I_1K_1 and a new projection A_1B_1 is made whose perspective center is U_1 and whose horizon point is H_1, the projection of H upon I_1K_1. A_1B_1 may then be regarded as a photograph of the same ground area as appeared on AB, but at a different tilt angle t_1, and which may be rectified to provide the same horizontalized perspection obtainable from AB, but more advantageously. To rectify A_1B_1, it is necessary only to choose a convenient perspective center such as U_2 in Figure 14b, which may be any point on the locus circle with center at H_1 and radius H_1U_1, an image plane such as I_2K_2, and project A_1B_1 thereupon to obtain the correct rectification A_2B_2. By careful selection of the position of the image planes I_1K_1 and I_2K_2, and also of course of the second perspective center U_2, it is thus possible to obtain a rectified print of the original tilted photograph of appreciably better quality than a single projection can provide.

To venture into the realm of conjecture, it seems entirely feasible that these two last-mentioned procedures, neither of which has as yet been exploited commercially, could be quite advantageously combined into what might be called "fixed-stage rectification." The result sought in attempting the combination of the two machines would be the production of rectified prints of high image quality with a minimum of manipulation. Since all rectifier settings would be eliminated, the technical qualifications required of operating personnel would be reduced and the time necessary to compute the usual settings would be saved, both of which represent major economies in the rectification process. To accommodate photographs of small tilt, a "tilter," or reversed rectifier, might be employed as an initial step whenever necessary. One apparent disadvantage is the number of plates required in the progress from stage to stage, but the total thereof might be reducible by using a fluorescent translucent screen as the intermediate image plane of the stage rectifier. It is not known whether experimentation is being actively carried forward along these lines, but the theoretical potentialities of the proposal commend it as a promising subject for future investigation.



FIGS. 14a and 14b


A Rectifying Camera of the U.S. Coast & Geodetic Survey

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PART V—CONCLUSION

In summation, this paper has endeavored, first to demonstrate the nature of the image displacements to which the aerial photograph as a perspective projection is inherently subject, to indicate the variable scalar and angular discrepancies introduced by tilt, and to establish the consequent necessity of eliminating this latter source of error in mapping by rectification. Second, there has been developed a general theorem concerning the relationships which must obtain between all essential elements of any optical apparatus intended to produce geometrically correct rectification. Finally, the application of these principles to presently known and proposed rectifiers has been exemplified.

The purpose of the foregoing exposition of the theoretical and practical aspects of the rectification process has been, as was enunciated at its inception, the evolution of a concise working hypothesis to explain the means by which the reduction of the tilted aerial photograph to an equivalent horizontalized perspection may be achieved. Simple and direct mathematical analysis has been employed in the accomplishment of this objective. More complex but not necessarily more rigorous methods are available and may prove useful in special circumstances. For every general requirement of instruction and investigation, however, the present treatment is considered to suffice.

NEWS NOTE

BAUSCH & LOMB OPTICAL COMPANY

ROCHESTER, N. Y.—A new series of motion picture lenses, comparable in performance to those used by Hollywood cameramen, are now available for the home movie maker.

Known as Animars, each lens is fitted with an ingenious seasonal exposure guide plus click and spread diaphragm stops to assure correct exposure. The standard lenses are for photography under average light conditions, while the telephotos are for candid and close-up shots of distant subjects.

The 16mm. camera telephoto lenses are equipped with a depth of field scale which gives the photographer a check on the focus range, enabling him to bring both foreground and background objects into sharp focus.

The new series consists of nine lenses: five standard and four telephoto. Focal lengths range from 12.7mm. for the smallest standard lens, to four inches for the largest telephoto. Speeds for both series of lenses range to f:1.9 in standard lenses, and f:3.5 in the telephotos.