ABOUT A NEW GRAPHIC METHOD OF ORIENTING A PAIR OF AERIAL PHOTOGRAPHS

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SECTION 1

PHOTOGRAMMETRY is one of the newest branches of scientific study. Although its first steps were taken long before photography was invented, it started as an independent branch of science about the beginning of this century. Very soon after the first World War, it achieved its greatest progress by solving the *principal problem of aerial photogrammetry*, that is to say, the relative orientation of two successive photographs, by applying the optical-mechanic method, and by improving the instruments thoroughly. But all great efforts made for these purposes contrasted strangely with the fact that important theoretical fundamentals, i.e., the *geometrical* ones, had not yet been cleared entirely. We readily admit that at this time the defect was not yet recognized. It seems to be characteristic that the first impulse to eliminate this deficiency arose from the *practical experience* with the newly developed apparatus. It was *R. Bosshardt* who, in 1933 (2), described for the first time an observation of the uncertainty of the relative orientation, due to the fact that the contemplated model points on each plane, being perpendicular to the base-line, were situated on a circle which passed through the base-line. A year before, however, *S. Finsterwalder* (9) had already mentioned the "*dangerous case*" (by computing) of the reciprocal adjustment (by means of five linear equations, the determinant of their factors may equal zero). But he still believed that this could be avoided by a suitable selection of the orientation points. Later on, *G. Poivilliers* (39) stated at the Fourth International Congress for Photogrammetry, held at Paris in 1934, that also such special ground forms are geometrically imaginable, which, if being surveyed, caused the principal problem to have *two quite different solutions*. Up to this date, the photogrammetrists, and well-known investigators of pure geometry too (38), were of the erroneous opinion that, in case of uneven ground, there may exist *only one solution;* furthermore, that this solution would be obtained indeed by putting at least *five pairs of collimating rays* in their position of intersection.

The above indicated statements have started discussions about what nowadays is called "critical surfaces" of aerial photogrammetry. An explanation of this problem was given by the author in a fundamental paper (16) completed in 1937, but without the knowledge of the above mentioned information. Chiefly, he gave an exhaustive consideration of the most general surfaces of the kind referred to. Especially, he indicated that besides the real orientation, representing the main solution, there still may exist one or two *additional solutions* essentially different. Further, he pointed out that all ground forms at hand, or other photographed objects (including the plane ones) which may show these qualities, belong to the so-called *orthogonal ruled surfaces of second degree* (or to certain degenerate forms). Therewith, it is necessary in each case, that the two surveying centers should lie on the surveyed surface (being geometrically complete), and should have certain special positions there. The cases of inexact adjustment mentioned by *S. Finsterwalder* and *R. Bosshardt* could now be ranged

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among those specialized cases of "critical surfaces" for which an additional solution is identical with the main one.

The author also examined the quadratic transformation between the model surfaces appearing in separate solutions. Furthermore he studied the variations of all five-parameter "critical surfaces" associated with two perspective centers. In three other papers (17, 18, 19), he dealt with remarkable special cases of these surfaces, and with a series of geometrical details resulting from his studies about the "critical surfaces"; e.g., it was shown that two different solutions of the principal problem are interchanged by rotating one of the pencils of rays around a distinct straight line of space.

In his summary of 1942 (20), the author discussed the "critical zones of space" for the first time, being in close connection with the "critical surfaces," and dealt with them later on in detail in his ten publications of 1947 and 1948, (21–30). Above all, he recognized that always *a well defined "critical zone of space" is attached to each movement of the two pencils* of collimating rays, such movements being performed in the execution of the optical-mechanical adjustment. Herewith, the useful effect of these investigations for photogrammetry became obvious.

SECTION 2

In the above mentioned papers, a "critical zone of space" was defined as *the totality of all those space points whose pairs of conjugate rays*, caused by an arbitrary displacement of the two pencils, *have y-parallaxes that are smaller than can be readily measured.* Each critical zone of this kind is limited by two surfaces of the second order, and this is valid for the angular or two-projector* method as well as for the single-projector method of orientation. Besides this, the measurement of parallaxes may be either actually on the spatial model (21–27) or one the picture plates, (28, 29, 30). In all these cases the complete spatial distribution of the y-parallaxes, caused by a displacement of the pencils of rays, is made very evident by a *linear pencil of surfaces of constant parallax* (all of them being of *second order*).

Further on, general formulas for the calculation of the *main sizes* of the "critical zones of space" have been derived (25, 28). One of these sizes has also been calculated by *J. Killian* (14) in another manner, but only for some typical special cases. These formulas show immediately that a *"critical zone of space" becomes the more extended*, the smaller the quantities of orientation are which produce the concerned displacement. This is of some practical importance regarding the fact that *each topographical surface, lying entirely inside of a "critical zone of space" is a product of the relative orientation of the same effect as a "critical surface."* Herewith, a very illustrative geometrical explanation was found for the fact, proved by *W. K. Bachmann* (1) and confirmed by *H. Kasper* (13), that the usual optical-mechanical procedure does not converge beyond a certain approximation (not at all a maximum one). The writer succeeded finally in showing the relations between the unknowns of orientation into simple mathematical relations which, between certain narrow limits, can also be understood as *equations of condition* between the quantities of orientation; these relations were formerly recognized by *R. Finsterwalder* (3, 4) and *E. Gotthardt* (10). Owing to numerous other details the reader has to refer to these related papers.

It would require a special geometrical knowledge to follow the explanation

* Editor's note—In the single-projector method, relative orientation is achieved by both the angular and translational motions of one projector, whereas in the two-projector method, only the angular motions of both projectors are employed.

of these various new results, especially a knowledge about the orthogonal ruled surfaces of second degree. Therefore, we confine ourselves to giving a short and easily comprehensive outline about the graphical method discovered by the author in connection with his investigations about the "critical zones of space." Finally, for a better understanding, a practical example is described.

SECTION 3

It is already known that in performing relative orientation of two aerial photographs, three different phases can be distinguished. The first one, called *pre-orientation*, includes all operations by means of which such a position of the two pencils of collimating rays can be reached that, relative to the remaining orientation, only the terms of the first order of the well-known types of observational equations have to be taken in consideration. The author has proposed the designation "main phase" (37) for the sum of the orientation movements which then are still to be accomplished. The largest part of all publications issued up to today about the principal problem of aerial photogrammetry, refers to this main-phase. In practice, the emerging of this phase is usually noticed by the y-parallaxes of the collimating rays which are measured either in the spatial model or on the picture plates, because, these quantities do not amount to certain maximum values (depending on the flying height and the model scale). The author denoted finally the last part of the main-phase wherein the orientation quantities are determined and applied from the already mentioned equations of condition as the "final phase" (32).

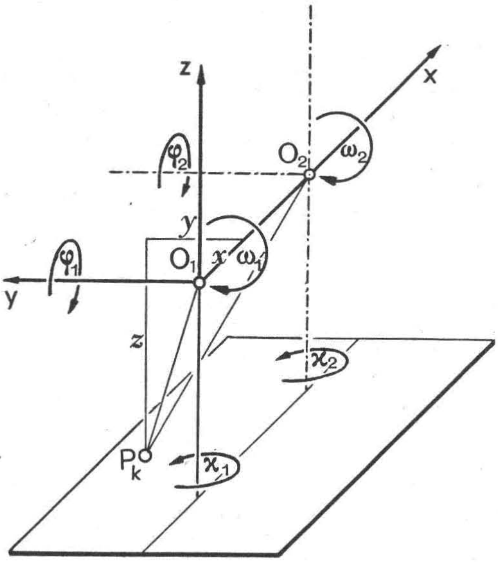

FIG. 1. The system of coordinates.

Now, we consider two air photographs of a topographical surface in which the normal case of stereo-photogrammetry is realized, at least approximately. Then, as usual, we assume a rectangular system of coordinates, the origin of which being identical with the center O_1 of the left photograph. Furthermore, the positive y-axis and the positive z-axis are directed backwards and upwards respectively. Of course, this system defines exactly the signs of the *translatory movements* of the two pencils of rays in the directions of the axes as well as the signs of the *rotary movements*, i.e., of the x-tilt ($y \rightarrow z$), y-tilt ($z \rightarrow x$), and swing ($x \rightarrow y$) (see also Figure 1). It is to be regretted that no standards have been adopted yet concerning these signs.

Once the main-phase is completed then the principal problem—the reestablishing of the original position of the two pencils during exposure—may be reduced to the following one: *There are given some model-points P_k; and the small residual parallaxes dp_k measured at these points are given as well. It is required to find out such displacements of the two pencils by which all these parallaxes disappear.*

As usual, we denote the small quantities of orientation, which produce a distinct change in y-parallax, $d\omega_1$, $d\omega_2$; $d\phi_1$, $d\phi_2$; $d\kappa_1$, $d\kappa_2$; db_{y1}, db_{y2}; db_{z1},db_{z2}. Let the coordinates of a model-point P_k be x, y, z. Then the parallaxes dp_k existing at P_k, will be changed in accord with the movement by the amount of (1, 3, 7, 13)

$$dp_y = -\frac{y^2+z^2}{z}(d\omega_1 - d\omega_2) - \frac{y}{z}[-xd\phi_1 + (x-b)d\phi_2 + db_{z1} - db_{z2}]$$
$$+ [xd\kappa_1 - (x-b)d\kappa_2 + db_{y1} - db_{y2}]. \tag{1}$$

Herein b indicates the *base-length* of the spatial model. Thus, for each model-point, it must be assured that:

$$dp_y = - dp_k. \tag{2}$$

If the parallax is not measured directly on the spatial model but *on the picture plates* (28, 29, 30), the observed picture parallaxes $dp_k{}^0$ should first be transformed into the corresponding space-parallaxes. For this purpose, we represent the focal length of the photographic camera by f; then, the plane of the picture plates may be represented by the equation $z = -f$, and we obtain

$$dp_k = \frac{z}{f} dp_k{}^0.$$

This transformation is easily determined by simple arithmetic or graphically.

SECTION 4

The term contained in the second bracket of equation (1) obviously indicates that the y-parallax dy, which is present for a pair of conjugate rays, being originally directed towards the point P_k, depends on only four quantities of orientation $d\kappa_1$, $d\kappa_2$, db_{y1}, db_{y2}. Likewise, the other bracket in (1) represents the parallax dz, being *measured in the z-direction*, and only originating from $d\phi_1$, $d\phi_2$, db_{z1} and db_{z2}. Accordingly, if we put

$$\begin{aligned} -x\cdot d\phi_1 + (x-b)d\phi_2 + db_{z1} - db_{z2} &= dz \\ x\cdot d\kappa_1 - (x-b)d\kappa_2 + db_{y1} - db_{y2} &= dy \end{aligned} \tag{3}$$

then, the equation (1) reads more simply:

$$dp_y = -\left(\frac{y^2}{z} + z\right)(d\omega_1 - d\omega_2) - \frac{y}{z}dz + dy.$$

In this general form, the equations (1), (3), (4)include the "two-projector" method as well as the "single-projector method," as two special cases.

It is evident that each of the quantities dy, dz is a constant for each point P_k of a plane ν (x=*constant*), being perpendicular to the base-line. Therefore, on each plane ν of this kind there exists a precisely defined point G with the coordinates:

$$X = x, \qquad Y = -\frac{dz}{d\omega_1 - d\omega_2}, \qquad Z = \frac{dy}{d\omega_1 - d\omega_2}; \tag{5}$$

we denote it as the "*base-point*" *of the plane* ν. The introduction of this point is of fundamental importance for our purpose.

If x is obtained for all possible values, it follows from equations (4) and (3) (the latter one being linear relative to x) that the base-points G of all planes ν will form a well determined straight line in space. Generally, this straight line g goes not intersect the base-line. On g, particularly, there are also situated the base-points G^0, and G^b of the planes ν^0 $(x=0)$ and ν^b $(x=b)$. The graphical locations of these points are essential for our method.

Indeed, as we obtain the coordinates of the points G^0, G^b from the equations (5) and (3):

$$X^0 = 0, \qquad Y^0 = \frac{-b\cdot d\phi_2(db_{z1} - db_{z2})}{d\omega_1 - d\omega_2}, \qquad Z^0 = \frac{b\cdot d\kappa_2(db_{y1} - db_{y2})}{d\omega_1 - d\omega_2};$$

$$X^b = b, \qquad Y^b = \frac{-b\cdot d\phi_1(db_{z1} - db_{z2})}{d\omega_1 - d\omega_2}, \qquad Z^b = \frac{b\cdot d\kappa_1(db_{y1} - db_{y2})}{d\omega_1 - d\omega_2},$$

we may also derive from the base-points and their coordinates, the following equations:

(A) In case of the "two-projector method" (for which we have to put:

$$d\omega_1 - d\omega_2 = d\omega,\ db_{y1} = db_{y2} = db_{z1} = db_{z2} = 0):$$

$$\begin{aligned} d\phi_1 &= \frac{Y^b}{b}\,d\omega, & d\kappa_1 &= \frac{Z^b}{b}\,d\omega, \\ d\phi_2 &= \frac{Y^0}{b}\,d\omega, & d\kappa_2 &= \frac{Z^0}{b}\,d\omega; \end{aligned} \tag{6}$$

(B) In case of the "single-projector method" (for which we may assume $d\omega_1 = d\phi_1 = d\kappa_1 = db_{y1} = db_{z1} = 0$):

$$\begin{aligned} d\phi_2 &= \frac{Y^b - Y^0}{b}\,d\omega_2, & d\kappa_2 &= \frac{Z^b - Z^0}{b}\,d\omega_2, \\ db_{y2} &= Z^b d\omega_2, & db_{z2} &= -\,Y^b d\omega_2. \end{aligned} \tag{7}$$

Thus, in both cases, the proportions of all the five unknowns of orientation are already precisely determined by the coordinates 0, Y^0, Z^0 and b, Y^b, Z^b of the base-points G^0, G^b.

Therefore, it is only essential to determine these two points G^0, G^b. If this cannot be done directly, as will be shown in Section 5, then, according to the previous statements, we have to determine the base-points G_1, and G_2 of any two different planes ν_1 and ν_2, which are perpendicular to the base-line, and to let the intersection of the spatial connecting line g of these points with the planes be ν^0 $(x=0)$ and ν^b $(x=b)$.

SECTION 5

Now, we will show how to obtain the base-point G of an arbitrary plane ν, that is perpendicular to the base-line by an easily-performable graphical procedure. For this purpose, we only have to assume that the main-phase is already accomplished, and that the proportions $dp_1:dp_2:dp_3$ of the parallaxes observed in three points P_1, P_2, P_3, being situated on ν, are known. We consider that the plane ν is viewed from the left, and we let this plane intersect the drawing plane. Furthermore, the system of coordinates used in ν may have its center at the point B $(x, 0, 0)$ lying on the base-line, whereas the positive y- and z-axes are directed towards the left and upward respectively (compare Section 3).

Thereupon, the model-points P_1, P_2, P_3 are plotted by using a suitable scale (see Figure 2). Next we draw through the point B the three straight lines p_1, p_2, p_3, being directed towards the points P_1, P_2, P_3; then, we erect at these points the straight lines n_1, n_2, n_3, being respectively perpendicular to p_1, p_2, p_3. *The lines n_1, n_2, n_3 form a triangle to which a second one, being similar and similarly situated with regard to the first one, is to be added.* For this purpose we draw through P_1, P_2, P_3 the rays, being parallel to the z-axis, and trace on them, beginning from the points P_1, P_2, P_3, any three distances K_1, K_2, K_3, whose lengths are of the same proportions as dp_1, dp_2, dp_3. The signs of these quantities and distances are to be respected carefully. The lines q_1, q_2, q_3 of the second triangle, being parallel to n_1, n_2, n_3, respectively, are drawn through the end points of these distances. Thereupon, *the center of similarity of both the triangles*, is obtained immediately, and it is already *identical with the requested base-point* G of the plane ν. This quick construction also remains performable if vertices of the triangles are extending over the edges of the drawing-sheet.

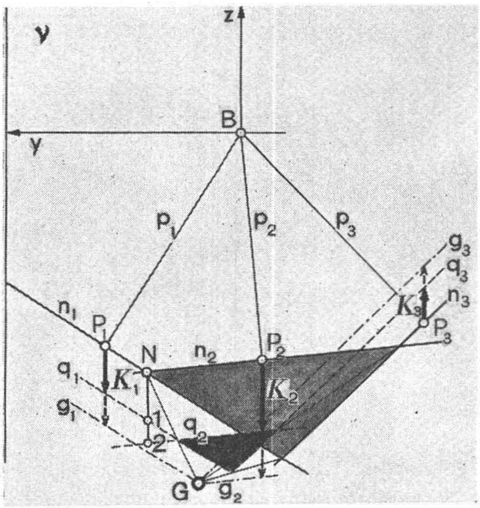

FIG. 2. Graphic solution for the base point.

From Figure 2 (where the distance $N1 = K_1$, further $N2 = K_2$, etc.) it is evident that the same point G is also obtained if the used distances K_1, K_2, K_3 are somehow proportionally altered, e.g. by multiplying them with a positive or negative factor. If one of the given parallaxes dp_k becomes zero, then the corresponding straight lines n_k and q_k are joined, whereupon G is situated on $n_k = q_k$ (see Figure 3).

In order to prove these results, first consider that the distances K_1, K_2, K_3 are altered proportionally in such a way that the straight lines g_1, g_2, g_3, being drawn through their new limits and being parallel to n_1, n_2, n_3, pass directly through the base-point G (see Figures 2 and 4; the latter one illustrating the same data as Figure 2). These distances, altered in such a manner, may be denoted by S_1, S_2, S_3. On the other hand, by replacing the values dy and dz in the equation (4) according to equation (5), equation (4) becomes

$$dp_y = S(d\omega_1 - d\omega_2), \tag{8}$$

whereby we give for purpose of abbreviation,

$$S = -\frac{y^2}{z} - z + \frac{y}{z}Y + Z. \tag{9}$$

As this is equivalent to

$$\frac{y}{z} = -\frac{(z + S) - Z}{y - Y}, \tag{10}$$

the result is the following important relation between a movement of the pencils and the parallaxes dp_y, being produced by this very movement:

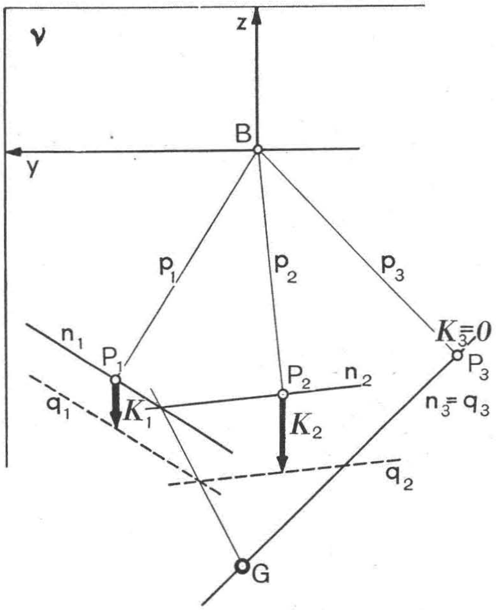

FIG. 3. Solution in the special case where one correction is zero.

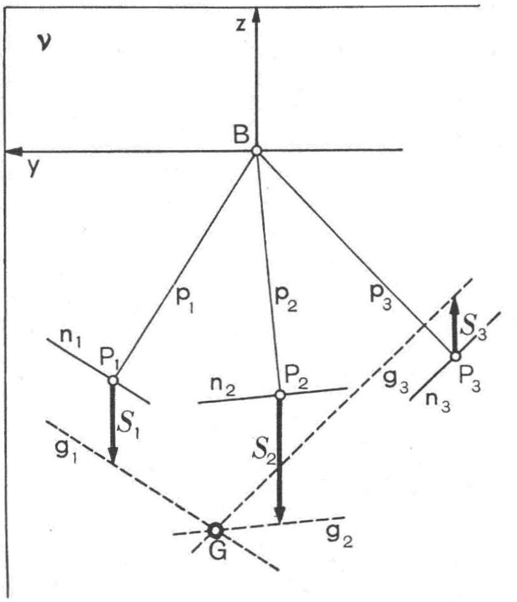

FIG. 4. Proof of the special solution.

A model-point P_k is given with its coordinates x, y, z and the parallax dp_y, produced at this plane by a certain pencil-movement (see Figure 5). *If, inside of the plane (x = constant) and in the direction of the z-axis, this point P_k is displaced along the distance $S_k = dp_y/d\omega_1 - d\omega_2$, and if we connect its new position Q_k (y, z+S) with the base-point G (Y, Z) by a straight line g_k, then, this line g_k is always perpendicular to the connecting line p_k (y/z = constant) of the two points: B (0, 0)* and *P_k (y, z).*

The shifting movement $P_k \rightarrow Q_k$ has to be accomplished either upwards or downwards depending whether S_k is a positive or negative quantity. In Figure 5 the quantities *z*, *S* and *Z* are of the *negative* sign.

We can also say: *If we displace the straight line n_k, being erected in the point P_k perpendicular to the ray p_k, in the z-direction along the distance* $S_k = dp_y/d\omega_1 - d\omega_2$ (having the proper sign), *then, this line always passes through the base-point G of the plane ν, being attached to P_k.*

Henceforth, we can easily understand that the base-point *G* of a plane (*x = constant*) (in which three model-points P_1, P_2, P_3 together with the proportions $dp_1:dp_2:dp_3$ of the corresponding parallaxes are given) coincides exactly with the center of similarity, being determined by the previous construction (Figures 2 or 3). It is only this center of similarity for which the relation just proved (considering all three points P_1, P_2, P_3) is true (compare also Figure 4).

SECTION 6

If the base-point *G* of a plane ν is obtained, then also the distances S_1, S_2, S_3 are given immediately, as was shown. Moreover, if we know not only the proportions of the parallaxes dp_1, dp_2, dp_3 to be eliminated, but also these parallaxes themselves, then on account of equation (2), it follows from equation (8):

$$d\omega_1 - d\omega_2 = \frac{-dp_1}{S_1} = \frac{-dp_2}{S_2} = \frac{-dp_3}{S_3} \cdot \tag{11}$$

Herewith, we obtain the differential tilt, being required for the elimination of the parallaxes dp_1, dp_2 and dp_3, by means of simple graphical construction. Or, in general:

With the base-point G of a plane ν perpendicular to the base-line, and with the real amount dp_k of the parallax that exists at a model-point P_k, being situated on ν, the differential tilt, by which this parallax shall be eliminated, is already clearly determined. It amounts to:

$$d\omega_1 - d\omega_2 = \frac{-dp_k}{S_k} \cdot \tag{12}$$

Herein, we put for the "angular method" $d\omega_1 - d\omega_2 = d\omega$, and for the "single-projector method," $d\omega_1 = 0$. By means of $d\omega$ or $d\omega_2$, according to (6) or to (7), the other unknowns of orientation are also determined.

The application of the construction, being described in Section 5, has no meaning if the base-point G of a plane ν lies outside of the drawing-sheet or at infinity. In the last case, all distances $S_k = \infty$, so that the *differential tilt vanishes*, How the existing parallaxes dp_k are to be eliminated in such a case, by using a graphical method, has already been explained by the author in another publication (35). Last, as we enter into details, it may be mentioned that in practice, such cases can easily be avoided by suitably tilting one of the projectors.

In practical applications, usually there is a large number of model-points P_k available, as well as their corresponding parallaxes dp_k. In using more than the required number of these points for a precise determination of the orientation movements, a new question arises: how the position of the base-points in the various planes perpendicular to the base-line is to be brought in best agreement with all given measurements, and with the connecting straight line g of these points as well (Section 4). It is not difficult to solve this problem owing to the well-known principles of least squares adjustment. But if these measurements are remarkably contradictory, then either the initial position of the pencils is still too far from the real orientation, or there are *systematic mistakes* caused by the instruments or by the photographic emul-

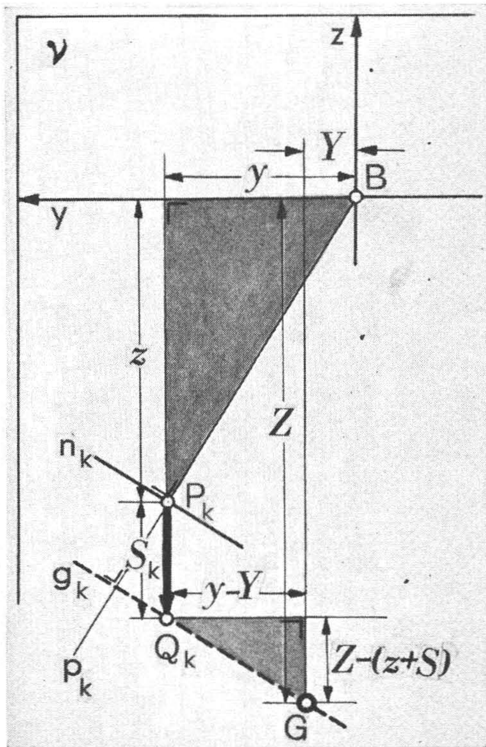

FIG. 5. Effect of displacing the model point.

sion. Especially, in view of equation (12), large mistakes can be recognized quickly from the drawing. But if both pencils are to some extent without mistakes, usually then a single application of this method meets the requirements. A second application is needed to establish the required relative orientation only if the pre-orientation is not sufficiently advanced.

SECTION 7

Another explanation remains: how a definite solution can also be obtained if only the proportions of the quantities of orientation are known (i.e. with sufficient accuracy). For this purpose, we use a diagram in which the different

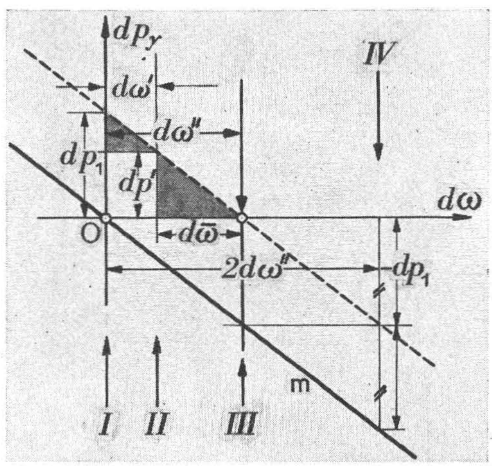

FIG. 6. Solution where the given orientation values are merely proportional to the actual values.

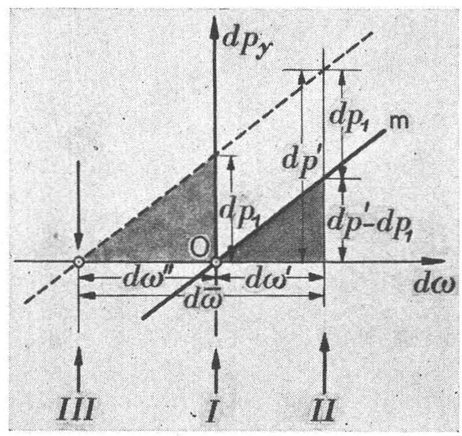

FIG. 7. Similar to Figure 6 except for a different value of slope.

values of $d\omega$ for the "angular method" (or of $d\omega_2$ for the "single-projector method") are entered as abscisses and the alterations of parallaxes dp_y are produced at any model-point P_k, possibly at P_1, as ordinates (Figure 6). After introducing the values (6) or (7), the equation (1) expresses a *linear relation* between $d\omega$ ($d\omega_2$) and dp_y, and this relation is represented by a straight line that passes through the origin. In Figure 6 the coefficient of slope of the line m is negative. In order to show that the derived formulas are independent from this circumstance, in a second diagram (Figure 7), the straight line m has been drawn rising to the right.

That the parallax dp_1, originally existing in P_1, is eliminated, $d\omega$ ($d\omega_2$) must be assumed in such a manner that the parallax dp_y, being produced at P_1, satisfies equation $dp_y = -dp_1$ (Section 3). At first, we take any (small) value $d\omega'$ of $d\omega$ ($d\omega_2$) and, accordingly to (6) or (7), we determine the orientation values. These are, together with $d\omega'$, to be introduced on the orientation instrument. If dp_1 has changed in parallax $dp' = dp_1 + dp_y$, then, we take for the correct value $d\omega''$ of the differential tilt from Figure 6 or 7:

$$d\omega'':(-d\omega') = dp_1:(dp' - dp'),$$

respectively,

$$d\omega'':d\omega' = dp_1:(dp_1 - dp')$$

or:

$$d\omega'' = \frac{dp_1}{dp_1 - dp'} d\omega'. \tag{13}$$

Introducing this value in (6) or (7), we obtain those quantities of orientation which eliminate (starting from the initial position) dp_1 in P_1 as well as all the other parallaxes dp_k. In order not to be obliged to recover the initial position of the two pencils, the situation, being obtained by $d\omega'$, may be corrected immediately with the quantities (6) or (7), being determined by

$$d\bar{\omega} = d\omega'' - d\omega' = \frac{dp'}{dp_1 - dp'} d\omega'. \tag{14}$$

With such linear interpolation being allowed during the main-phase, the possibility exists of overcoming the formerly (Section 2) mentioned "non-convergence" of the relative orientation, as it is *to penetrate into the interior of the "critical domains of space."* Up to now, this has not yet been kept in view with any other method of orientation.

Furthermore, these interpolations ensure a very valuable *new criterion* for the quality of orientation of the pencils. Indeed, except for ground-control, up to now we had to be satisfied with the *absence of all observed parallaxes*, which exist in the field of view. Now, the following re-examination of an adjustment can be executed:

We double the quantities of orientation, which had produced the last displacement of the pencils, and introduce these doubled values on the plotting apparatus. If the first orientation is correct, then in the new position, all parallaxes should be equal to and opposite from the original ones. In fact, only if $dp' = -dp_1$, do we satisfy equations (13) (14) (compare also Figures 6, 7):

$$d\omega'' = \tfrac{1}{2} d\omega', \qquad d\bar{\omega} = -\tfrac{1}{2} d\omega',$$

i.e., in this case the initial adjustment itself represents the best approximation to the perfect solution of the principal problem.

More generally, this control may also be carried out in the following manner. After the application of the quantities of orientation, being determined by $d\omega'$ and eliminating all the measurable parallaxes dp_k, the values (6) or (7) are additionally multiplied with a factor n and are introduced on the plotting apparatus. Afterwards, the original parallaxes dp_k should appear in their n-fold values. If other ones emerge, possibly the m-fold values of these quantities $(m+n)$, then (beginning from the initial position) the $(n+1/m+1)$-fold values of the first used quantities of orientation have to be applied. For better use, we may also directly correct the just obtained position of control, by applying the $[-m(n+1)/m+1]$-fold values of the previously used quantities on the apparatus. The factors m and n may adopt any arbitrary values, even negative ones.

The author has recently reported further possibilities of development concerning this graphical method in another paper. The essential question is to determine the *most probable position of the base-point* of a plane ν (x = *constant*), in which no measurable y-parallaxes can be observed any more. Other variations, based on practical experiments with different modern instruments, are still to be expected.

SECTION 8

Regarding the use of this new method for practical purposes, in most of the cases it will be possible to select in each of the planes ν^0 $(x=0)$ and ν^b $(x=b)$ at least three model-points, so that by these six points the base-points G^0 and G^b may be obtained directly (see 4, 5). In case of an independent pair of pictures, it should be assured even before the pre-orientation, that the axes of rotation of the projectors are, insofar as possible, parallel to the axes of the photographic pair. On the other hand, this precaution will hardly be possible with the single-projector method because one of the projectors must remain stationary. However, the practical experiments conducted in this office have shown the following fact: Discrepancies in the parallelism of corresponding axes have no significant effect if the discrepancies are less than 1° for normal-angle photographs, or 2° for wide-angle photographs. Therefore, it can be stated that deviations that do not exceed these limits of the angular errors during the main-phase have no harmful effect on our graphical method.

If bigger deviations of axes occur, the required rotations of the pencils by small angles could also be done by resolving them into components according to the rules of the *vector analysis;* namely, each into three rotations around the axes of the projectors. We may also resolve the translatory displacements of the pencils in the same manner, if necessary.

Finally we describe a practical example of an orientation according to the "angular or two-projector method." This example was worked out using a multiplex instrument. The selected orientation points had the following coordinates (expressed in mm.):

point	x	y	z	dp_k	point	x	y	z	dp_k
3	0	158.5	−427.5	0.1	4	151.5	232	−426.5	0
1	0	0	−428.5	0	2	151.5	0	−427.6	0
5	0	−244.2	−428.5	0.1	6	151.5	−213.5	−426.5	0.25

(15)

The last columns contain the parallaxes dp_k, which were measured at these places. The construction of the base-points G^0, G^b was then made on graph paper at the actual or full scale of the spatial model, which may be gathered from Figures 8 and 9 (reduced for publication). The coordinates of the points G^0, G^b, being obtained according to Section 5, Figure 2, were read as follows:

$$Y^0 = -86.5, \quad Z^0 = -428.5; \qquad Y^b = 229.2, \quad Z^b = -427.6 \text{ mm.}$$

Thereupon, the distances

$$S_3 = S_5 = 89 \text{ mm}, \quad \text{and} \quad S_6 = 222.5 \text{ mm.}$$

were determined (to compare to Figure 2, 4 or 5). According to equation (12), afteewards, we obtained the differential tilt:

$$d\omega \approx 7^{\grave{}} \text{ (the sign } ^{\grave{}} \text{ stands for centisimal minutes)}$$

and, according to equation (6), the other angles of orientation were:

$$\begin{aligned} d\phi_1 &\approx 1.51 d\omega \approx -10.5^{\grave{}}; & d\phi_2 &\approx 0.57 d\omega \approx 4^{\grave{}} \\ d\kappa_1 &\approx 2.82 d\omega \approx 21^{\grave{}}; & d\kappa_2 &\approx 2.83 d\omega \approx 21^{\grave{}}. \end{aligned} \tag{16}$$

Corresponding to the signs of these quantities, the principal ray of the left projector, being directed downwards, was to be turned to the right, but the principal ray of the right projector to the left, while both the pencils were to be swung counterclockwise. The tilting $d\omega$ was only applied on the left pencil (its principal ray was moved to the front). Thereafter, all parallaxes had in fact completely disappeared.

In order to assure accuracy, according to Section 7, the above mentioned

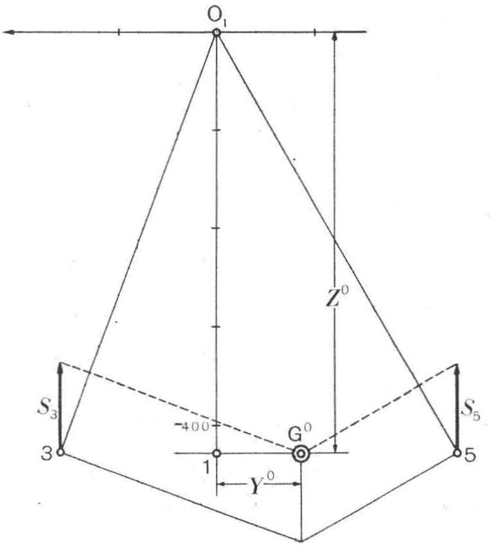

FIG. 8. Sample graphic solution in the zero plane.

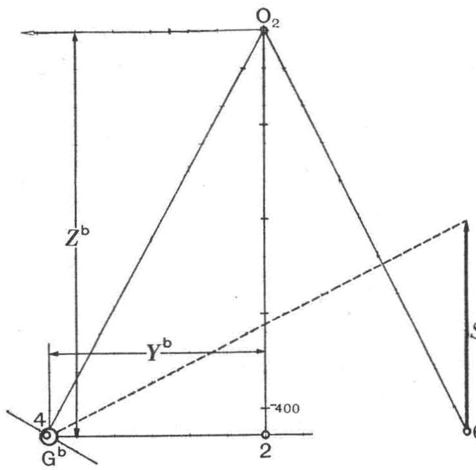

FIG. 9. Sample graphic solution in the $x=b$ plane.

angles of orientation were applied a second time. Then the following parallax values appeared:

$$dp_3 = -0.15 \text{ mm} \qquad dp_4 = 0$$
$$dp_1 = 0 \qquad dp_2 = 0$$
$$dp_5 = -0.15 \text{ mm} \qquad dp_6 = -0.4 \text{ mm}.$$

As these values were 50% larger than the expected negative of the values of dp_k given in (15), $-\frac{3}{5}$-fold values of the angles (16) were introduced in addition on the instrument. This last correction was performed according to equation (14).

Although operating with such small quantities of orientation and parallax values by means of the multiplex cannot be measured accurately, *all practical experiments with the new method have brought results that are in complete accord with the theoretical considerations and the conclusions expected of them.*

By his graphic method, representing the subject of this article, the author has demonstrated the practical importance of his extensive research in photogrammetry. In any case, his method seems to be more suitable and more easily adaptable (34, 35) than the method suggested by *G. Poivilliers* at the 6th International Congress for Photogrammetry, held in The Hague in 1948 (40). Where his method results in an approximate solution of the so-called "*plane resection problem*," the method described in this paper is more exact, because by it the operation with very small distances is avoided. Moreover, *it permits complete utilization of the accuracy of the coordinates and the parallax values that form the given data*, and finally, *the method seems to be so clear and simple that it can be understood immediately and applied correctly by any skilled operator.*

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LARGE AND INTERMEDIATE SCALE MAPPING OF EXTENSIVE AREAS WITH APPLICATION OF SPATIAL AERIAL TRIANGULATION. MAPPING EXAMPLE OF ISRAEL

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THERE is an ever growing need for the mapping of extensive areas (which may, in some instances, cover the whole territory of a country) at large or intermediate scales such as 1/2,500 and 1/5,000. As a matter of fact, this type of work is by no means limited to such highly developed areas as make up most of Europe, but also applies to regions that have yet to be fully developed.

Whatever the detailed specifications may be, such projects always entail major surveying assignments which must be completed as quickly as possible while meeting very exacting precision requirements. Large scale maps have many uses and must be kept up to date systematically, so that this precision of the techniques and the final accuracy of the results are essential requisites if the whole operation is not to be repeated after a short time.

Bearing the above in mind, there has been a tendency to supplement the photogrammetric work with a great deal of ground control when using such scales for the actual mapping, which would increase costs to a variable extent, depending on the nature of the terrain and on some local factors. Nevertheless with the many refinements embodied in modern photogrammetric cameras and plotting equipment, it becomes advisable briefly to review some of the techniques currently used in aerial surveying.

Attention will be drawn, in particular, to use of spatial aerial triangulation in the determination of geodetic control points, with the simplification and economies which this entails in large and intermediate scale mapping, especially in those regions which have to be surveyed for the first time.

In the case under review, two methods could be contemplated:

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