WEIGHT DISTRIBUTION IN STEREOSCOPIC MODELS AFTER ADJUSTMENT OF COORDINATES IN PLAN AND HEIGHT

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AFTER the stereoscopic model has been produced by the reciprocal orientation, and has been located with reference to the coordinate system in the projector room by the absolute orientation, which comprises the determination of the scale, three angular displacements and three linear displacements, the coordinates of any arbitrary point of the model can be determined both in plan and in elevation.

An important question which arises in this connection is to find the accuracy of the determination of coordinates, since high quality is the principal prerequisite to the applicability of the photogrammetric methods.

According to the investigations which have so far been made in this respect, the accuracy that can be reached in the photogrammetric determination of coordinates is usually expressed by a function of the flight altitude, i.e. the height of the aeroplane above ground. It is assumed that the mean error in the determination of the coordinates is H/5,000 in plan and H/4,000 in elevation. These values have been obtained from investigations made by means of normal-angle lens cameras a relatively long time ago.

Furthermore, it is evident that the data on accuracy given in the above cannot be considered to be sufficiently exhaustive since they do not take into account the characteristics of the given points, i.e. the number of points and their positions in the model. Moreover, these data take no account of the distribution of accuracy in the stereoscopic model, although it is obvious that this distribution cannot be uniform.

Investigations of the accuracy in this respect must, of course, be made with the aid of supernumerary observations in the determination of the coordinates. In principle, the photogrammetric determination of the coordinates consists in finding the coordinates of a number of points of the model by setting up the collimating mark over the points in question and by reading the scales of the coordinate register. At the same time, readings are taken of the coordinates of at least two points whose ground coordinates are known. After that, since the model is similar to the ground, all model coordinates can be transformed into ground coordinates. The data required for this transformation are calculated by means of those points whose ground coordinates are given. If there are any supernumerary observations—and, as a general rule, they should always be available—, recourse must be had to methods of adjustment which ensure that the errors found by means of supernumerary observations are treated in a theoretically correct manner.

The only method for dealing with such problems that is generally recognized as theoretically correct, is the method of least squares. Therefore, this method should be used as a basis of investigations concerning the propagation of errors.

Approximate methods, which result in more rapid and less intricate calculations, have been advanced for the transformation of coordinates and the adjustment of supernumerary observations. These methods cannot be regarded as equivalent to the method of least squares. A method which gives good results, while involving a relatively small amount of labor in the calculations is given in: Hallert, "Ueber die Herstellung photogrammetrischer Pläne" Stockholm 1944, p. 54.

The principle of this method is briefly outlined in what follows: The transformation of coordinates is carried out with reference to the centers of gravity of the two systems of points. Consequently, these centers of gravity are calculated in the first place. Then the data on the angular displacements and on the enlargement are computed from the coordinates of the centers of gravity and each of the given points. After that, the mean values of these data are calculated by using the distances from the center of gravity to the respective points as weights. The mean values of the angular displacements and enlargement data are used for calculating the ground coordinates of each point whose coordinates are to be transformed. However, it can be shown that it is more correct, from a theoretical point of view, to assume that the weights are equal to the squares of the distances from the center of gravity to the respective points. This assumption simplifies the calculations still more, since these squares enter into those terms which are employed for computing the transformation data.

The equations are:

$$X_v = V \sin \alpha \Delta y_v + V \cos \alpha \Delta x_v + X_s$$

$$Y_v = V \cos \alpha \Delta y_v - V \sin \alpha \Delta x_v + Y_s$$

where X_v, Y_v are the coordinates to be computed Δx_v, Δy_v are the differences between the points in the model and their center of gravity. X_s, Y_s are the coordinates of the center of gravity of the given points

V is the enlargement factor and

α is the angle of rotation.

The corresponding differential expressions are:

$$dX_v = dX_s + \Delta Y_v d\alpha + \Delta X_v \frac{dV}{V}$$

$$dY_v = dY_s - \Delta X_v d\alpha + \Delta Y_v \frac{dV}{V}.$$

These equations are in the form of observation equations:

$$v_{Xv} = dX_s + \Delta Y_v d\alpha + \Delta X_v dV' + X_v' - X_v$$

$$v_{Yv} = dY_s - \Delta X_v d\alpha + \Delta Y_v dV' + Y_v' - Y_v$$

$$v = 1, 2, \cdots, n \ (n \geqq 4).$$

$X_v' - X_v$ and $Y_v' - Y_v$ denote the differences between the preliminarily calculated values and the given values of the coordinates of those points which are used in the transformation of the coordinates. ΔX and ΔY are the differences between the coordinates of the given points and the coordinates of their center of gravity.

The observation equations are tabulated as shown at top of facing page.

From these equations, we deduce the normal equations in usual manner and obtain the unknowns, together with their weight coefficients and the expression for $[vv]$:

$$dX_s = -\frac{[X' - X]}{n} \qquad [\alpha\alpha] = \frac{1}{n}$$

$$dY_s = -\frac{[Y' - Y]}{n} \qquad [\beta\beta] = \frac{1}{n}$$

	Point	dX_s	dY_s	$d\alpha$	dV'	$X_v'-X_v$
	1	1	0	ΔY_1	ΔX_1	$X_1'-X_1$
	2	1	0	ΔY_2	ΔX_2	$X_2'-X_2$
v_X	·	·	·	·	·	·
	n	1	0	ΔY_n	ΔX_n	$X_n'-X_n$
....	....	....	....	....	....	....
	1	0	1	$-\Delta X_1$	ΔY_1	$Y_1'-Y_1$
	2	0	1	$-\Delta X_2$	ΔY_2	$Y_2'-Y_2$
v_Y	·	·	·	·	·	·
	n	0	1	$-\Delta X_n$	ΔY_n	$Y_n'-Y_n$

$$d\alpha = -\frac{[(X'-X)\Delta Y]-[(Y'-Y)\Delta X]}{[\overline{\Delta X^2}]+[\overline{\Delta Y^2}]} \qquad [\gamma\gamma] = \frac{1}{[\overline{\Delta X^2}]+[\overline{\Delta Y^2}]}$$

$$dV' = -\frac{[(X'-X)\Delta X]+[(Y'-Y)\Delta Y]}{[\overline{\Delta X^2}]+[\overline{\Delta Y^2}]} \qquad [\delta\delta] = \frac{1}{[\overline{\Delta X^2}]+[\overline{\Delta Y^2}]}$$

$$[vv] = [(X'-X)^2]+[(Y'-Y)^2] - \frac{[X'-X]^2+[Y-Y]^2}{n}$$

$$-\frac{\{[(X'-X)\Delta Y]-[(Y'-Y)\Delta X]\}^2+\{[(X'-X)\Delta X]+[(Y'-Y)\Delta Y]\}^2}{[\overline{\Delta X^2}]+[\overline{\Delta Y^2}]}.$$

The radial mean error at any point is obtained as

$$m^2 = \mu^2\left(\frac{1}{n}+\frac{\overline{\Delta X^2}+\overline{\Delta Y^2}}{[\overline{\Delta X^2}]+[\overline{\Delta Y^2}]}\right)+i^2$$

where i is the mean error in the new measurement.

The equation above can be written in the form:

$$m = \pm\,\mu\sqrt{\frac{1}{n}+\frac{S^2}{[ss]}+k} \quad \text{where} \quad k = \frac{i^2}{\mu^2}\cdot$$

It is evident that the formula takes into account the number of given points (n), their relative positions ($[ss]$, i.e. the sum of the squares of the distances from the given points to their center of gravity), and the position of that point whose radial mean error is to be determined (S, i.e. the distance from this point to the center of gravity of the given points).

THE MEAN ERROR OF A STANDARD OBSERVATION

The determination of the mean error of a standard observation is obviously a momentous problem since this error is to be regarded as the most important constant in the determination of the accuracy of the coordinates. This error can be calculated from the well-known formula

$$\mu = \pm \sqrt{\frac{[vv]}{\ddot{o}}}$$

where $[vv]$ can be obtained from the normal equations and $\ddot{o}$ is the number of supernumerary observations in addition to the necessary two points or four coordinates.

From a large number of models, in scale varying from 1:20,000 to 1:1,500, such calculations were carried out. The photographs were taken by the Zeiss' wide angle lens camera RMK 20, 30/30 and the plotting instrument was the Zeiss stereoplanigraph.

When the mean value of the mean error of a standard observation was calculated from measurements on those models which contain specially marked points, it was found to be ± 0.05 mm. on the scale of the photograph. The same mean value, computed from the measurements on those models in which the corresponding points consisted of natural features, (e.g. rocks, corners of fields and the like), was found to be ± 0.065 mm. on the scale of the photograph. This shows that the use of specially marked points brings about a considerable increase in accuracy.

On the assumption that use is made of specially marked points, the error function can be written as follows

$$m = \pm\, 0.05\sqrt{\frac{1}{n} + \frac{S^2}{[ss]} + k}$$

mm. on the scale of the photographs or, on the scale of 1 to N

$$m = \pm\, 0.05N\sqrt{\frac{1}{n} + \frac{S^2}{[ss]} + k}\,.$$

If $i = \pm 0.02$ mm. k is found to be 0.16. If the camera constant is 200 mm., μ can be written as $H/4{,}000$.

This function can be employed for computing the mean error to be feared in different portions of a model when a number of points is given in given positions e.g. for adjustment between overlapping models. Conversely, if a predetermined accuracy is required, this function can be used for determining the scale of the photograph (i.e. the flight altitude), the necessary number of points of control, and the most appropriate positions of these points.

In actual fact, the accuracy of the photogrammetric determination of coordinates may be expected to be slightly higher than that which is indicated by the figures given in the above, since these figures also include the errors in the terrestrial measurements. In all cases under consideration, these measurements were made in accordance with the relevant Swedish regulations for high precision measurements, but even under such conditions, the unavoidable errors in the terrestrial measurements are noticeable in photogrammetric operations carried out on a comparatively large scale.

WEIGHT DISTRIBUTION IN STEREOSCOPIC MODELS AFTER ADJUSTMENT OF ELEVATION MEASUREMENTS BY ROTATION OF MODEL AND CHANGES IN SETTING OF HEIGHT REGISTER.

Under normal conditions, the elevated points are given in entirely arbitrary positions. Therefore, the calculations required for adjusting the elements of absolute orientation so as to take into account the deformations of the model would involve too much labor. Furthermore, if the number of the given elevated points is less than five, the complete adjustment of the deformations of the model is not possible at all.

In practice, the elevation measurements must therefore usually be adjusted without taking into account the deformations of the model. In other words, the model must be regarded as a solid, and the position of this solid in elevation is adjusted by rotation about two axes and by linear displacement in one direction so that the sum of the squares of the errors at the given points becomes a minimum. Under such conditions, the mean error of a standard observation should be determined on the given assumptions, that is to say, the effects of the deformations of the model should also be taken into account in calculating the mean error of a standard observation.

The differential expression for small changes in the respective elements of orientation is obtained as:

$$dh = dh_0 + x \cdot d\eta - y \cdot d\xi$$

where dh is the change in elevation in a model point with the coordinates x and y
dh_0 is the change in setting of height register.
$d\eta$ and $d\xi$ are the rotations of model.

If supernumerary observations are available, this equation is written in the form of an observation equation:

$$v_v = dh_0 + x_v d\eta - y_v d\xi + h_v$$

where h_v is the resultant error in elevation i.e. $h_{\text{observed}} - h_{\text{given}}$.

In order to simplify the normal equations as far as possible, it is advantageous to reckon the coordinates from the center of gravity of the given points.

Consequently, we obtain

$$X_1 = x_1 - \frac{[x]}{n} \qquad Y_1 = y_1 - \frac{[y]}{n}$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$X_n = x_n - \frac{[x]}{n} \qquad Y_n = y_n - \frac{[y]}{n}$$

$$[X] = 0 \qquad [Y] = 0$$

The observation equations are tabulated as follows

Point	dh_0	$d\eta$	$d\xi$	h
1	1	X_1	$-Y_1$	$+h_1$
2	1	X_2	$-Y_2$	$+h_2$
. . .	. . .	. . .	. . .	. . .
n	1	X_n	$-Y_n$	$+h_n$

From these equations we deduce the normal equations in usual manner and find the unknowns and the weight coefficients

$$dh_0 = -\frac{[h]}{n} \qquad [\alpha\alpha] = \frac{1}{n}$$

$$d\eta = \frac{[XY][Yh] - [Xh][YY]}{[XX][YY] - [XY]^2} \qquad [\beta\beta] = \frac{[YY]}{[XX][YY] - [XY]^2}$$

$$d\xi = \frac{[XX][Yh] - [XY][Xh]}{[XX][YY] - [XY]^2} \qquad [\gamma\gamma] = \frac{[XX]}{[XX][YY] - [XY]^2}$$

$$[\beta\gamma] = \frac{[XY]}{[XX][YY] - [XY]^2}.$$

Furthermore the expression for $[vv]$ is

$$[vv] = [hh] - \frac{[h]^2}{n} \frac{[XX][Yh]^2 + [YY][Xh]^2 - 2[XY][Xh][Yh]}{[XX][YY] - [XY]^2}.$$

The mean error of a standard observation is calculated as

$$\mu = \pm\sqrt{\frac{[vv]}{n-3}}.$$

Since the measured heights of new points under the made conditions are functions of the adjusted quantities, we find the weight coefficient of an arbitrary point with the coordinates X_v, Y_v as

$$Q_v = \frac{1}{n} + \frac{X_v^2[YY] + Y_v^2[XX] - 2X_vY_v[XY]}{[XX][YY] - [XY]^2}.$$

If i is the mean error of the new measurement, the mean error of the height in an arbitrary point can be written as

$$m^2 = \mu^2\left(\frac{1}{n} + \frac{X_v^2[YY] + Y_v^2[XX] - 2X_vY_v[XY]}{[XX][YY] - [XY]^2}\right) + i^2$$

and, if

$$\frac{i^2}{\mu^2} = k$$

$$m = \pm\mu\sqrt{\frac{1}{n} + \frac{X_v^2[YY] + Y_v^2[XX] - 2X_vY_v[XY]}{[XX][YY] - [XY]^2} + k}.$$

The mean error μ of a standard observation can readily be calculated by means of the expression for $[vv]$ from tests made on models whose absolute orientation is not definitively determined in respect of the angles of rotation about the horizontal axes and in reference to the linear displacement along the vertical axis.

In those investigations concerning the mean error of a standard observation which have so far been carried out, the absolute orientation of the models was first adjusted in appropriately located and sharply identifiable points. Then the setting of the height register was changed by an arbitrary amount, and the elevations of all given points were read. The constant error, due to the change in setting of the height register, was calculated and reduced by comparing the elevations of the points measured in this manner, with their elevations determined by ground surveying. The residual errors were regarded as accidental, and the mean error was computed from the usual formulae.

Of course, this mean error also includes those errors which are due to the deformations of the model and the residual errors caused by the other operations of absolute orientation. However, it is this mean error of a standard observation that must be reckoned with in practice, since the square deformations of the model are in general not adjusted rigorously.

The mean error obtained by means of this method was 0.04 mm. on the scale of the photograph. The camera was Zeiss wide-angle camera RMK 20, 30/30 and the plotting instrument was Zeiss stereoplanigraph.

The function above can be employed for computing the mean error to be feared in different portions of a model when a number of points is given in given positions. Conversely, if a predetermined accuracy is required, this function can be used for determining the scale of the photograph (i.e. the flight altitude), the necessary number of the points of control, and the most appropriate positions of these points.

NEWS OF PHOTOGRAMMETRISTS

The many friends of *William H. Meyer, Jr.* will be interested and pleased to know that since November 1948 he has been associated with Jack Ammann Photogrammetric Engineers. He is now Manager of the Eastern Branch and has his office at 34 Webster Avenue, Box 411, Manhasset, N. Y.

Alonzo C. Hammon, *Herbert A. Jensen* and *Arnold F. Wallen*, all of whom have had extensive experience in photogrammetry for Federal and State Governments, have entered into a partnership to offer a wide range of mapping and forestry services. The firm's headquarters will be in Oakland, California.

James J. Mongrain is now in Saudi Arabia making a survey for the Arabian American Oil Company.

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