DEFORMATION OF A STEREOGRAM*

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Introduction

IN THIS brief paper it is proposed to study the deformation of a normal stereoscopic model when the photos are affected by tilt.

Bi-Projective Transformation

If the photo-coordinates on one photo be ξ_1, ξ_2, those on the adjacent one ξ_3, ξ_4, then in the case of perfectly vertical photographs the equations of transformation between ξ_1, ξ_2, ξ_3 and the ground or model coordinates x_1, x_2, x_3 are

$$\xi_1 = \frac{a_1x_1 + a_2x_2 + a_3x_3 + a_4}{r_1x_1 + r_2x_2 + r_3x_3 + 1}, \quad \xi_2 = \frac{b_1x_1 + \cdots}{r_1x_1 + \cdots}, \quad \xi_3 = \frac{c_1x_1 + \cdots}{r_1x_1 + \cdots} \tag{1}$$

This is a projective transformation between the ξ-space and the x-space.

In the case of a tilted pair, however, the equations of transformation are of the form

$$\xi_1 = \frac{\alpha_1e + \alpha_2n + \alpha_3h + \alpha_4}{\theta_1e + \theta_2n + \theta_3h + 1}, \quad \xi_2 = \frac{\beta e + \cdots}{\theta_1e + \cdots}, \quad \xi_3 = \frac{\gamma_1e + \cdots}{\psi_1e + \cdots} \tag{2}$$

where $(\theta_1, \theta_2, \theta_3)$ & (ψ_1, ψ_2, ψ_3) are the direction-ratios of the respective camera axes. Such a transformation may be termed "bi-projective" as distinguished from the projective transformation (1) which holds in the case of a normal stereoscopic model formation.

Deformation

Let the photographs be tilted and let the equations (2) be the transformation between the (ξ_1, ξ_2, ξ_3)-space and the ground-space (enh) and let the equation (1) be the transformation between the (ξ_1, ξ_2, ξ_3)-space and the model-space (x_1, x_2, x_3).

Considering now any plane, without loss of generality, as the plane of reference for the third coordinate h, its equation is $h=0$ and consequently,

$$\xi_1 = \frac{\alpha_1e + \alpha_2n + \alpha_4}{\theta_1e + \theta_2n + 1}, \qquad \xi_2 = \frac{\beta_1e + \beta_2n + \beta_4}{\theta_1e + \theta_2n + 1}.$$

Solving for e, n

$$e = \frac{\lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3}{\nu_1\xi_1 + \nu_2\xi_2 + 1}, \qquad n = \frac{\mu_1\xi_1 + \mu_2\xi_2 + \mu_3}{\nu_1\xi_1 + \nu_2\xi_2 + 1}.$$

Substituting in

$$\xi_3 = \frac{\gamma_1e + \gamma_2n + \gamma_4}{\psi_1e + \psi_2n + 1},$$

$$\xi_3 = \frac{h_1\xi_1 + h_2\xi_2 + h_3}{n_1\xi_1 + n_2\xi_2 + 1}.$$

Hence the equation of the stereoscopic model is

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$$\frac{c_1x_1 + \cdots}{r_1x_1 + \cdots} = \frac{h_1 \dfrac{a_1x_1 + \cdots}{r_1x_1 + \cdots} + h_2 \dfrac{b_1x_1 + \cdots}{r_1x_1 + \cdots} + h_3}{n_1 \dfrac{a_1x_1 + \cdots}{r_1x_1 + \cdots} + n_2 \dfrac{b_1x_1 + \cdots}{r_1x_1 + \cdots} + 1}.$$

or

$$\sum_i \sum_j \alpha_{ij} x_i x_j = 0,$$

which is a second degree equation in x_1, x_2, x_3 and therefore represents a conicoid.

The converse is also true, namely, what appears as a plane surface under a normal stereoscope is really a conicoid in ground-space when the photographs are tilted.

For, let $\lambda x_1 + \mu x_2 + \nu x_3 = 1$, be the plane in model and let the inverse transformation of (1) be

$$x_1 = \frac{A_1\xi_1 + A_2\xi_2 + A_3\xi_3 + A_4}{R_1\xi_1 + R_2\xi_2 + R_3\xi_3 + 1}, \quad x_2 = \frac{B_1\xi_1 + \cdots}{R_1\xi_1 + \cdots}, \quad x_3 = \frac{C_1\xi_1 + \cdots}{R_1\xi_1 + \cdots}$$

then,

$$\lambda \frac{A_1\xi_1 + \cdots}{R_1\xi_1 + \cdots} + \mu \frac{B_1\xi_1 + \cdots}{R_1\xi_1 + \cdots} + \nu \frac{C_1\xi_1 + \cdots}{R_1\xi_1 + \cdots} = 1$$

or,

$$U\xi_1 + V\xi_2 + W\xi_3 = 1$$

or,

$$U \frac{\alpha_1 e + \cdots}{\theta_1 e + \cdots} + V \frac{\beta_1 e + \cdots}{\theta_1 e + \cdots} + W \frac{\gamma_1 e + \cdots}{\psi_1 e + \cdots} = 1,$$

by (2) which is a second degree surface in the (enh)-space.

Graphical Demonstration

The properties of the distorted surfaces may be analysed from the equations considering different sets of direction-ratios θ&ψ. The distortions may also be studied graphically as follows.

Let the photographs be considered as bundles of rays emanating from the perspective centers, and let the datum plane be supposed to be split up by two sets of straight lines, one set being parallel to the orthogonal projection of the airbase on it, and the second set at right angles to the first.

In particular, let the projection of the airbase be considered. Originally, in space, the two pencil of rays intersecting at the projection of the airbase are 'perspective' in the sense of projective geometry. In the case of fore-and-aft tilt (tip) in the photographs, these pencils are relatively swung round under the normal stereoscopic vision to form 'projective' pencils and the locus of intersections of corresponding rays becomes either an ellipse or hyperbola, according as the relative swing is inwards or outwards and this conic passes through the perspective centers. The same holds for every other line of the first set. The lines of the second set, however, still remain parallel straight lines. Hence in the case of 'tip' the deformed surface is an elliptic or hyperbolic cylinder.

In the case of lateral tilt, however, the two sets of straight lines are changed into two sets of generators of a ruled surface as for a single-sheeted hyperboloid.