AN ORIENTATION ANALYSIS OF THE MULTIPLEX MODEL*

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INTRODUCTION

THE multiplex model is constructed with two projectors, each containing a diapositive. If each diapositive is in absolute orientation, the model is in every respect, excepting scale, an exact replica of the terrain. The absolute orientation can be disturbed by displacing the diapositive from its interior orientation, exterior orientation, or both. The behavior of the multiplex model will be examined under the simple displacements which disturb the interior or exterior orientation.

The simple displacements which disturb the exterior orientation will be treated first. These consist of the six primary motions of the projector: *bx*, *by*, *bz*, *x*-tilt, *y*-tilt, and swing. For these motions, the characteristic curves will be considered for both the vertical and oblique wide-angle projectors. The wide-angle oblique projector here considered is adapted from the wide-angle vertical projector by simply tilting the optical system with respect to that part of the assembly which controls the *bx*, *by*, and *bz* motions.

CHAPTER I, DISPLACEMENT IN EXTERIOR ORIENTATION

ARTICLE 1. THE CHARACTERISTIC CURVES

A KNOWLEDGE of the characteristic curves of the primary motions provides a first step toward understanding the procedures in orienting the model, and also toward determining the effects of a displacement on the model. In deriving the characteristic curves, only one projector is used. Let the lens, O, of the projector be the origin of a rectangular Cartesian coordinate system in three dimensions, as shown in Figure 1 below.

The optical axis of the vertical projector coincides with the OZ axis. For the oblique projector, the optical axis is given by its direction angles α, β, γ:

$$\alpha = \frac{\pi}{2}, \qquad \beta = \frac{\pi}{6}, \qquad \gamma = \frac{2\pi}{3}.$$

The plane $Z=c$ is selected as a datum plane. Every ray which emerges from the lens intersects the datum plane in a definite point. In fact, there is a one-to-one correspondence between the image rays and the points of the plane $Z=c$. Under a given displacement of the projector, any image point in the datum plane traces a curve in that plane. If the displacement is one of the six primary motions, the curve is called a characteristic curve for that motion. An arbitrary displacement can be obtained by a combination of one or more of the primary motions performed sequentially.

1.1 The Translations.

The characteristic curves for the *bx*, *by*, and *bz* motions are independent of the orientation of the optical axis of the projector, so that the introduction of

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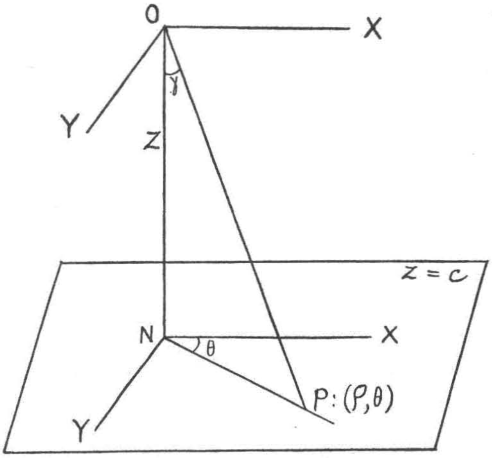


FIG. 1

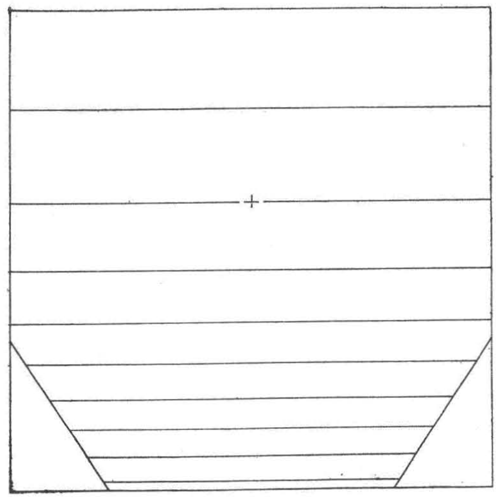


FIG. 2. Characteristic Curves for BX Motion.

the oblique projector offers no additional problem to the study of the translations. For bx and by motion the characteristic curves are evident and their equations are respectively, $y=b$ and $x=a$ where a and b are constants which depend on the image point selected. These lines constitute a one-parameter family of curves. It will be seen that the characteristic curves for every primary motion is a one-parameter family of curves.

The bz motion will be considered now. Let $P:(\rho, \theta)$ be an image point in the datum plane, where ρ and θ are the polar coordinates for the plane (Figure 1). Let γ be the direction angle the image ray to P makes with the OZ axis. Then $P:(x, y, z)$ is determined by the following equations:

$$\begin{aligned} x &= \rho \cos \theta \\ y &= \rho \sin \theta \\ z &= C. \end{aligned} \tag{1.1; 1}$$

Under bz motion, the lens point O moves along the OZ axis, giving rise to a moving coordinate system. The direction angle γ remains constant and therefore

$$\rho = z \tan \gamma.$$

Then making use of this relation in equations (1.1; 1) we have:

$$\begin{aligned} x &= z \tan \gamma \cos \theta \\ y &= z \tan \gamma \sin \theta. \end{aligned} \tag{1.1; 2}$$

These are the parametric equations of the characteristic curve passing through P. Now eliminating z and γ between these equations yields the one-parameter family of characteristic curves for bz motion:

(A) $$y = x \tan \theta.$$

This family of curves consists of all the straight lines in the datum plane passing through the nadir point $N:(O, O, c)$.

In the illustrations of the characteristic curves, the curves shown are those generated by rays making angles with the optical axis which are multiples of five degrees. The characteristic curves for the vertical projector are not included in these illustrations as they are accessible elsewhere.[1]

1.2 The Rotations.

To determine the characteristic curves for the rotations, the following procedure will be adopted:

1. Obtain the family of curves generated by the vertical projector
2. Determine the family of conical surfaces which pass through these curves and the lens point.
3. Rotate these surfaces about the lens point and intersect them with the datum plane. The curves of intersection will then be the characteristic curves for the oblique projector.

The procedure outlined above will now be applied to y-tilt motion. Let M denote the angle of rotation about the OY axis. Then if we consider this rotation as a rigid motion, a point $Q:(x, y, z)$ is displaced to $Q':(x', y', z')$. The transformation of coordinates is given by:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & O & \sin\theta \\ O & 1 & O \\ -\sin\theta & O & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad (1.2; 1)$$

where the coordinates of Q and Q' are considered to be column vectors. Let $P:(O, b, c)$ be a point on the NY axis (Figure 2). Under the rotation, P is displaced to P'. Then employing (1.2; 1) P' is given by:

$$P' = \begin{bmatrix} \cos\theta & O & \sin\theta \\ O & 1 & O \\ -\sin\theta & O & \cos\theta \end{bmatrix} \begin{bmatrix} O \\ b \\ c \end{bmatrix} = \begin{bmatrix} c\sin\theta \\ b \\ c\cos\theta \end{bmatrix}.$$

The equations of the line through O and P' are:

$$\frac{x}{c\sin\theta} = \frac{y}{b} = \frac{z}{c\cos\theta}.$$

Intersecting this line with the datum plane $z=c$ yields:

$$\frac{x}{c} = \tan\theta; \qquad \frac{y}{b} = \sec\theta.$$

The elimination of θ between these equations gives:

$$\text{(B)} \qquad \frac{y^2}{b^2} - \frac{x^2}{c^2} = 1.$$

These are the characteristic curves for y-tilt in the vertical projector.

Let (α, β, c) be the coordinates of an arbitrary point on the curve given by (B). The equations of the line through O and (α, β, c) are:

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{c}.$$

[1] *U. S. War Department, Multiplex Mapping Equipment,* TM 5-244 (June 1943), pp. 52–53.

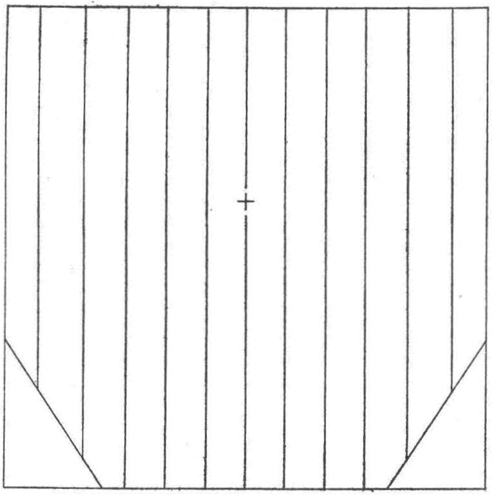


FIG. 3. Characteristic Curves for BY Motion.

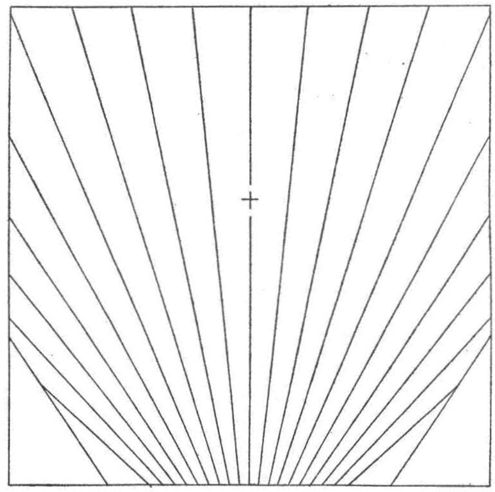


FIG. 4. Characteristic Curves for BZ Motion.

Then solving for α and β and substituting into the equation:

$$\frac{\beta^2}{b^2} - \frac{\alpha^2}{c^2} = 1,$$

we arrive at the equation of the conical surface determined by O and the curve representing (B):

$$\frac{y^2}{b^2} - \frac{x^2}{c^2} = \frac{z^2}{c^2}. \qquad (1.2;\ 2)$$

This surface is now rotated about the OX axis. The equations of transformation are given by:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & O & O \\ O & \cos\theta & \sin\theta \\ O & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad (1.2;\ 3)$$

where $\theta = 60°$.

Equation (1.2; 3) can be solved for the unprimed column vector by multiplying both members on the left by the inverse of the square matrix, yielding:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & O & O \\ O & \cos\theta & -\sin\theta \\ O & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}. \qquad (1.2;\ 4)$$

The application of this transformation to equation (1.2;2) gives the equation of the rotated conical surface:

$$\frac{(y - \sqrt{3}z)^2}{4b^2} - \frac{x^2}{c^2} = \frac{(\sqrt{3}y + z)^2}{4c^2}. \qquad (1.2;\ 5)$$

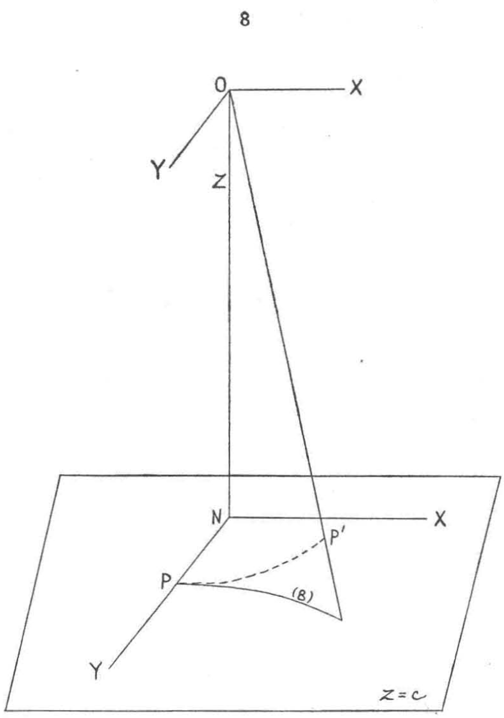


FIG. 5

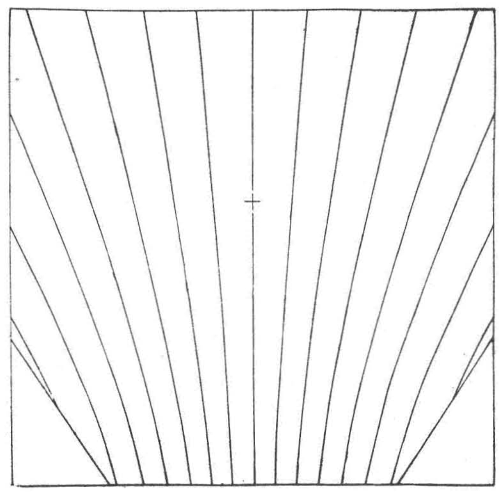


FIG. 6. Characteristic Curves for X-Tilt Motion.

The intersection of this surface with the datum plane $z=c$ gives the equation of the characteristic curves for y-tilt in the oblique projector:

$$\text{(B')} \qquad \frac{(y - c\sqrt{3})^2}{4b^2} - \frac{x^2}{c^2} = \frac{(\sqrt{3}y + c)^2}{4c^2} \, .$$

Further information about these curves can be obtained by introducing a change of parameter. Let γ be the direction angle OP makes with the OZ axis. Then $\tan \gamma = b/c$. Through this change of parameter, equation (B') becomes, after grouping the terms:

$$\frac{y^2}{4}(1 - 3 \tan^2 \gamma) - x^2 \tan^2 \gamma$$

$$- \frac{\sqrt{3}cy}{2}(1 + \tan^2 \gamma) + \frac{c^2}{4}(3 - \tan^2 \gamma) = 0. \qquad (1.2; 6)$$

It is evident from this equation that a member of this family of curves is a hyperbola, parabola, or ellipse according as γ is less than 30°, equal to 30°, or greater than 30°. For $\gamma = 0°$ the equation yields the line

$$y = \frac{\sqrt{3}}{2} c$$

The characteristic curves for swing motion will now be determined. Let P be a point in the plane $z=c$. For the vertical projector, it is evident that point P moves in a circle about the origin as center. The equation of this circle is:

(C) $$x^2 + y^2 = b^2$$

where b is a parameter. Let (α, β, c) be the coordinates of a point on this circle. The equations of the line through P and the lens point are:

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{c}.$$

Then, after solving for α and β and substituting into the equation

$$\alpha^2 + \beta^2 = b^2$$

the equation of the circular cone is obtained:

$$x^2 + y^2 = \frac{b^2}{c^2} z^2. \qquad (1.2;\ 7)$$

Now rotate this cone about the OX axis through an angle of 60°. We do this by making use of the transformation (1.2; 4) and obtain the equation of the rotated cone:

$$x^2 + \frac{(y - \sqrt{3}z)^2}{4} = \frac{b^2}{c^2} \frac{(\sqrt{3}y + z)^2}{4}. \qquad (1.2;\ 8)$$

The intersection of this surface with the plane $z=c$ yields the characteristic curves for swing in the oblique projector:

(C′) $$x^2 + \frac{(y - \sqrt{3}c)^2}{4} = \frac{b^2}{c^2} \frac{(\sqrt{3}y + c)^2}{4}.$$

We now effect a change of parameter by employing the relation $\tan \gamma = b/c$. Then equation (C') becomes:

$$x^2 + y^2 \frac{(1 - 3\tan^2 \gamma)}{4} - \sqrt{3}cy \frac{(1 + \tan^2 \gamma)}{2} + \frac{c^2}{4}(3 - \tan^2 \gamma) = 0. \quad (1.2;\ 9).$$

From (1.2; 9) it is clear that a curve of the family is an ellipse, parabola, or hyperbola according as γ is less than 30°, equal to 30°, or greater than 30°. For $\gamma = 0°$ the equation yields a point ellipse.

We consider finally x-tilt motion. Let P: (a, O, c) be a point in the plane $z=c$. Under x-tilt motion, the point Q: (x, y, z) is displaced to Q': (x', y', z'), the transformation being given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & O & O \\ O & \cos\theta & \sin\theta \\ O & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad (1.2;\ 10)$$

where θ is the angle of rotation about the OX axis. Therefore, under x-tilt motion P is displaced to P':

$$P' = \begin{bmatrix} 1 & O & O \\ O & \cos\theta & \sin\theta \\ O & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ O \\ c \end{bmatrix} = \begin{bmatrix} a \\ c\sin\theta \\ c\cos\theta \end{bmatrix}.$$

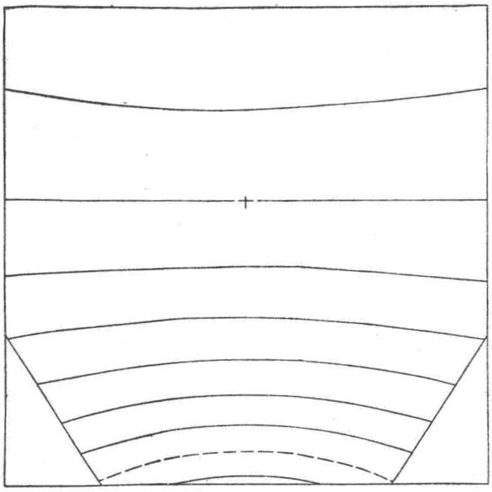


FIG. 7. Characteristic Curves for Y-Tilt Motion.

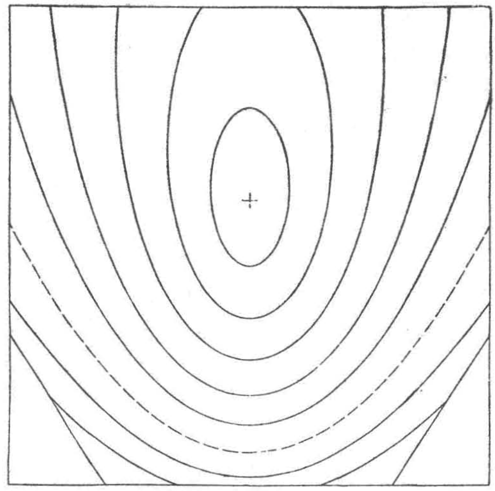


FIG. 8. Characteristic Curves for Swing Motion.

The equations of the line through O and P' are:

$$\frac{x}{a} = \frac{y}{c \sin \theta} = \frac{z}{c \cos \theta}.$$

The intersection of this line with the plane $z = c$ gives:

$$\frac{x}{a} = \sec \theta; \qquad \frac{y}{c} = \tan \theta.$$

The elimination of θ between these two equations gives the equation of the characteristic curves for x-tilt in the vertical projector:

$$(D) \qquad \frac{x^2}{a^2} - \frac{y^2}{c^2} = 1.$$

Since the orientation of the oblique projector is the same as that of the vertical projector displaced 60° by an x-tilt motion, it is clear that the characteristic curves for x-tilt motion are the same for both projectors and are given by (D).

In the illustrations of the characteristic curves, the principal point is located at the plus sign. The nadir point is easily located in Figure 4 since it lies on the common intersection of all the characteristic curves of bz motion.

ARTICLE 2. APPLICATIONS OF THE CHARACTERISTIC CURVES

2.1 General Analysis.

Let $O_1:(O, O, O)$ and $O_2:(a, O, O)$ be the lens points of projectors 1 and 2 (Figure 9). We suppose O_1 to be the origin of an ijk-system of base vectors. Let $P_0:(x_0, y_0, z_0)$ be a point in the multiplex model and let Γ be the horizontal plane through P_0.

When projector 2 undergoes a displacement, not necessarily a translation, the ray to P_0 moves and traces a curve C in the plane Γ:

$$x = x(\tau) \quad ; \qquad (2.1;\ 1)$$
$$y = y(\tau)$$

where we suppose that $x(\tau)$ and $y(\tau)$ are functions of τ which are continuous and which have as many continuous derivatives as may be required, and such that

$$x_0 = x(\tau_0); \qquad y_0 = y(\tau_0).$$

Under a small displacement, let τ vary from τ_0 to $\tau_0+\Delta r$ as the ray moves from P_0 to P_1: (x_1, y_1, z_1). We now define δx and δy by the following equations:

$$x_1 = x(\tau_0 + \Delta\tau) = x_0 + \delta x$$
$$y_1 = y(\tau_0 + \Delta\tau) = y_0 + \delta y$$

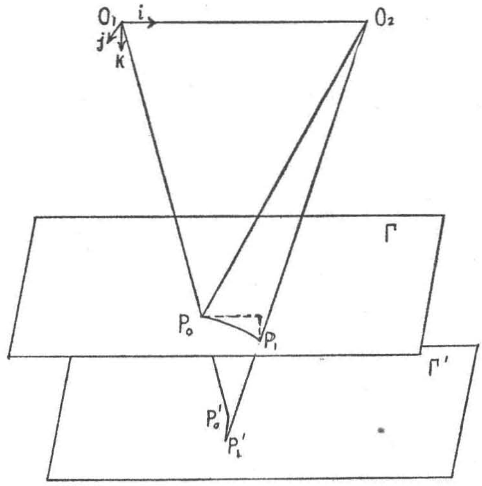


FIG. 9

Let B, R_1, R_2, R_2' be vectors defined as follows:

$$B = \overrightarrow{O_1O_2}; \qquad R_1 = \overrightarrow{O_1P_0}; \qquad R_2 = \overrightarrow{O_2P_0}; \qquad R_2' = \overrightarrow{O_2P_1},$$

where R_1 and R_2 are the conjugate rays before the displacement and R_1, R_2' are the conjugate rays after the displacement. Then employing the base vectors it is evident that we can obtain:

$$\begin{aligned} B &= ai \\ R_1 &= x_0 i + y_0 j + z_0 k \\ R_2 &= R_1 - B = (x_0 - a)i + y_0 j + z_0 k \\ R_2' &= R_2 + \delta x i + \delta y j. \end{aligned} \qquad (2.1;\ 2)$$

Now consider another horizontal plane Γ' and suppose the line determined by O_2P_1 intersects Γ' at P_1' and the line determined by O_1P_0 intersects Γ' at P_0'. In multiplex mapping, when the conjugate rays fail to intersect, the image point is taken to lie in that horizontal plane for which $P_0'P_1'$ is in the j-direction. Let the scalars λ, μ be defined by the equations:

$$\overrightarrow{O_1P_0'} = \lambda R_1; \qquad \overrightarrow{O_2P_1'} = \mu R_2'.$$

We now determine λ and μ such that:

$$\overrightarrow{P_0'P_1'} = pj.$$

where p is a scalar denoting the y-parallax. This condition is satisfied if and only if:

$$\overrightarrow{O_1P_1'} = \overrightarrow{O_1O_2} + \overrightarrow{O_2P_1'} = \overrightarrow{O_1P_0'} + \overrightarrow{P_0'P_1'}, \quad \text{or}$$

$$B + \mu R_2' = \lambda R_1 + pj. \qquad (2.1;\ 3)$$

Now making use of the relations in (2.1; 2) and the fact that the base vectors i, j, k are linearly independent, (2.1; 3) yields three scalar equations:

$$\begin{aligned} a + \mu(x_0 + \delta x - a) &= \lambda x_0 \\ \mu(y_0 + \delta y) &= \lambda y_0 + p \\ \mu z_0 &= \lambda x_0. \end{aligned} \tag{2.1; 4}$$

The solution of these equations yields:

$$\begin{aligned} \lambda &= \frac{a}{a - \delta x} = \mu \\ p &= \lambda \delta y. \end{aligned} \tag{2.1; 5}$$

The change in elevation, $-\Delta Z$, is given by:

$$- \Delta z = (1 - \lambda) z_0. \tag{2.1; 6}$$

Furthermore, if λ is eliminated between the two equations of (2.1; 5), the parallax can be expressed by:

$$p = \frac{a \delta y}{a - \delta x} = \frac{\delta y}{1 - \dfrac{\delta x}{a}} = \delta y \left[\sum_{i=0}^{\infty} \left(\frac{\delta x}{a} \right)^i \right]. \tag{2.1; 7}$$

The continuity of $x(\tau)$ and $y(\tau)$ insures that:

$$\underset{\Delta\tau \to 0}{\text{limit}}\, \delta x = \underset{\Delta\tau \to 0}{\text{limit}}\, \delta y = 0$$

and, except for isolated points,

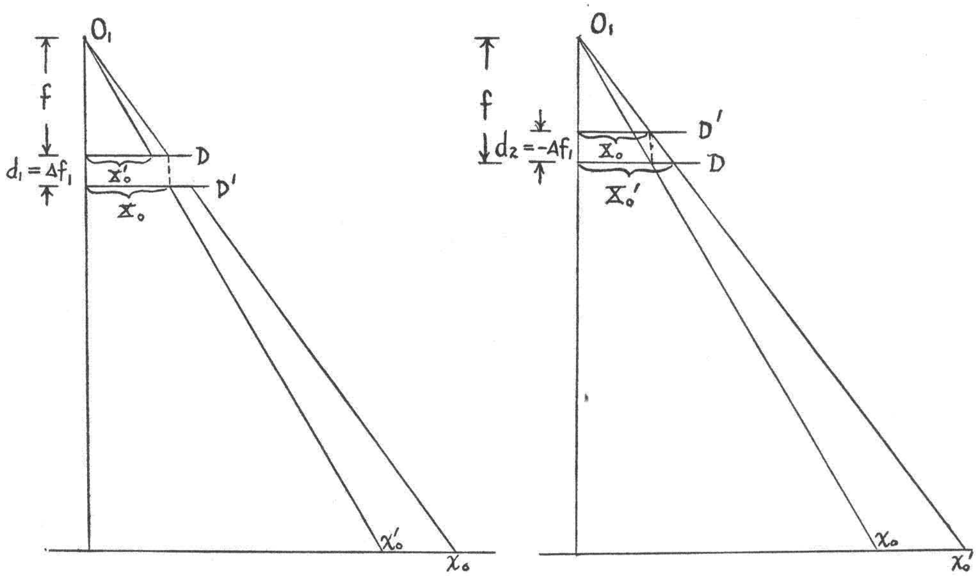
$$\underset{\Delta\tau \to 0}{\text{limit}} \frac{\delta y}{\delta x} = \left(\frac{dy}{dx} \right)_{P_0} \neq 0.$$

Therefore δy and δx are infinitesimals of the same order. Equation (2.1; 7) shows that p is an infinitesimal of the same order as δy. Then substituting the value of λ given by (2.1; 5) into equation (2.1; 6), the following may be obtained:

$$\begin{aligned} - \Delta z &= z_0 \left(1 - \frac{a}{a - \delta x} \right) = z_0 \left\{ 1 - \frac{1}{1 - \dfrac{\delta x}{a}} \right\} = z_0 \left\{ \frac{- \dfrac{\delta x}{a}}{1 - \dfrac{\delta x}{a}} \right\} \\ &= - z_0 \left[\sum_{i=1}^{\infty} \left(\frac{\delta x}{a} \right)^i \right]. \end{aligned} \tag{2.1; 8}$$

The last equation shows that Δz and δz are infinitesimals of the same order. Due to the presence of y-parallax, the new position of the image point is not definite, but it may be defined for practical purposes, as the mid-point of the line segment $P_0'P_1'$. Then Δy, the change in the y-coordinate of the image point, is given by:

$$\Delta y = (\lambda - 1) y_0 + \frac{p}{2} = \frac{p}{2} + y_0 \left[\sum_{i=1}^{\infty} \left(\frac{\delta x}{a} \right)^i \right]. \tag{2.1; 9}$$



Principal Distance Increased **Principal Distance Decreased**

FIG. 10

Similarly, the change in the x-coordinate, Δx, is given by:

$$\Delta x = (\lambda - 1)x_0 = x_0\left[\sum_{i=1}^{\infty}\left(\frac{\delta x}{a}\right)^i\right]. \tag{2.1; 10}$$

These results suggest an immediate application of the characteristic curves, and indicate the usefulness of the parametric representation. Now assuming there is given an absolutely oriented and parallax-free model, let us investigate the effects of a small but definite displacement in one projector, the displacement being one of the primary motions. We shall consider first the translations, where the results are equally valid for both the vertical and oblique projectors.

2.2 The Translations.

The equations of the characteristic curves for the translations were obtained in Article 1. In order to apply the results of section 2.1, these equations will be put in parametric form. Let us first consider bx motion. The parametric equations are:

$$x = \tau \qquad y = b, \tag{2.2; 1}$$

where b is a constant which defines a particular curve of the family and τ is the continuous parameter for the representation. Then the differentials of x and y are given by:

$$dx = d\tau \qquad dy \overset{b}{\equiv} 0, \tag{2.2; 2}$$

where $d\tau$ is a small but definite linear displacement. Now applying equation (2.1; 5) we find that λ is constant and the parallax vanishes for every point in the plane $\Gamma: z=z_0$. Similarly we find, by means of equation (2.1; 6), that the change in elevation is constant for every point in Γ. Therefore, a bx displacement carries the plane Γ into the plane $\Gamma': z=z+\Delta z$. Moreover, by applying (2.1; 6) we find that Δz is proportional to z_0. This implies a uniform change in the vertical scale. For let $P_1:(x_1, y_1, z_1)$ and $P_2:(x_2, y_2, z_2)$ be any two points in the model. The difference in elevation between P_1 and P_2 is $|z_2-z_1|$. Now suppose a small bx displacement is introduced. Then P_1 moves to $P_1':(x_1', y_1', z_1')$ and P_2 moves to $P_2':(x_2', y_2', z_2')$. In the transformed model, the difference in elevation between the corresponding points P_1' and P_2' is given by $|z_2'-z_1'|$. Then we have $z_2'=z_2+\Delta z=z_2+(\lambda-1)z_2=\lambda z_2$, and similarly, $z_1'=\lambda z_1$. Therefore it follows that $z_2'-z_1'=\lambda(z_2-z_1)$, and since λ is constant a uniform vertical scale is established. The same argument may be applied, using the x and y coördinates, to show that there is a uniform scale in the i and j directions. Thus it follows that the scale is independent of direction and is equal to λ times the original scale. For let $D(A, B)$ denote the distance between the points A and B.
Then

$$\begin{aligned} D(P_1', P_2') &= \sqrt{(x_2'-x_1')^2+(y_2'-y_1')^2+(z_2'-z_1')^2} \\ &= \sqrt{\lambda^2(x_2-x_1)^2+\lambda^2(y_2-y_1)^2+\lambda^2(z_2-z_1)^2} \\ &= \lambda\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \\ &= \lambda D(P_1, P_2). \end{aligned}$$

We proceed now to consider by motion. The parametric equations of the characteristic curves are:

$$\begin{aligned} x &= a \\ y &= \tau, \end{aligned} \tag{2.2; 3}$$

where a is the constant defining a curve of the family and τ is the continuous parameter. Then taking differentials, we obtain:

$$\begin{aligned} dx &\overset{a}{\equiv} 0 \\ dy &= d\tau. \end{aligned} \tag{2.2; 4}$$

The application of (2.1; 5) yields $\lambda=1$ and $p=d\tau$, which is constant for the model. Furthermore, from (2.1; 6) we see that there is no elevation change. Multiplex mappers make use of this fact when they remove y-parallax with the by motion of one projector without affecting elevations.

We conclude this section with the analysis of bz motion. Let τ denote the continuous parameter, and γ the direction angle defined in section 1.1. Then using the polar coordinates for the plane, we have:

$$\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta. \end{aligned} \tag{2.2; 5}$$

Now if we consider a particular point during a bz displacement, we obtain the relation:

$$\rho = \tau\tan\gamma.$$

Therefore the parametric equations of the characteristic curves are:

$$\begin{aligned} x &= \tau \tan \gamma \cos \theta \\ y &= \tau \tan \gamma \sin \theta. \end{aligned} \qquad (2.2;\ 6)$$

These equations then yield the differentials of x and y:

$$\begin{aligned} dx &= \tan \gamma \cos \theta d\tau = x\frac{d\tau}{\tau} \\ dy &= \tan \gamma \sin \theta d\tau = y\frac{d\tau}{\tau}\cdot \end{aligned} \qquad (2.2;\ 7)$$

Here $d\tau$ is taken to be a constant displacement for the entire model. For a horizontal plane in the model, τ is a constant. Then making use of (2.1; 7) and (2.1; 8), we find that the change in elevation is proportional to x and the parallax is proportional to y. For the vertical line $x=x_0$, $y=y_0$, the elevation change and the parallax are both inversely proportional to the z-coordinate, since $\tau=z$. This demonstrates that "shear" is introduced into the model under a bz displacement. However, for the small displacements which occur in practice, this effect may be safely neglected.

2.3 The Rotations in the Vertical Projector.

We shall consider swing motion first. The parametric equations of the characteristic curves are:

$$\begin{aligned} x &= R \cos \omega \\ y &= R \sin \omega, \end{aligned} \qquad (2.3;\ 1)$$

where R is a constant. The differentials are then given by:

$$\begin{aligned} dx &= - R \sin \omega d\omega = - y d\omega \\ dy &= R \cos \omega d\omega = x d\omega. \end{aligned} \qquad (2.3;\ 2)$$

If we make use of the series expansions given by (2.1; 7) and (2.1; 8), neglecting terms higher than the first, equations (2.3; 2) show that for the horizontal plane Γ the parallax is proportional to x and the change in elevations is proportional to y. It therefore follows that this transformation rotates the horizontal plane about a line parallel to the OX axis.

Let us now examine y-tilt motion. The characteristic curves are given by these parametric equations:

$$\begin{aligned} x &= z_0 \tan \omega \\ y &= b \sec \omega, \end{aligned} \qquad (2.3;\ 3)$$

where z_0 is a constant determined by the horizontal plane $\Gamma: z=z_0$, b is a constant defining a curve of the family, and ω is the continuous parameter of the rotation. The differentials are then found to be:

$$\begin{aligned} dx &= z_0 \sec^2 \omega d\omega = (z_0{}^2 + x^2) \frac{d\omega}{z_0} \\ dy &= b \sec \omega \tan \omega d\omega = xy \frac{d\omega}{z_0} \end{aligned} \qquad (2.3;\ 4)$$

where $d\omega$ is taken to be a small angular displacement, so that $d\omega/z_0$ is constant. The elevation change does not vary with y and thus is constant for any line parallel to the OY axis. The variation of Δz with respect to x shows that this transformation warps the horizontal plane Γ into a parabolic cylinder. This effect is quite appreciable, since a displacement of only two minutes of arc results in elevation changes which can be detected. The variation of the parallax over Γ is easily read from the second equation of (2.3; 4) and the implications are rather obvious.

Let us finally consider x-tilt motion. The parametric equations of the characteristic curves are:

$$\begin{aligned} x &= a \sec \omega \\ y &= z_0 \tan \omega, \end{aligned} \tag{2.3; 5}$$

which yields the following differentials:

$$\begin{aligned} dx &= a \sec \omega \tan \omega d\omega = xy \frac{d\omega}{z_0} \\ dy &= z_0 \sec^2 \omega d\omega = (z_0^2 + y^2) \frac{d\omega}{z_0} \cdot \end{aligned} \tag{2.3; 6}$$

These equations indicate that the distribution of parallax is symmetric with respect to the x-axis of the plane, and that the warped surface is a ruled surface. This transformation makes it possible for the multiplex operator to level a model by warping the corners on one side of the model.

2.4 The Rotations in the Oblique Projector.

The analysis of a rotation in the oblique projector is more difficult than for the vertical projector. In this section, swing motion will be the only rotation to be investigated. However, the methods adopted may be applied with the same facility to the other rotations. We may consider the swing motion in this projector as resulting from a rotation about the OZ axis (given by the transformation matrix A) followed by a rotation about the OX axis (given by the transformation matrix B), where A and B are as follows:

$$A: \begin{bmatrix} \cos \omega & -\sin \omega & O \\ \sin \omega & \cos \omega & O \\ O & O & 1 \end{bmatrix}, \qquad B: \begin{bmatrix} 1 & O & O \\ O & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ O & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \tag{2.4; 1}$$

The transformation B carries the OZ axis into the optical axis of the oblique projector. Since B is an orthogonal matrix, B^{-1}, the inverse of B, is obtained by interchanging rows and columns. The resultant transformation is then given by[1]:

$$BAB^{-1}: \begin{bmatrix} \cos \omega & -\frac{1}{2} \sin \omega & \frac{\sqrt{3}}{2} \sin \omega \\ -\frac{1}{2} \sin \omega & \frac{1}{4} \cos \omega + \frac{3}{4} & \frac{\sqrt{3}}{4} \cos \omega - \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{2} \sin \omega & \frac{\sqrt{3}}{4} \cos \omega - \frac{\sqrt{3}}{4} & \frac{3}{4} \cos \omega + \frac{1}{4} \end{bmatrix}. \tag{2.4; 2}$$

Let $P:(a, b, c)$ be any point in the plane $\Gamma: z=c$. Under swing motion this point moves to $P':(a', b', c')$, which is given by:

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = BAB^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} =$$

$$\begin{bmatrix} a \cos\omega + \frac{b}{2}\sin\omega + \frac{c}{2}\sqrt{3}\sin\omega \\ -\frac{a}{2}\sin\omega + \frac{b}{4}(\cos\omega + 3) + \frac{c}{4}(\sqrt{3}\cos\omega - \sqrt{3}) \\ -\frac{a}{2}\sqrt{3}\sin\omega + \frac{b}{4}(\sqrt{3}\cos\omega - \sqrt{3}) + \frac{c}{4}(3\cos\omega + 1) \end{bmatrix} \qquad (2.4;\ 3)$$

Now $x/a' = y/b' = z/c'$ are the equations of the ray through P'. The intersection of this ray with the plane Γ gives:

$$x = \frac{a'}{c'}c, \qquad y = \frac{b'}{c'}c. \qquad (2.4;\ 4)$$

Since ω is small, we taken $\cos\omega = 1$ and $\sin\omega = \omega$ as satisfactory approximations. Then substituting these quantities into (2.4; 4), we obtain:

$$x = \frac{a + \frac{b}{2}\omega - \frac{\sqrt{3}}{2}c\omega}{c - \frac{\sqrt{3}}{2}a\omega}\cdot c$$

$$y = \frac{-\frac{a}{2}\omega + b}{c - \frac{\sqrt{3}}{2}a\omega}\cdot c. \qquad (2.4;\ 5)$$

We approximate further by finding the tangent line to the curve given by (2.4; 5).
Let

$$\frac{a + \alpha\omega}{c + \beta\omega} \equiv K_0 + K_1\omega + \cdots \quad \text{where} \quad \alpha = \frac{b}{2} + \frac{\sqrt{3}}{2}c, \quad \beta = -\frac{\sqrt{3}}{2}a.$$

Then $a + \alpha\omega \equiv (c + \beta\omega)(\kappa_0 + \kappa_1\omega + \cdots)$. Now equating coefficients, we obtain $a = c\kappa_0$, $\alpha = \beta\kappa_0 + c\kappa_1$. Then solving these equations for K_0 and K_1 and substituting into $x = (\kappa_0 + \kappa_1\omega)c$ we obtain, after repeating the procedure for the second equation of (2.4; 5).

$$x = a + \left(\frac{\sqrt{3}}{2}\frac{a^2}{c} + \frac{b}{2} + \frac{\sqrt{3}}{2}c\right)\omega$$

$$y = b + \left(\frac{\sqrt{3}}{2}\frac{ab}{c} - \frac{a}{2}\right)\omega \qquad (2.4;\ 6)$$

The differentials are then given by:

$$dx = \left(\frac{\sqrt{3}}{2}\frac{a^2}{c} + \frac{b}{2} + \frac{\sqrt{3}}{2}c\right)d\omega$$

$$dy = \left(\frac{\sqrt{3}}{2}\frac{ab}{c} - \frac{a}{2}\right)d\omega \qquad (2.4;\ 7)$$

Chapter II, Displacement of Interior Orientation

In the multiplex projector, there is a fixed orientation of the diapositive plate with respect to the lens. The word plate will be used hereafter to mean diapositive plate. The optical axis of the lens system must be normal to the plate and pass through the principal point of the plate. The principal distance of the lens will be denoted by f. As before, we shall suppose O_1, the lens point of projector 1, to be the origin of an ijk-system of base vectors (Figure 9). We shall consider several simple types of displacements which have been known to occur in practice, beginning with the translations.

ARTICLE 1. THE TRANSLATIONS

In this article we shall consider two types of translations. First, the translation where the only element of interior orientation which is disturbed is the principal distance. The other translation which is examined, carries the principal point of the plate away from the optical axis without affecting the other elements of orientation.

1.1 Variation of the Principal Distance.

We suppose the principal distances of projectors 1 and 2 to be in error by Δf_1 and Δf_2 respectively, where Δf is positive or negative according as the principal distance is increased or decreased. Let ρ_1 and ρ_2 be defined by:

$$\rho_1 = \frac{f}{f + \Delta f_1}, \qquad \rho_2 = \frac{f}{f + \Delta f_2}.$$

In the diagram (Figure 10), the plane of the paper represents the vertical section through the lens of the projector which contains the OX and OZ axes. The point O_1 is the lens point, D is the plate in correct position, and D' is the plate whose principal distance is in error. The coordinates X_0 and X_0' are the plate coördinates, while x_0 and x_0' are the corresponding coordinates in the plane $\Gamma: z = z_0$. For the case where the principal distance is increased we have:

$$\frac{x_0'}{x_0} = \frac{X_0'}{X_0} = \frac{f}{f + d_1} = \frac{f}{f + \Delta f_1} = \rho_1.$$

For the case where the principal distance is decreased we have:

$$\frac{x_0'}{x_0} = \frac{X_0'}{X_0} = \frac{f}{f - dz} = \frac{f}{f + \Delta f_1} = \rho_1.$$

Thus, in either case, $X_0' = \rho_1 X_0$. A similar demonstration shows that $Y_0' = \rho_1 Y_0$. Now let O_1, O_2, P_0, R_1, R_2, and B be defined as in section 2.1. Then, due to the error in the principal distance, the rays R_1 and R_2 are carried into R_1' and R_2' which are given by:

$$R_1' = \rho_1 x_0 i + \rho_1 y_0 j + z_0 K$$
$$R_2' = \rho_2(x_0 - a)i + \rho_2 y_0 j + z_0 K. \qquad (1.1;\ 1)$$

We seek to determine scalars λ_1 and λ_2 such that $\lambda_1 R_1'$ and $\lambda_2 R_2'$ are the position vectors for the points P_1 and P_2 which satisfy the vector equation $\overrightarrow{P_1P_2} = Pj$, where p is a scalar defining the parallax. This condition is satisfied if and only if:

$$\overrightarrow{O_1P_2} = \overrightarrow{O_1P_1} + \overrightarrow{P_1P_2} = \overrightarrow{O_1O_2} + \overrightarrow{O_2P_2} \quad \text{or}$$
$$B + \lambda_2 R_2' = \lambda_1 R_1' + pj. \qquad (1.1;\ 2)$$

Then substituting for B, R_1', and R_2' we obtain:

$$ai + \lambda_2[\rho_2(x_0 - a)i + \rho_2 y_0 j + z_0 K] = \lambda_1[\rho_1 x_0 i + \rho_1 y_0 j + z_0 K] + pj.$$

Since the base vestors are linearly independent, we obtain three scalar equations:

$$a + \lambda_2\rho_2(x_0 - a) = \lambda_1\rho_1 x_0$$
$$\lambda_2\rho_2 y_0 = \lambda_1\rho_1 y_0 + p \qquad (1.1;\ 3)$$
$$\lambda_1 z_0 = \lambda_2 z_0.$$

The solution of these equations is given by:

$$\lambda_1 = \lambda_2 = \frac{a}{x_0(\rho_1 - \rho_2) + \rho_2 a}$$
$$p = \frac{a y_0(\rho_2 - \rho_1)}{x_0(\rho_1 - \rho_2) + \rho_2 a}. \qquad (1.1;\ 4)$$

The change in elevation, $-\Delta z$, is given by:

$$-\Delta z = (1 - \lambda_1)z_0. \qquad (1.1;\ 5)$$

If $\Delta f_1 = \Delta f_2$ then $\rho_1 = \rho_2$ and (1.1;4) shows that $p = 0$. This indicates that when the principal distances are both increased or both decreased by the same amount, no parallax is introduced. Furthermore, if $\rho_1 = \rho_2$, then $(1 - \lambda_1)$ is constant and equation (1.1; 5) shows that a new vertical scale is obtained which is λ_1 times the original scale, where $\lambda_1 = 1/\rho_1$. The horizontal scale of the model is not affected in the i-direction, but the scale in the j-direction is multiplied by the factor ρ_1.

1.2 Displacement of the Principal Point from the Optical Axis

This kind of displacement ordinarily occurs when the centering microscope is out of adjustment. For simplicity, let us assume that only the plate in projector 1 is displaced, and suppose the principal point to be displaced by amounts ϵ_x and ϵ_y in the i and j directions respectively. Then the ray R_1 to the image point P_0: (x_0, y_0, z_0) is carried into the ray R_1':

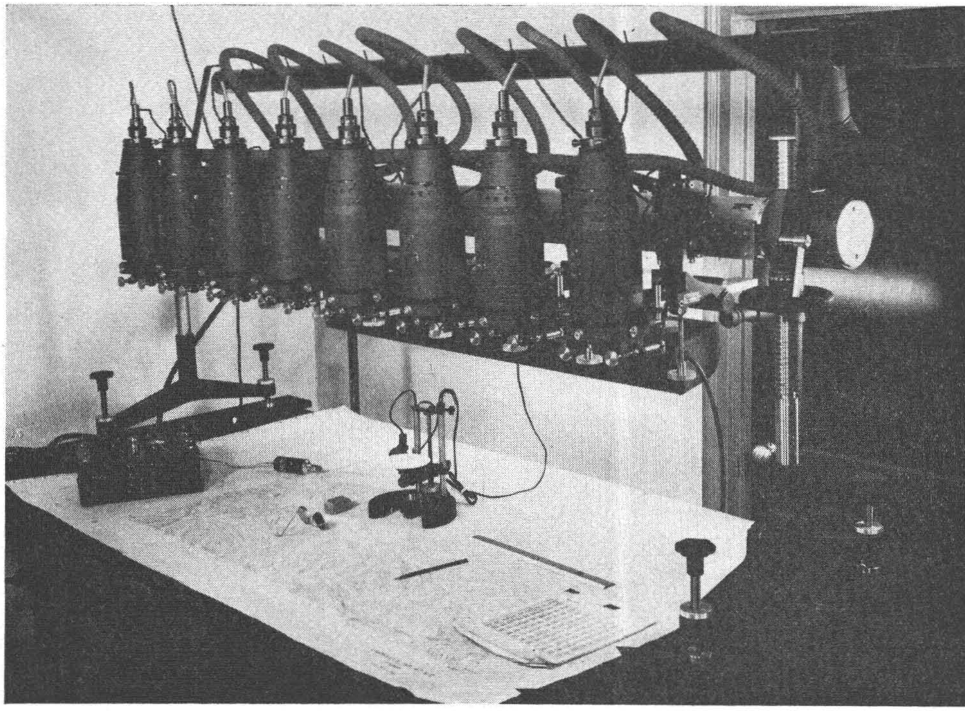
$$R_1' = (x_0 + hz_0)i + (y_0 + Kz_0)j + z_0 K,$$

where $h = \epsilon_x/f$ and $K = \epsilon_y/f$. Now defining λ_1 nd λ_2 and p as in section 1.1, we are led to the following equation:

$$B + \lambda_2 R_2 = \lambda_1 R_1' + pj. \qquad (1.2;\ 1)$$

When the vector quantities are written in terms of the base vectors, the equation above becomes:

$$ai + \lambda_2[(x_0 - a)i + y_0 j + z_0 K] = \lambda_1[(x_0 + hz_0)i + (y_0 + Kz_0)j + z_0 K] + pj,$$



Multiplex

which yields as before three scalar equations:

$$\begin{aligned} a + \lambda_2(x_0 - a) &= \lambda_1(x_0 + hz_0) \\ \lambda_2 y_0 &= \lambda_1(y_0 + Kz_0) + p. \\ \lambda_2 z_0 &= \lambda_1 z_0. \end{aligned} \tag{1.2; 2}$$

Then solving for λ's and p we obtain:

$$\begin{aligned} \lambda_1 = \lambda_2 &= \frac{a}{a + hz_0} \\ p &= -\lambda_1 K z_0. \end{aligned} \tag{1.2; 3}$$

We observe immediately that if $\epsilon_y = 0$, no parallax is introduced. More generally, the parallax effect and the change in elevation depend only on z_0. Now λ_1 can be expanded into the convergent series:

$$\lambda_1 = \sum_{i=0}^{\infty}\left(-\frac{hz_0}{a}\right)^i. \tag{1.2; 4}$$

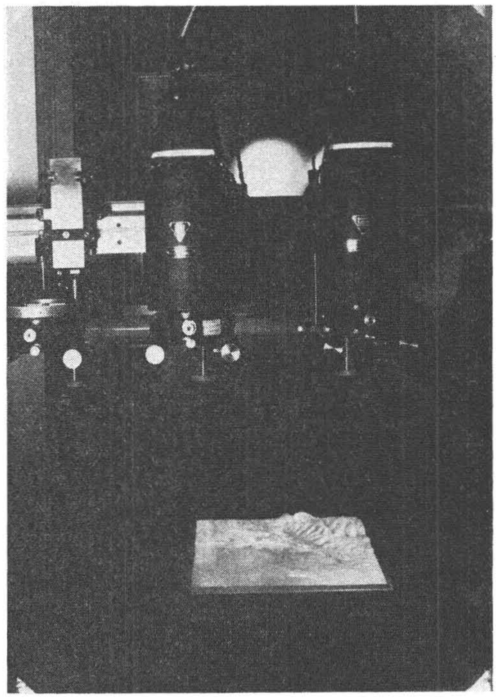
Then employing (2.1; 6) of Chapter I and (1.2; 3) above, p and $-\Delta z$ have the following series representations:

$$\begin{aligned} -\Delta z &= z_0 \sum_{i=1}^{\infty} (-1)^{i+1}\left(\frac{hz_0}{a}\right)^i, \\ p &= -Kz_0 \sum_{i=0}^{\infty}\left(-\frac{hz_0}{a}\right)^i. \end{aligned} \tag{1.2; 5}$$

We observe that p and $-\Delta z$ are independent of the x and y coördinates. The displacement of a point in the i and j directions is proportional to the x and y coördinates of the point. If $h \neq 0$ there is "shear" in the model, so that the image of a vertical object has a positive angle of inclination with respect to the OZ axis.

In the special case of a flat model, the change in elevation is constant and the model can be leveled by adjusting the index on the plotting instrument. Furthermore, the parallax is constant and the analysis of by motion in section 2.2 of Chapter I shows that this parallax may be removed by applying by motion in one projector.

With the proper modifications, the case of displacement in the plates of both projectors can be similarly treated. In fact, it appears that this case might be treated simply as the resultant of a displacement in projector 1 followed by a displacement in projector 2. Then, as a first approximation, the parallax and elevation effects would be obtained by ordinary algebraic addition.



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ARTICLE 2. ROTATION

In this Article, we shall consider the rotations of the plate about a line in the plate passing through the principal point. The optical axis, though passing through the principal point, is now no longer normal to the plate. We shall assume for the sake of simplicity that the plate in projector 2 has the proper interior orientation. Suppose the plate in projector 1 is rotated through an angle ω about a line through the principal point. Let O, the principal point of the plate, be the origin of a Cartesian coördinate system whose x, y, and z axes are parallel respectively to the i, j, and k directions and which is so oriented that the positive direction of the OZ axis is upward. The plane of the plate is the XY-plane. Let L denote the line in the plane about which the plate is rotated, and let α denote the angle of inclination of L with respect to the OX axis. Now treating this rotation as a rigid body displacement, we consider the transformation of coördinates as resulting from a rotation about the OX axis (matrix A) followed by a rotation about the OZ axis (matrix B). The latter carries the OX axis into the line L. Matrices A and B are given by:

$$A: \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{bmatrix}, \qquad B: \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (2;\,1)$$

Then the resultant transformation is given by the matrix:

$$BAB^{-1}: \begin{bmatrix} \cos^2\alpha + \sin^2\alpha\cos\omega & \sin\alpha\cos\alpha - \sin\alpha\cos\alpha\cos\omega & \sin\alpha\sin\omega \\ \sin\alpha\cos\alpha - \sin\alpha\cos\alpha\cos\omega & \sin^2\alpha + \cos^2\alpha\cos\omega & -\cos\alpha\sin\omega \\ -\sin\alpha\sin\omega & \cos\alpha\sin\omega & \cos\omega \end{bmatrix} \qquad (2;\,2)$$

Now let P_0: (x_0, y_0, z_0) be a point in the model. The plate coördinates of P_0 are given by (a_0, b_0, O) where a_0 and b_0 are determined by:

$$x_0 = \frac{z_0}{f} a_0$$
$$y_0 = \frac{z_0}{f} b_0. \tag{2; 3}$$

Due to the displacement of the plate, the point (a, b, O) is carried into (a, b, c):

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = BAB^{-1} \begin{bmatrix} a_0 \\ b_0 \\ O \end{bmatrix}. \tag{2; 4}$$

We now obtain the equations of the line through (O, O, f) and (a, b, c):

$$\frac{x}{a} = \frac{y}{b} = \frac{z - f}{c - f}.$$

The intersection of this line with the plane $z = O$ is the point (a_0', b_0', O) where

$$a_0' = \frac{af}{f - c}, \qquad b_0' = \frac{bf}{f - c}.$$

Then

$$R_1' = x_0'i + y_0'j + z_0'k \quad \text{where} \quad x_0' = \frac{z_0}{f} a_0' \quad \text{and} \quad y_0' = \frac{z_0}{f} b_0'.$$

Now defining λ and p as before, we obtain the vector equation:

$$B + \lambda R_2 = \lambda R_1' + pj. \tag{2; 5}$$

The solution of the resulting scalar equations can then be completed as in the preceding article. With obvious modifications, rotation in both plates may be investigated. However, under ordinary operating conditions, it is highly improbable that both plates would have this error in orientation.

NEWS NOTE

"WINGS FOR THE TRANSIT" PICTURES AERIAL MAPPING

Basic principles of modern techniques employed in producing aerial topographic maps are explained in a new brochure, "Wings for the Transit," published by Lockwood, Kessler & Bartlett, Inc., 32 Court St., Brooklyn 2, N. Y. Profusely illustrated with striking photographs, this publication reveals various stages in the preparation of accurate maps from precision aerial photographs. It also exhibits samples of these photographs, stereoscopic pairs, mosaics, and photogrammetric maps drawn with the aid of precise stereoplotting instruments.

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