AN IMPROVEMENT IN ACCURACY OF THE ORIENTATION OF A PHOTOGRAMMETRIC CAMERA BY MEANS OF CONDITION EQUATIONS*

Dr. Hellmut Schmid, Geodesist, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland

ABSTRACT

In certain cases of ground photogrammetry the orientation of the cameras is determined by means of recorded control points. The least squares adjustment is numerically convenient if algebraic expressions are used which relate two arbitrarily oriented central projections. However, these computations may lead to systematically erroneous results if the number of freedoms in such a system of formulas is not in agreement with the geometry of the problem. In order to improve the accuracy of such calculations, condition equations which exist between the unknowns are introduced into the solution of the orientation problem.

I. INTRODUCTION

THE spatial triangulation of airborne targets from photographs taken on the earth may be considered as a special case of ground photogrammetry. Because of the extremely high precision requirements, and partly due to the lack of suitable precision phototheodolites, the interior and exterior orientations of the individual photograph are usually computed from recorded control points. Hence the spatial coordinates of the target points are obtained from the measured plate coordinates by the method of intersection photogrammetry, combining the individual spatial directions of at least two measuring stations. The control points are either ground based targets or stars. From their known coordinates the spatial directions within a local coordinate system relative to the center of projection can be computed or directly measured in the field. In case of astronomical reference data the spherical coordinates of the stars—Right ascension and Declination—must first be reduced to a form consistent with the local net using the geographic coordinates of the stations and the time of exposure. A set of corresponding plane coordinates for the control points is obtained by projecting the control points according to the principle of a stereographic projection from the center of the photographic projection onto a suitably chosen tangent plane on the unit sphere. (Figure 1 shows the projection plane at the zenith.)

II. THE ORIENTATION BY A LEAST SQUARES ADJUSTMENT NEGLECTING THE CONDITION EQUATIONS

The orientation of the plate is computed from a least squares adjustment based on formulas which give the relation between the coordinates in the plane of the photograph and the projection of the control points onto the chosen tangent plane. Both projections are considered to be exact central projections with a common center of projection. Of course, corresponding corrections for distortion, refraction and systematic comparator errors must be introduced. The problem of orientation is thus based on such geometrical relations as exist between photographs taken from the same point. For the numerical reduction von Gruber's formulas (7)[1]† are usually considered as an adequate basis for the least squares adjustment.

* A more detailed discussion of this subject and a numerical application are included in Ballistic Research Laboratories Report No. 784, "Spatial Triangulation by Intersection Photogrammetry."

† The indices are related to the Bibliography at the end of this paper.

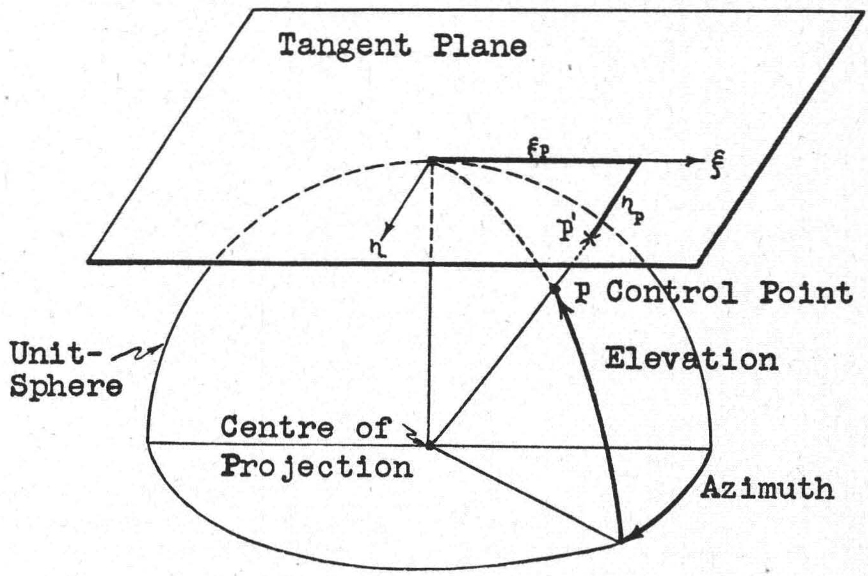

FIG. 1

The plane rectangular coordinates of the projection of a control point ξ, η (often called standard coordinates) are expressed as functions of the plane plate coordinates x, y and the plate constants a_1, b_1, c_1, a_2, b_2, c_2, a_0 and b_0 as follows:

$$\xi = \frac{a_1x + b_1y + c_1}{a_0x + b_0y + l} \quad \text{and} \quad \eta = \frac{a_2x + b_2y + c_2}{a_0x + b_0y + l} \tag{1}$$

and by introducing the measured plate coordinates l and l', where

$$x = l + v \quad \text{and} \quad y = l' + v'. \tag{2}$$

the corresponding observation equations are:

$$\begin{aligned} v(a_0\xi - a_1) + v'(b_0\xi - b_1) &= la_1 + l'b_1 + c_1 - \xi la_0 - \xi l'b_0 - \xi \\ v(a_0\eta - a_2) + v'(b_0\eta - b_2) &= la_2 + l'b_2 + c_2 - \eta la_0 - \eta l'b_0 - \eta \end{aligned} \tag{3}$$

Introducing approximation values for the plate constants a_0^0, b_0^0, a_1^0, b_1^0, c_1^0, a_2^0, b_2^0, and c_2^0 the observation equations are

$$\begin{aligned} \rho &= l\Delta a_1 + l'\Delta b_1 + \Delta c_1 - l\xi\Delta a_0 - l'\xi\Delta b_0 - L \text{ with weight } p \\ \rho' &= l\Delta a_2 + l'\Delta b_2 + \Delta c_2 - l\eta\Delta a_0 - l'\eta\Delta b_0 - L' \text{ with weight } p' \end{aligned} \tag{4}$$

where

$$p = \frac{l}{(\xi a_0^0 - a_1^0)^2 + (\xi b_0^0 - b_1^0)^2} \quad \text{and} \quad p' = \frac{l}{(\eta a_0^0 - a_2^0)^2 + (\eta b_0^0 - b_2^0)^2}.$$

The introduction of weighting factors becomes necessary because in each of the observation equations (4) there appears more than one observation and consequently more than one residual.

$$\begin{aligned} -L &= la_1^0 + l'b_1^0 + c_1^0 - l\xi a_0^0 - l'\xi b_0^0 - \xi \\ -L' &= la_2^0 + l'b_2^0 + c_2^0 - l\eta a_0^0 - l'\eta b_0^0 - \eta. \end{aligned} \tag{5}$$

III. The Least Squares Adjustment with Condition Equations

The least squares adjustment of the system of observation equations (4) leads to an incorrect answer due to an unjustified number of freedoms present in the system. The problem of determining the interior and exterior orientation of a camera at a given point has six freedoms—namely three translations and three rotations. The relations between the coordinates of projection and the plate coordinates are expressed for instance by the well known von Gruber's formulas (12a), (12b), and (13a), (13b).[2] With $h=1$ for the unit sphere these formulas reduce to:

$$\xi=\frac{\{[(y-\Delta y)\cos\eta+(x-\Delta x)\sin\eta]\cos\nu+d\sin\nu\}\cos A+[(x-\Delta x)\cos\eta-(y-\Delta y)\sin\eta]\sin A}{[(y-\Delta y)\cos\eta+(x-\Delta x)\sin\eta]\sin\nu-d\cos\nu}$$

$$\eta=\frac{\{[(y-\Delta y)\cos\eta+(x-\Delta x)\sin\eta]\cos\nu+d\sin\nu\}\sin A-[(x-\Delta x)\cos\eta-(y-\Delta y)\sin\eta]\cos A}{[(y-\Delta y)\cos\eta+(x-\Delta x)\sin\eta]\sin\nu-d\cos\nu}$$

or (6)

$$\xi=\frac{[(y-\Delta y)\sin\eta+(x-\Delta x)\cos\eta]\cos\alpha+\{d\cos\omega-[(y-\Delta y)\cos\eta-(x-\Delta x)\sin\eta]\sin\omega\}\sin\alpha}{\{d\cos\omega-[(y-\Delta y)\cos\eta-(x-\Delta x)\sin\eta]\sin\omega\}\cos\alpha-[(x-\Delta x)\cos\eta+(y-\Delta y)\sin\eta]\sin\alpha}$$

$$\eta=\frac{d\sin\omega+[(y-\Delta y)\cos\eta-(x-\Delta x)\sin\eta]\cos\omega}{\{d\cos\omega-[(y-\Delta y)\cos\eta-(x-\Delta x)\sin\eta]\sin\omega\}\cos\alpha-[(x-\Delta x)\cos\eta+(y-\Delta y)\sin\eta]\sin\alpha}.$$

A comparison between formulas (1) and (6) shows that there are 8 unknowns in the system (1) and only 6 unknowns in the system (6). Consequently two additional condition equations must exist between the plate constants. By a suitable regrouping of formulas (6) it is possible to eliminate these condition equations by a comparison with formulas (1). They are:

1.) $$a_0b_0+a_1b_1+a_2b_2=0$$

2.) $$a_0^2+a_1^2+a_2^2-b_0^2-b_1^2-b_2^2=0. \qquad (7)$$

Introducing these condition equations in the observation equations (5) the number of unknowns will be reduced to six independent variables. By means of substitutions made in formulas (4) and by the Taylor series, neglecting terms of second and higher order, the condition equations (7) may be written in the form:

1.) $$b_1^0\Delta a_1+a_1^0\Delta b_1+b_2^0\Delta a_2+a_2^0\Delta b_2+b_0^0\Delta a_0+a_0^0\Delta b_0+\lambda_1=0,$$
$$\lambda_1=a_0^0b_0^0+a_1^0b_1^0+a_2^0b_2^0$$

2.) $$a_0^0\Delta a_0+a_1^0\Delta a_1+a_2^0\Delta a_2-b_0^0\Delta b_0-b_1^0\Delta b_1-b_2^0\Delta b_2+\lambda_2=0,$$
$$\lambda_2=\frac{a_0^{0^2}+a_1^{0^2}+a_2^{0^2}-b_0^{0^2}-b_1^{0^2}-b_2^{0^2}}{2}.$$

Choosing such approximation values for the plate constants which satisfy the condition equations, (7), λ and λ_2 are equal to zero and therefore

$$l\Delta a_1+l'\Delta b_1=\Delta a_2(la'+l'b')+\Delta b_2(l'a'-lb')+\Delta a_0(lc'+l'd')+\Delta b_0(l'c'-ld') \qquad (8)$$

with the auxiliaries:

$$a'=\frac{-a_1^0a_2^0-b_1^0b_2^0}{a_1^{0^2}+b_1^{0^2}} \qquad c'=\frac{-b_0^0b_1^0-a_0^0a_1^0}{a_1^{0^2}+b_1^{0^2}}$$

$$b'=\frac{-b_1^0a_2^0-a_1^0b_2^0}{a_1^{0^2}+b_1^{0^2}} \qquad d'=\frac{-a_1^0b_0^0+a_0^0b_1^0}{a_1^{0^2}+b_1^{0^2}}.$$

Substituting the equations (8) in the observation equations (4) the new observation equations are:

$$\Delta c_1 + \alpha\Delta a_2 + \beta\Delta b_2 + \gamma\Delta a_0 + \delta\Delta b_0 - L = \rho$$

and

$$\Delta c_2 + l\Delta a_2 + l'\Delta b_2 - l\eta\Delta a_0 - l'\eta\Delta b_0 - L' = \rho' \qquad (9)$$

where

$$\alpha = (la' + l'b'), \qquad \beta = (l'a' - lb'),$$

$$\gamma = [l(c' - \xi) + l'd'], \qquad \delta = [l'(c' - \xi) - ld']$$

and finally by eliminating Δc_1 and Δc_2 the reduced observation equations are:

$$A\Delta a_2 + B\Delta b_2 + C\Delta a_0 + D\Delta b_0 - L^* = \rho \text{ with weight } p$$

and

$$A'\Delta a_2 + B'\Delta b_2 + C'\Delta a_0 + D'\Delta b_0 - L^* = \rho' \text{ with weight } p' \qquad (10)$$

where

$$A = \left(\alpha - \frac{[\alpha]}{n}\right) \quad D = \left(\delta - \frac{[\delta]}{n}\right) \quad A' = \left(l - \frac{[l]}{n}\right) \quad D' = \left(- l'\eta + \frac{[l'\eta]}{n}\right)$$

$$B = \left(\beta - \frac{[\beta]}{n}\right) \quad L^* = \left(L - \frac{[L]}{n}\right) \quad B' = \left(l' - \frac{[l']}{n}\right) \quad L^* = \left(L' - \frac{[L']}{n}\right)$$

$$C = \left(\gamma - \frac{[\gamma]}{n}\right) \qquad C' = \left(- l\eta + \frac{[l\eta]}{n}\right).$$

With this step the rigorous least squares adjustment of the orientation problem is solved. The computation requires the reduction of four normal equations. A reliability factor for the final result may be obtained by substituting the result in the original condition equations. In case of an unfavorable influence of the neglected second order terms an iteration must be made until the condition equations are sufficiently satisfied.

BIBLIOGRAPHY

O. v. Gruber, *Ferienkurs in Photogrammetrie.* Stuttgart (1930) (translated into English with the title: *Photogrammetry, collected lectures and essays*) (1) page 17 (2) pages 27 and 29, (3) page 17.