From a qualitative standpoint one may consider the amount of effort expended to do a particular job. If this is an important factor then, one can see the advantages of placing lines of field control along roads or easily accessible routes. In the Camp Grayling Project the control disposition was planned in this manner. The advantages of being able to do this for the planner and for the geodetic field parties are self evident. These practical considerations being for all intensive purposes the main considerations in efficient map production, I believe that one can assert, that a modest degree of progress has been made toward more efficient production of maps.

AFFINE TRANSFORMATIONS APPLIED TO THE MULTIPLEX AERO PROJECTOR

*William A. Allen**

IN HIS last paper[1] the author derived certain closed expressions specifying the effects in the multiplex model due to errors of relative orientation. These results can be easily generalized to include the effects of errors in interior orientation, and that is the purpose of this paper.

ERRORS OF INTERIOR ORIENTATION INTERPRETED AS AFFINE TRANSFORMATIONS

Dr. Wang has called attention to the role of affine transformations in photogrammetry.[2] As might be inferred from his paper, every error of interior orientation can be investigated by the use of these transformations. As applied to the multiplex, these errors consist of three types:

1. Improper centering of a diapositive.
2. An error in the principal distance of a projector
3. A diapositive lying in a plane not perpendicular to the optical axis of a projector.

These three types of errors can also result from maladjustments in the aerial camera and/or the reduction printer. Moreover, as Dr. Wang suggests, errors of type 2 are sometimes tolerated for the advantage of utilizing long focal-length photographs.

We consider a projector whose perspective center is at the point (a^i), $i=1, 2, 3$. We select the system so that the x^3-axis is parallel to the optical axis of the projector. The vector equation of a ray from this projector can be represented by the relation

$$\boldsymbol{r}_2 = \mu(a^i - x_1{}^i)\boldsymbol{e}_i, \tag{1}$$

where the repeated indices imply summation. The vectors, $\boldsymbol{e}^i$, represent a unitary orthogonal basis; the point, $(x_1{}^i)$, is an arbitrary image point in the model; μ is an arbitrary scalar. In this case, the effect of an error of type 1 can be

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[1] "Analysis of the Multiplex Model," William A. Allen, PHOTOGRAMMETRIC ENGINEERING, June, 1946.

[2] "Affine Transformation in Stereophotogrammetry," Dr. Chih-Cho Wang, PHOTOGRAMMETRIC ENGINEERING, June 1947.

represented by the relation

$$\mathbf{r}_2' = \mu(a^i - x_1^i)\mathbf{e}_i', \tag{2}$$

where

$$\mathbf{e}_i' = \alpha_i^j \mathbf{e}_j$$

represents an affine transformation. Denoting the focal length of the projector as f and translating the diapositive the amount

$$\mathbf{x} = f(k_1^3\mathbf{e}_1 + k_2^3\mathbf{e}_2),$$

then

$$\alpha_i^j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k_1^3 & k_2^3 & 1 \end{bmatrix},$$

where $\mathbf{e}_j'$, $\mathbf{e}_j$ are interpreted as column vectors.

The effect of an error of type 2 can also be represented by equation (2) which can be written in the form

$$\mathbf{r}_2' = \mu(a^j - x_1^j)\mathbf{e}_j',$$

where

$$\mathbf{e}_j' = \beta_j^k \mathbf{e}_k$$

is an affine transformation. If the diapositive is translated the amount

$$\mathbf{y} = fk_3^3\mathbf{e}_3,$$

then

$$\beta_j^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + k_3^2 \end{bmatrix},$$

where $\mathbf{e}_j'$, $\mathbf{e}_k$ are interpreted as column vectors.

The effect of a general displacement of the diapositive can also be represented by equation (2),

$$\mathbf{r}_2' = \mu(a^i - x_1^i)\mathbf{e}_i',$$

where

$$\mathbf{e}_i' = \gamma_i^k \mathbf{e}_k$$

is an affine transformation. If the diapositive is translated the amount

$$\mathbf{z} = f\sum^{3} k_i^3\mathbf{e}_i,$$

then

$$\gamma_i^k \equiv \alpha_i^j\beta_j^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k_1^3 & k_2^3 & 1 + k_3^3 \end{bmatrix}$$

where $\mathbf{e}_j'$, $\mathbf{e}_k$ are interpreted as column vectors.

Provided there is no danger of ambiguity, this relation can be written in the form

$$\gamma_i^k = \delta_i^k + k_i^k, \tag{3}$$

where

$$k_i^k = k_i^l \delta_l^3. \tag{4}$$

We consider the effects of an error of type 3. We assume that the diapositive is rotated about some arbitrary fixed line. This rotation can be resolved into a rotation and a translation. We consider the rotation as being about a fixed line passing through the perspective center of the projector. This rotation may be ignored because its effects are equivalent to a rotation of the projector. The effects of the translation already have been considered.

We conclude that every error of interior orientation can be specified by means of an affine transformation.

STATEMENT AND SOLUTION OF THE PROBLEM

Let us formulate the problem. We have an oriented model where both projectors may be tilted. We denote the positions of the projectors by the points (0), (a^i), and the position of an arbitrary image point in the model by the point (x_1^i). We introduce, into both projectors, an arbitrary error of interior orientation. Moreover, we displace one of the projectors arbitrarily. What effect will be produced in the model?

In tensor notation we have the relation

$$a^i\mathbf{e}_i - x_1^i\mathbf{e}_i - (a^i - x_1^i)\mathbf{e}_i = 0. \tag{5}$$

We introduce two new unitary orthogonal bases $\bar{\mathbf{e}}_j$, $\bar{\mathbf{e}}_j'$ determined by the tilted projectors such that $\bar{\mathbf{e}}_3$, $\bar{\mathbf{e}}_3'$, respectively, are parallel to the optical axes.
We have the relations

$$\begin{aligned} \bar{\mathbf{e}}_j &= \alpha_j^i\mathbf{e}_i, \\ \bar{\mathbf{e}}_j' &= \alpha_j'^i\mathbf{e}_i. \end{aligned} \tag{6}$$

Solving for $\mathbf{e}_j$, we obtain the relations

$$\begin{aligned} \mathbf{e}_i &= A_i^j\bar{\mathbf{e}}_j, \\ \mathbf{e}_i &= A_i'^j\bar{\mathbf{e}}_j', \end{aligned} \tag{7}$$

where A_i^j, $A_i'^j$ denote the cofactors of the elements α_j^i, α'^i_j in the determinants α, α'. We specify that

$$\alpha = \alpha' = +1. \tag{8}$$

Substituting equations (7) into equations (5) we obtain the relations

$$a^i\mathbf{e}_i - x_1^iA_i^j\bar{\mathbf{e}}_j - (a^i - x_1^i)A_i'^j\bar{\mathbf{e}}_j' = 0. \tag{9}$$

Introducing errors of interior orientation into both projectors, we obtain the relations

$$a^i\mathbf{e}_i - \lambda x_1^iA_i^k\bar{\mathbf{e}}_k^* - \mu(a^i - x_1^i)A_i'^k\mathbf{e}_k'^* = P\delta_2^i\mathbf{e}_i, \tag{10}$$

where λ, μ, P, are arbitrary scalars. P is interpreted as the magnitude of the

y-parallax at the point x^i into which the point x_1^i has been transformed. We assume the affine transformations

$$\begin{aligned} \bar{\mathbf{e}}_k^* &= \beta_k{}^j \bar{\mathbf{e}}_j, \\ \bar{\mathbf{e}}_k'^* &= \beta_k'{}^j \bar{\mathbf{e}}_j'. \end{aligned} \tag{11}$$

Substituting equations (11) into equations (10), we obtain the relations

$$a^i \mathbf{e}_i - \lambda x_1{}^i A_i{}^k \beta_k{}^j \bar{\mathbf{e}}_j - \mu(a' - x_1{}^i) A_i'{}^k \beta_k'{}^j \bar{\mathbf{e}}_j' = P\delta_2{}^i \mathbf{e}_i. \tag{12}$$

Substituting equations (6) into equations (12) we obtain the relation

$$a^i \mathbf{e}_i - \lambda x_1{}^l A_l{}^k \beta_k{}^j \alpha_j{}^i \mathbf{e}_i - \mu(a^l - x_1{}^l) A_l'{}^k \beta_k'{}^j \alpha_j'{}^i \mathbf{e}_i = P\delta_2{}^i \mathbf{e}_i. \tag{13}$$

If we define

$$\begin{aligned} \bar{\beta}_l{}^i &= A_l{}^k \beta_k{}^j \alpha_j{}^i, \\ \beta_l'{}^i &= A_l'{}^k \beta_k'{}^j \alpha_j'{}^i, \end{aligned} \tag{14}$$

as the *transforms* of β, β' by α, α', respectively, we obtain the relations

$$a^i \mathbf{e}_i - \lambda x_1{}^j \bar{\beta}_j{}^i \mathbf{e}_i - \mu(a^j - x_1{}^j) \bar{\beta}_j'{}^i \mathbf{e}_i = P\delta_2{}^i \mathbf{e}_i. \tag{15}$$

Relations (15) represent three linear equations in three unknowns λ, μ, P. The solution is sufficient to specify the state of the model due to the introduced errors of interior orientation. It is little more difficult, however, to include the effects of relative orientation. Rotating the movable projector and translating it from point (a^i) to point $(a^i + \Delta x^i)$, we obtain the relations

$$(a^i + \Delta x^i)\mathbf{e}_i - \lambda x_1{}^j \bar{\beta}_j{}^i \mathbf{e}_i - \mu(a^k - x_1{}^k) \bar{\beta}_k'{}^i \mathbf{e}_j' = P\delta_2{}^i \mathbf{e}_i, \tag{16}$$

where

$$\mathbf{e}_j' = \gamma_j'{}^i \mathbf{e}_i \tag{17}$$

is a properly orthogonal transformation. Substituting equations (17) into equations (16), we obtain the relations

$$x_1{}^j \bar{\beta}_j{}^i \lambda + (a^k - x_1{}^k) \bar{\beta}_k'{}^j \gamma_j'{}^i \mu + \delta_2{}^i P = a^i + \Delta x^i. \tag{18}$$

Solving, we obtain the equations

$$\lambda = D^{-1} \left| \, a' + \Delta x^i \quad (a^k - x_1{}^k) \bar{\beta}_k'{}^j \gamma_j'{}^i \quad \delta_2{}^i \, \right|, \tag{19}$$

$$P = D^{-1} \left| \, x_1{}^j \bar{\beta}_j{}^i \quad (a^k - x_1{}^k) \bar{\beta}_k'{}^j \gamma_j'{}^i \quad a^i + \Delta x^i \, \right|, \tag{20}$$

where

$$D \equiv \left| \, x_1{}^j \bar{\beta}_j{}^i \quad (a^k - x_1{}^k) \bar{\beta}_k'{}^j \gamma_j'{}^i \quad \delta_2{}^i \, \right|.$$

The new point into which $x_1{}^i$ has been transformed can be defined by the relation

$$x^i = \lambda x_1{}^j \bar{\beta}_j{}^i. \tag{21}$$

A FEW EXAMPLES

The power of equations (19) and (20) can best be demonstrated by a few examples. We shall consider first three cases:

1. The effects of an increment of y-movement.
2. The effects of an increment of x-tilt
3. The effects of an increment of y-tilt.

For these cases we assume that the elements of interior orientation of both projectors remain unaltered. For definiteness, we shall specify the problem by assuming the following values:

$$\begin{aligned} a^i &= \delta_1{}^i a, \\ \bar{\beta}_j{}^i &= \delta_j{}^i, \\ 0_k{}'^i &= \delta_k{}^i. \end{aligned} \tag{22}$$

These restrictions imply that the coordinate system has been selected in such a manner that the projectors are located at the points (0, 0, 0), $(a, 0, 0)$, respectively.

THE EFFECTS OF AN INCREMENT OF *Y*-MOVEMENT

Let us determine the effects of an increment of y-movement. Into equations (19), (20), we make the substitutions

$$\begin{aligned} \Delta x^i &= \delta_2{}^i \Delta x^2, \\ \gamma_j{}'^i &= \delta_j{}^i. \end{aligned} \tag{23}$$

These restrictions imply that only the increment of y-movement will be considered, the increments of the five other elements of relative orientation have been set equal to zero.

In order to conform with standard notation we hereafter also make the substitutions

$$\begin{pmatrix} x^1 & x^2 & x^3 \\ x & y & z \end{pmatrix}. \tag{24}$$

We can write directly

$$\begin{aligned} \lambda_y &= \frac{1}{D}\begin{vmatrix} a & a - x_1 & 0 \\ \Delta y & -y_1 & 1 \\ 0 & -z_1 & 0 \end{vmatrix}, \\ P_y &= \frac{1}{D}\begin{vmatrix} x_1 & a - x_1 & a \\ y_1 & -y_1 & \Delta y \\ z_1 & -z_1 & 0 \end{vmatrix}, \end{aligned} \tag{25}$$

where

$$D = \begin{vmatrix} x_1 & a - x_1 & 0 \\ y_1 & -y_1 & 1 \\ z_1 & -z_1 & 0 \end{vmatrix}.$$

Reducing the determinants, we obtain the relations

$$\begin{aligned} \lambda_y &= 1 \\ P_y &= \Delta y. \end{aligned} \tag{26}$$

We conclude that y-parallax, in this case, can be defined as the magnitude of the y-movement that generates it. From equations (21) we conclude that the coordinates of the image points are invariant with respect to an increment of y-movement.

THE EFFECTS OF AN INCREMENT OF X-TILT

As a slightly more difficult example, we consider the effects of an increment of x-tilt. Into equations (19), (20), we make the substitutions

$$\Delta x^i = 0,$$

$$\gamma_j'^i = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & -\alpha & 1 \end{vmatrix}. \tag{26}$$

These restrictions imply that the movable projector at $(a, 0, 0)$ is rotated through the infinitesimal angle α about the x-axis, the increments of the five other elements of relative orientation have been set equal to zero. We can write directly

$$\lambda_\alpha = \frac{1}{D}\begin{vmatrix} a & A_1 & 0 \\ 0 & B_1 & 1 \\ 0 & C_1 & 0 \end{vmatrix},$$

$$P_\alpha = \frac{1}{D}\begin{vmatrix} x_1 & A_1 & a \\ y_1 & B_1 & 0 \\ z_1 & C_1 & 0 \end{vmatrix}, \tag{27}$$

where

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & -\alpha & 1 \end{bmatrix}\begin{bmatrix} a - x_1 \\ - y_1 \\ - z_1 \end{bmatrix},$$

$$D = \begin{vmatrix} x_1 & A_1 & 0 \\ y_1 & B_1 & 1 \\ z_1 & C_1 & 0 \end{vmatrix}.$$

Reducing the determinants, and ignoring the infinitesimals in the denominator, we obtain the usual relations

$$\lambda_\alpha - 1 = -\frac{y_1}{z_1}\alpha,$$

$$P_\alpha = \frac{y_1^2 + z_1^2}{z_1}\alpha. \tag{28}$$

THE EFFECTS OF AN INCREMENT OF Y-TILT

We consider the effects of an increment of y-tilt. Into equations (19), (20), we make the substitutions

$$\Delta x^i = 0,$$

$$\gamma_j'^i = \begin{bmatrix} 1 & 0 & -\beta \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}. \tag{29}$$

These restrictions imply that the movable projector is rotated through the infinitesimal angle β about the y-axis, the increments of the five other elements of relative orientation have been set equal to zero. We can write directly

$$\lambda_\beta = \frac{1}{D}\begin{vmatrix} a & A_1 & 0 \\ 0 & B_1 & 1 \\ 0 & C_1 & 0 \end{vmatrix}, \qquad P_\beta = \frac{1}{D}\begin{vmatrix} x_1 & A_1 & a \\ y_1 & B_1 & 0 \\ z_1 & C_1 & 0 \end{vmatrix}, \tag{30}$$

where, in this case,

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\beta \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}\begin{bmatrix} a - x_1 \\ -y_1 \\ -z_1 \end{bmatrix},$$

$$D = \begin{vmatrix} x_1 & A_1 & 0 \\ y_1 & B_1 & 1 \\ z_1 & C_1 & 0 \end{vmatrix}.$$

Reducing the determinants and ignoring the infinitesimals in the denominator, we obtain the usual relations

$$\lambda_\beta - 1 = -\frac{a - x_1}{z_1}\beta, \qquad P_\beta = \frac{y_1(a - x_1)}{z_1}\beta. \tag{31}$$

THE EFFECTS OF AN ERROR IN FOCAL LENGTH

From the combination of effects due to increments in the elements of interior orientation, we shall consider only one case; that is, the case in which the principal distances of both projectors are in error by the same amount. Into equations (19), (20), we substitute the following values:

$$a^i = \delta_1{}^i a, \quad \Delta x^i = 0, \quad \gamma_j{}'^i = \delta_j{}^i, \quad \bar{\beta}_j{}^i = B, \quad \bar{\beta}_k{}'^i = T^{-1}BT, \tag{32}$$

where

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+k \end{bmatrix},$$

$$T = \begin{bmatrix} 1 & -\gamma_1 & \beta_1 \\ \gamma_1 & 1 & -\alpha_1 \\ -\beta_1 & \alpha_1 & 1 \end{bmatrix}.$$

These restrictions imply that only the elements of interior orientation are altered, that the stationary projector is untilted, that the initial state of the movable projector can be expressed by the infinitesimal rotation T, where α_1, β_1, γ_1, respectively, represent infinitesimal rotations about the x, y, z axes. The matrix $\bar{\beta}_k'^j$ can be computed readily and we have

$$\bar{\beta}_k'^j = \begin{bmatrix} 1 & 0 & -k\beta_1 \\ 0 & 1 & k\alpha_1 \\ -k\beta_1 & k\alpha_1 & 1+k \end{bmatrix}.$$

We can write directly

$$\lambda_f = \frac{1}{D}\begin{vmatrix} a & A_1 & 0 \\ 0 & B_1 & 1 \\ 0 & C_1 & 0 \end{vmatrix}, \tag{33-a}$$

$$P_f = \frac{1}{D}\begin{vmatrix} X & A_1 & a \\ Y & B_1 & 0 \\ Z & C_1 & 0 \end{vmatrix}, \tag{33-b}$$

where, in this case,

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -k\beta_1 \\ 0 & 1 & k\alpha_1 \\ -k\beta_1 & k\alpha_1 & 1+k \end{bmatrix} \begin{bmatrix} a - x_1 \\ -y_1 \\ -z_1 \end{bmatrix},$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+k \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix},$$

$$D = \begin{vmatrix} X & A_1 & 0 \\ Y & B_1 & 1 \\ Z & C_1 & 0 \end{vmatrix}.$$

The preceding treatment was based upon the assumptions that the model initially was correctly oriented and that the same change of focal length occurred subsequently in both projectors. In practice, however, the error in focal length usually occurs either in the aerial camera or in the reduction printer. Consequently, the coordinates of the representative image point, (x_1, y_1, z_1), are usually fictitious quantities which cannot be measured experimentally. It is necessary, then, to express these values in terms of the final coordinates, (x, y, z), into which they are transformed. Ignoring infinitesimals, equation (33-a) becomes

$$\lambda_f = 1. \tag{34}$$

Substituting this value into equation (21) we obtain the relation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+k \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \tag{35}$$

from which we conclude that only the z-coordinate of the image point, (x_1, y_1, z_1), is altered. This change can be expressed by the relation

$$z = (1 + k)z_1. \tag{36}$$

Substituting relation (36) into equations (33), and ignoring infinitesimals in the denominator, we obtain the relations

$$\lambda_f - 1 = \frac{y}{z}(k\alpha_1) + \frac{a - x}{z}(k\beta_1), \tag{37-a}$$

$$P_f = -\frac{y^2 + z^2}{z}(k\alpha_1) - \frac{y(a - x)}{z}(k\beta_1) + z\frac{2 + k}{1 + k}(k\alpha_1). \tag{37-b}$$

We observe from equations (28), (31), that

$$\lambda_f - 1 = (\lambda_\alpha - 1) + (\lambda_\beta - 1),$$

$$P_f = P_\alpha + P_\beta + z\frac{2 + k}{1 + k}(k\alpha_1),$$

provided that

$$(x \quad y \quad z) = (x_1 \quad y_1 \quad z_1),$$

$$(\alpha \quad \beta) = (-k\alpha_1 \quad -k\beta_1).$$

From the principal of superposition of effects, we conclude that the transformed model is in the same state as one in which x-tilt and y-tilt, of these amounts $K\alpha_1$, $K\beta_1$, has been introduced. If we proceed to clear this y-parallax by the usual methods, we shall be left with an undistorted model with only the z-dimension altered. The elements of rotation will be transformed

$$(\alpha_1, \beta_1, \gamma_1) = \left| \frac{\alpha_1'}{1 + k}, \quad \frac{\beta_1'}{1 + k}, \quad \gamma_1' \right|.$$

We conclude that the elements of rotation x-tilt, y-tilt are transformed in the same manner as the z-dimension. We also conclude that if x-tilt is present, we have irreducible y-parallax present specified by the equation

$$P = z\frac{2 + k}{1 + k}(k\alpha_1). \tag{38}$$

CONCLUSIONS

Sufficient examples have been worked to demonstrate that equations (20), (21), actually constitute a rigorous generalized solution of the multiplex model. The equations afford an immediate answer to the simpler problems in this phase of photogrammetry; the more advanced problems offer only computational difficulties.

The complicated nature of the problem necessitated the use of the tensor notation, a mathematical tool which is becoming fairly standard in many fields of physics and engineering.[3]

[3] "Vector Analysis With An Introduction To Tensor Analysis," J. H. Taylor, Prentice-Hall, Inc., New York, 1939.