THE CALCULUS OF SCALE*

Amrom H. Katz, Chief Physicist, Research Studies Office, Photo Reconnaissance Lab, Wright Air Development Center, Wright-Patterson Air Force Base, Dayton, Ohio

ABSTRACT

The limitations to the now prevalent arithmetic notions of scale as a constant and a fraction are discussed and are shown to arise from the fictitious vertical photograph. An alternative proposal is made for the use of "scale numbers" which have analytic properties and which yield insight into the role of scale, especially in obliques.

Four specific aerial photographic problems are analyzed. These deal with the problem of getting the range at which air-ground guns or rockets are fired, the problem of getting accurate scale over unknown territory and without barometric altimeter correction, and the problems of determining altitude and depression angle in forward oblique photography made in level flight and in a dive.

In all these problems, use is made of scale as a changing parameter; and the usefulness of this approach to these and other problems is demonstrated.

CONTENTS

I. INTRODUCTION

THIS paper consists of two unequal parts. The first part deals with the concept of scale; the second with a series of (it is hoped) practical and useful problems.

The second part of this paper was actually started in connection with the author's work on determination of sea-wall heights for the Inch'on invasion (Reference 1). The origin of this problem was summarized in that paper, from which the following section is reprinted:†

"The description of the analysis used in this problem rightly emphasizes that our basic data (ground distances on the vertical photos, Figures 4 and 5), were derived through use of a 1:12,500 map of Inch'on. Perhaps we could have used high-altitude photographs and made either easy assumptions re verticality or tedious corrections to verticality. There was neither virtue nor time enough for this procedure. Low-altitude barometric altimeter errors in jet aircraft are too large for even rough photogrammetric procedures.

"It became obvious that what was needed (for future problems) was a method of making ground measurements which would be completely independent of both any (prior-known) ground data and of altitude.

"This sounds like a tough specification. Actually it is both fairly simple and fairly

* Paper read at Eighteenth Annual Meeting of the Society, Hotel Shoreham, Washington, D. C., January 9 to 11, 1952.

† This quotation is from Reference 1, pp. 97–98.

obvious. A verbal argument will illustrate the system which the author conceived after the main problem was solved.

"Suppose we have a camera mounted at 0° depression angle with respect to the longitudinal axis of the aircraft (i.e., a bore-sighted camera). Suppose flying either at very low altitude (or in a dive), an object of lateral dimension x is photographed at a certain time t. If at a known interval later the same object is photographed, then the ratio of image sizes is a direct function of the distances away from the object. The trick is to get the difference in the two distances without knowing either distance. This can be done if we know both true ground speed and the exact interval between photographs. Knowing this (readily available) information will enable calculations of the true size of the ground object. Putting the same argument in another way, the rate of change of S_x, dS_x/dt, is a simple function of the distance from the target. We can obtain dS_x/dt from two successive photographs taken at times which are not necessarily close, and easily make the required ground measurements."

II. The Concept of Scale and Its Use

Scale—as it is used almost universally—refers to the ratio of photographic image size (or map dimension) to the size of the original ground object. The numbers used are always given as, say 1:5,000, or 1:40,000—which is read as "one to five thousand or one to forty thousand." That this usage is well nigh universal and traditional does not insure that it is either good or useful. In fact, the author believes that this usage is neither good nor useful.

Within the author's experience, and that of numerous photogrammetrists, aerial photographers, and photo-interpreters, to whom he has talked, the sole use of "scale" has been to go from the map or photograph (the image) to the corresponding ground dimension. Thus a small number (image) is multiplied by a large number to get the ground distance. This multiplier is obviously the reciprocal of the number defined above as scale.

In an earlier paper (Reference 2), scale numbers S_i were defined so that an image measurement multiplied by the appropriate scale number yields the corresponding ground measurement. The scale numbers used in that paper were S_v, S_x, S_y, S_h, S_A for the truly vertical photograph, for x and y directions in the oblique photograph, for height, and for area, respectively.

The important point is that these scale numbers S_i are the ones that are actually used by everyone; thus the definition is an operational one. In Reference 1 this point is discussed as follows (p. 83):

> "The scale of Figures 4 and 5 was thus determined to be 1:3,270 and 1:3,290. (This is conventional representation of scale. The author much prefers to use vertical scale numbers S_v, which would be 3,270 and 3,290. These numbers are easier to write and to use, for their definition is an operational one. The major use to which a map is put is to calculate ground distances. Therefore S_v is defined as the number by which a map distance is multiplied to get a ground distance. This usage does away with the confusion always present when discussing "large-scale," "small-scale," etc.)"

This matter of operational definitions is very important, and the particular example herein treated is simple; for a comprehensive discussion on operational concepts and definitions, the reader is referred to Nobel Prize-Winner P. W. Bridgman's, "The Logic of Modern Physics" (Reference 3), wherein, among other things, he says (p. 7) ". . . For, of course, the true meaning of a term is to be found by observing what a man does with it, not what he says about it. . . ."

Some may feel that this point has been labored or belabored more than enough. This is not so. In the next paragraphs we shall see that the concept of scale as a fraction, and other pernicious notions, have had a stultifying effect on photogrammetric analysis.

Let us now examine another underlying tradition entering into the nature and use of scale.

The arithmetic concept of scale—derived from consideration of the vertical photograph—is a convenient fiction. It would indeed be a lovely (photogrammetric) world if all aerial photographs were truly vertical, thus yielding a constant scale number in all directions on the photographs. However, the truly vertical photograph (meaning that the lens axis was perpendicular to the surface of the earth immediately below the aircraft, and that this axis is also perpendicular to the film plane) is a rare event, even when such photography is attempted. The vast body of photogrammetric literature, techniques, and equipment designed to deal with tilt is ample testimony to the rarity of the vertical photograph.

Thus the vertical photograph is but a very special case of the oblique photograph. When the aerial photograph is not even a near-vertical, it is even more obvious that the arithmetic concept of uniform and constant scale—so handy for a vertical—is of no avail.

In the oblique photograph the scale numbers are constantly changing, and at any given point the scale numbers are (in general) different in all directions. This point is even more obvious when examining a map covering a good fraction of the world. In conformal maps, the scale is the same in all directions at any point, but varies continuously from point to point. In other types of maps, the scale is different in different directions at any point. Were one to build a theory of scale (and photogrammetry) starting with the oblique photograph, it is clear that scale (or preferably "scale numbers") must necessarily be represented by a continuous point function. These expressions must clearly be differential equations.

An earlier paper by the author (Reference 2) could well have been subtitled, "The Differential Calculus of Scale," for in this paper, the scale numbers S_i for elements of interest in an oblique photograph were derived. These expressions were differential equations and will be repeated here.

In all these equations x and y refer to the conventional cartesian coordinate directions when the photograph is so held that its top and bottom edges are parallel to the horizon and the foreground is at the bottom of the photograph.

The symbols used are:

H = flying height
f = focal length
X = ground distance in x direction
Y = ground distance in y direction
h = height of an object on the ground
I = image measurement (for x, y, h, this is a length)
A = ground area
a = image area
ϕ = angle off axis, measured down from horizon
θ = depression angle of camera axis, measured down from the horizon

The scale numbers S_v, S_x, S_y, S_h, S_A were defined above.

$$S_v = \frac{dX}{dI} = \frac{dY}{dI} = \frac{H}{f}$$

$$S_x = \frac{dX}{dI} = \frac{H}{f} \frac{\cos\phi}{\sin(\theta+\phi)}$$

$$S_y = \frac{dY}{dI} = \frac{H}{f} \frac{\cos^2\phi}{\sin^2(\theta+\phi)}$$

$$S_h = \frac{dh}{dI} = \frac{2H}{f} \frac{\cos^2\phi}{\sin 2(\theta+\phi)}$$

$$S_A = \frac{dA}{da} = S_x S_y.$$

In both References 1 and 2 methods were developed indicating the usefulness of these scale numbers for making small measurements from photographs, measurements which either assumed that the appropriate S_i remained sensibly constant over the small interval used, or employed an approximate or average value of the appropriate S_i over larger areas on the photograph. It is clear that the more or less orthodox and cumbersome geometric analysis of the oblique photograph could solve these problems only with cumbersome, stiff, inelegant machinery. Most important, the insight into what is really happening in an oblique photograph is either lost or is very difficult to achieve. Examination of the effects of small changes in angle of depression, in small excursions over the image plane,—these are practically impossible to evaluate by ordinary geometric and arithmetic analysis.

Scale is a changing function; it can best be handled by the differential and integral calculus. Photogrammetric analysis has, by and large, and with only infrequent exceptions, ignored the potentialities and power of the analytic approach.

There remains to be developed the integral calculus of scale—wherein any ground measurement, especially over large distances and areas on the photograph, can be made. One can at least write the expressions for which simple and practicable working systems must be developed. Sample expressions are:

For Y distances:

$$Y = \int_{y_1}^{y_2} S_y dy.$$

For areas:

$$A = \int_{x_1}^{x_2} \int_{y_1}^{y_2} S_x S_y dy dx.$$

As an example of the power of the analytic approach and the comparative sterility of the orthodox geometric analysis, the author cannot refrain from mentioning that in discussions with some few photogrammetrists, they failed completely to understand what is meant by "scale at a point" or "two scales (x and y) at a point in an oblique photo." These concepts are of course elementary, and differ not one whit from the most elementary concept in the differential calculus, that of slope at a point.

It is not claimed that these notions solve all problems, for even in Reference 1, the author ignored more topics than were then treated. It is a beginning, however, and it is hoped that other workers will continue from this modest beginning.

III. Ground Measurements Without Altitude or Ground Data

PROBLEM ONE—ESTIMATING THE RANGE OF AIR-GROUND GUN OR ROCKET FIRE

It is of occasional interest and importance to be able to evaluate fighter performance. One clue in this evaluation may be offered by determination of the range at which rockets or guns were fired. Consider the case of the aircraft making a ground attack, and assume that the aircraft mounts a motion-picture camera whose axis is essentially parallel to the longitudinal axis of the aircraft. The camera will take photographs at various positions on the closure path, such as H, H_1, and H_2. Assume now that some ground object G (which can be any two convenient points) lies on a line perpendicular to the flight path and intersects that line. Thus G lies in a plane perpendicular to Figure 1 at P and contains P. In this case there are no cosine corrections to make. The image remains on axis and expands as the aircraft closes in on the target. The following simple formula applies

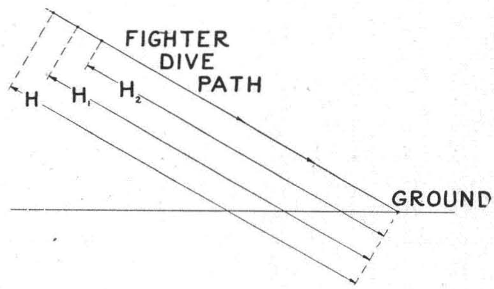

FIG. 1

$$I = \frac{Gf}{H} \tag{1}$$

where

I = image size

f = focal length

G = ground object size

H = distance from aircraft to ground object H

From equation (1), we have

$$\log I = \log Gf - \log H.$$

Whence

$$\frac{dI}{Idt} = \frac{-dH}{Hdt} \tag{2}$$

and

$$H = -\left(\frac{dH}{dt}\right)\left(\frac{1}{\frac{dI}{Idt}}\right). \tag{3}$$

Because dH/dt is negative, and is the rate of closure, it is equal to the negative of aircraft speed; equation (3) can be rewritten

$$H = V_{a/c}\left(\frac{1}{\frac{dI}{Idt}}\right). \tag{4}$$

To use equation (4) for determination of range H it is first necessary to interpret the term in parentheses on the right hand side of the equation. This term is the reciprocal of the rate of change of image size per second. This quantity can be determined from successive measurements on the motion picture film. An example will demonstrate this:

Let

$$V_{a/c} = 600 \text{ feet/sec.} \qquad (=412 \text{ mph}).$$

Then at $H=2{,}000'$, we have

$$\frac{dI}{Idt} = \frac{+dH}{Hdt} = \frac{600}{2{,}000} = 0.30.$$

This means that at this point ($H=2{,}000'$), the image I is growing at the rate of 30% per second.

If frame speed is known to within $X\%$ and aircraft speed to within $Y\%$, it is clear that an upper limit on the accuracy of determination of the range H is certainly $(X+Y)\%$, for we have not as yet estimated the accuracy of determination of dI/Idt. A rough statement of over-all accuracy can be made, however. This method is a photo-interpretation class measurement process. It is in the 5–10% class, giving such accuracy easily, but capable, under unusual conditions or with special equipment, of accuracies of the order of 1%. To the obvious question which may be raised "Why not put known objects on the target range?" the obvious answer is that we are discussing doing this under combat conditions, which severely limit ground access for the experimenter.

Another approach for this identical problem begins with equation (1), and photographs taken at the two positions H_1 and H_2. If H_1 is the range at which firing starts we have immediately

$$\frac{I_1}{I_2} = \frac{H_2}{H_1} \tag{5}$$

and, if $H_1 > H_2$

$$H_1 - H_2 = V_{a/c} \cdot t \tag{6}$$

where t is the interval between photographs. From equations (5) and (6), we obtain immediately

$$H_1 = \frac{V_{a/c} \cdot t}{\left(1 - \frac{I_1}{I_2}\right)}. \tag{7}$$

Thus, without knowing the size of any ground object, but from measuring the ratio of successive images of the ground object, H is determined. The time of firing of a gun or rocket does not necessarily (and in fact will likely not) coincide with any particular exposure; nor are gun cameras so mounted as to record the very first part of a trajectory. These factors mean only that care and further improvisation must be taken in application of the methods given here, and investigation of the accuracy of the method must be considered an integral part of the problem in each application.

PROBLEM TWO: ACCURATE ALTITUDE DETERMINATION IN VERTICAL PHOTOGRAPHY BY A METHOD OF DIFFERENCES

The thinking on the following interesting problem will be recognized as intimately related to the preceding problem.

Suppose a photographic aircraft is flying at a great distance from a friendly meteorological station, and over territory whose height above sea level is unknown (Figure 2). The aircraft carries a sensitive altimeter of the conventional type. Again, by a method essentially the same as in the previous problem, it is possible to make fairly accurate scale determinations.

This solution to the problem depends on the aircraft's taking two photographs over the same area at different altitudes.

Now, unless the proper sea-level pressure setting is known, absolute altitude H_1 above sea level cannot be computed from the (temperature-corrected) altim-

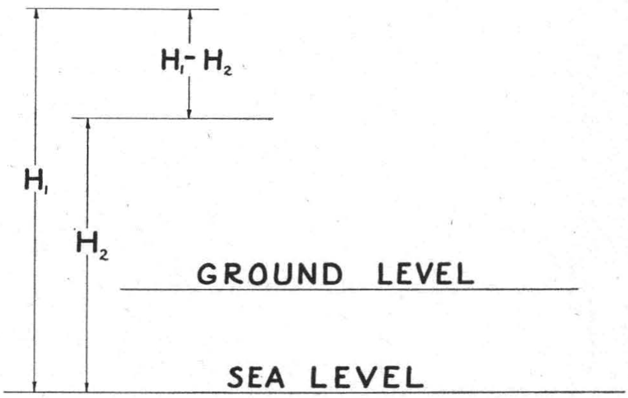

FIG. 2

eter reading. But this correction is in the nature of a dial rotation only, so that the same absolute error will be made at H_2. This very fact insures that the difference $H_1 - H_2$ can be accurately computed. Of course temperature corrections to the readings at H_1 and H_2 must be made, and one must be sure that there is no inversion between H_1 and H_2. At considerable altitudes, say 20,000′ and higher, in photographic weather, such inversions are extremely improbable.

The solution in this case is much the same as in the preceding problem, and yields very simply that

$$H_1 = \frac{H_1 - H_2}{\left(1 - \frac{I_1}{I_2}\right)}. \tag{8}$$

Note that in measurement of I_1 and I_2, the selected points need not be physically related and should be preferably some distance apart. For example, if H_1 is about 20,000 and $H_1 - H_2$ about 2,000′, then I_1/I_2 will be 0.90.

If the two points on the print are about six or seven inches apart, the difference in image sizes between I_2 and I_1, and therefore their ratio can be determined quite accurately. (It must be remembered that I_2 and I_1 are image measurements of the same ground distance.)

There are some interesting implications and possibilities in this differencing method. It affords an extra parameter as it were, and may prove very useful in related work, such as tilt analysis. Some work along this line, using some of these ideas, has been started by Mr. Eldon D. Sewell.

A word about the accuracy of this method. It is obviously and intimately a function of the accuracy of the sensitive altimeter; the estimates of the Equipment Laboratory altimeter experts, which the author obtained through Dr. Sam Burka, are about $1\frac{1}{2}\%$ of altitude. It is entirely possible, and needs investigation, that a goodly portion of such error may cancel out when taking differences, over relatively small changes. But this hopeful notion has no sound basis as yet, and must be regarded as a teasing prospect.

It is clear from a description of this method that certain aspects of practical problems have been ignored—tilts, relief, etc. All this means is that further developments are needed. In their absence, this must be relegated to the class of P.I. techniques, which are characterized by accuracies, real and required, of several per cent.

PROBLEM THREE: ALTITUDE, CAMERA DEPRESSION ANGLE AND SCALE DETERMINATION FROM LEVEL-FLYING FORWARD OBLIQUE PHOTOGRAPHY

Consider now the following interesting problem. A modern high speed aircraft has installed in it a forward oblique camera. It makes a level pass over a target area, and now we wish to determine dimensions, distances, etc. of ground objects captured on the photography. If we can get, or know speed and camera interval data, this problem is easily solved; and as part of the solution, the exact angle of depression of the camera is found. True ground speed in a low flying high speed aircraft can be obtained to within several per cent, and if care is exercised, more accurately than this. Calibrated intervalometers, or watches mounted in cameras and photographed on the film can yield the interval between photographs to within a per cent or two.

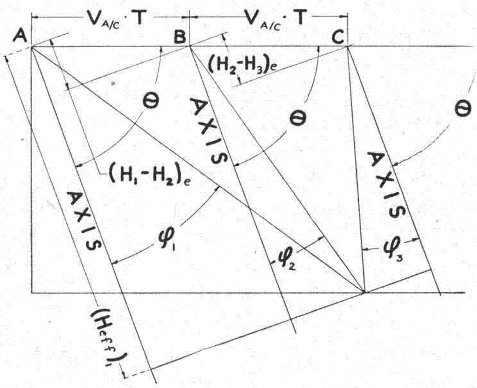

FIG. 3

Reference to Figure 3 will explain the geometry of such photography.

In this figure, θ is the camera axis depression angle, ϕ_1 and ϕ_2 successive positions of the ground object G with respect to the axis as the aircraft advances from A to B.

As before G is required to lie on a line perpendicular through G.

Then if I_1 is the length of the image of G when the aircraft was at A, I_2 the length of the image when the aircraft was at B, these image lengths are related to the actual length G by the equations

$$\begin{aligned} S_{x_1}\cdot I_1 &= G \\ S_{x_2}\cdot I_2 &= G \end{aligned} \tag{9}$$

where the S_{X_i}'s are the x scale numbers

$$S_{x_i} = \frac{H}{f}\,\frac{\cos\phi_i}{\sin(\theta+\phi_i)}\,. \tag{10}$$

Now using the concept of *effective altitude* H_e, which was defined in Reference 2 as that number which, when divided by focal length, yields the appropriate scale, we have

$$S_{x_i} = \frac{H_{e_i}}{f}\,. \tag{11}$$

H_{e_1} and H_{e_2} are drawn on Figure 3.
From equations (9) and (11), we have

$$\frac{S_{x_1}}{S_{x_2}} = \frac{I_2}{I_1} = \frac{H_{e_1}}{H_{e_2}}. \tag{12}$$

Also from Figure 3,

$$H_{e_1} - H_{e_2} = V_{a/c} \cdot t \cos \theta \tag{13}$$

where t is the interval between photographs, and $V_{a/c}$ the aircraft speed.
Equations (12) and (13) together yield

$$H_{e_2} = \frac{V_{a/c} \cdot t \cos \theta}{\left(\frac{I_2}{I_1} - 1\right)}. \tag{14}$$

If we now knew θ, we would have H_{e_2} and from equations (10) and (11), the actual height H. Observe that ϕ_1, ϕ_2, I_2, I_1 are numbers which are obtainable directly from the photographs. $V_{a/c}$ and t must be obtained from other sources. This problem was discussed above in connection with Problem One.

A brief discussion about the role of θ in this problem is in order. If the depression angle θ is of the order of 10° or less, equation (14) clearly demonstrates that we can be no more than 1.5% off by assuming $\cos \theta = 1$. *This is true for the particular object at G only*. Remember that G was selected for convenience only, and that we may be interested in other objects as well. The true role of θ, and its importance, especially when small, is more easily seen from equation (10). Here we have $\sin \theta$ in the denominator (assume for the moment that ϕ is very small) where it exerts a powerful lever on the value of S_x. Of course, if the flying characteristics of the aircraft at various altitudes and fuel loads are well known, its altitude (the angle between the a/c longitudinal axis and the flight path) is determinable and careful ground measurements may be made of the camera installation angle; thus, the angle of depression θ may be found. Nothing is wrong with this procedure. However, the following solution for θ is quite simple.

From equations (10) and (12)

$$\frac{I_2}{I_1} = \left[\frac{\cos \phi_1}{\cos \phi_2}\right]\left[\frac{\sin (\theta + \phi_2)}{\sin (\theta + \phi_1)}\right]. \tag{15}$$

In this equation θ is the only unknown.
For convenience, set

$$K = \frac{I_2}{I_1} \frac{\cos \phi_2}{\cos \phi_1}. \tag{16}$$

Equation (15) now can be written

$$F(\theta) = \sin (\theta + \phi_2) - K \sin (\theta + \phi_1) = 0. \tag{17}$$

This is readily solved by Newton's method. A full exposition of the method will be found in any text book on the theory of equations or numerical methods. A brief description follows:

The Taylor's expansion for $F(\theta_0 + \Delta\theta)$ is given by

$$F(\theta_0 + \Delta\theta) = F(\theta_0) + \Delta\theta F'(\theta_0) + \frac{(\Delta\theta)^2}{2} F''(\theta_0) + \cdots$$

where $F'(\theta_0)$, $F''(\theta_0)$, $\cdots$ are the first, second, etc. derivatives of $F(\theta)$ evaluated at θ_0.

If $\Delta\theta$ is reasonably small, we may drop the higher terms; we have

$$F(\theta_0 + \Delta\theta) = F(\theta_0) + \Delta\theta F'(\theta_0). \tag{18}$$

Now let θ_0 be the first approximation to the root of equation (17); $\Delta\theta$ is the correction sought after, so that $F(\theta_0+\Delta\theta)=0$. From equation (18), setting $F(\theta_0+\Delta\theta)=0$,

$$\Delta\theta = -\frac{F(\theta_0)}{F'(\theta_0)}. \tag{19}$$

This method is easily demonstrated graphically.

It is clear from Figure 4 that if $F(\theta)$ is plotted as a function of θ and if θ_0 is the first approximation to θ_R, that

$$F'(\theta_0) = -\frac{F(\theta_0)}{\Delta\theta}$$

whence comes equation (19) for $\Delta\theta$.

If necessary, the process may be repeated. In general it will not be necessary.

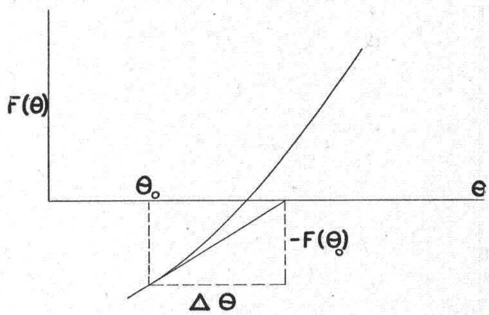

FIG. 4

From equation (17)

$$F'(\theta) = \cos(\theta + \phi_2) - K\cos(\theta + \phi_1). \tag{20}$$

Two examples will illustrate the simplicity and power of this method of solution.

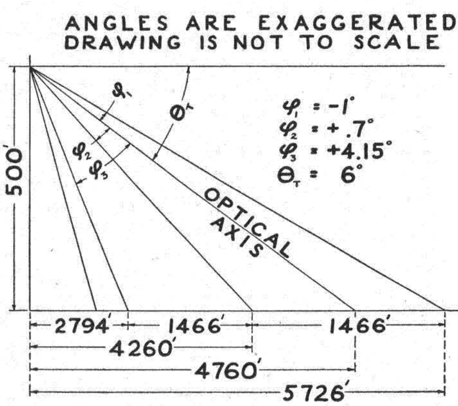

FIG. 5

Example A

Let us set up the following conditions, illustrated in Figure 5 (which, because of the small angles, is not to scale). A 24″ focal length 9×9 inch camera is mounted at a true depression angle $\theta_T=6°$. The true ground speed of the aircraft is 500 mph and its altitude above terrain is 500′. The camera cycles in 2.00 seconds. Assume photo #1 of the series we are analyzing is made at $\phi_1=-1°$ (i.e., 1° above the axis). Then, under the above conditions,

$$\phi_2 = +0.70° \quad \text{and} \quad \phi_3 = +4.15°.$$

Observe now that the angles ϕ_1, ϕ_2, ϕ_3 are determinable from the photographs, and being determined with reference to the principal point, are found independent of any knowledge about θ.* I_2/I_1 will be determined from photographs

* A word is in order about the sensitivity of the measurement of the ϕ_i. The 24″ 9×9 camera has a half side angle of 10.5°. Thus, assuming linearity, there is roughly 2.33°/inch on the focal plane, or 0.433 inches/degree.

1 and 2. The value of this ratio may be calculated from the data at hand, using the true value $\theta_T = 6°$ and the values of ϕ_1 and ϕ_2. Equation (15) yields

$$\frac{I_2}{I_1} = \left[\frac{\cos(-1°)}{\cos(0.7°)}\right]\left[\frac{\sin 6.7°}{\sin 5°}\right]$$
$$= 1.338.$$

Let us assume, as the first approximation to ϕ_T, $\theta_1 = 9°$. Equation (17) yields (after dropping the cosine terms)

$$F(\theta_1) = \sin 9.7° - 1.338 \sin 8° = -0.0179.$$

Similarly

$$F'(\theta_1) = \cos 9.7° - 1.338 \cos 8° = -0.337.$$

Whence, from equation (19)

$$\Delta\theta = -\left(\frac{-0.0179}{-0.337}\right) = -0.0532 \text{ radians} = -3.05°.$$

Hence, the corrected first approximation is

$$\theta_1 + \Delta\theta = 5.95°.$$

Thus the first approximation to θ_T, over but 1.7° of image travel, converges to within 0.05°. Part of the reason for this, of course, is the high scale ratio of 1.338 between the two photographs.

We may do this same analysis for photographs 1 and 3, with ϕ_1 and ϕ_3. In this case,

$$\frac{I_3}{I_1} = 2.025.$$

For assumed $\theta_1 = 90°$

$$F(\theta_1) = \sin 13.15° - 2.020 \sin 8° = -0.0535$$

and

$$F'(\theta_1) = \cos 13.15° - 2.020 \cos 8° = -1.021.$$

Whence

$$\Delta\theta = -\left(\frac{-0.0535}{-1.021}\right) \text{radians} = -3.00°$$

which, to two significant decimal places, gives the correct or true value of θ as 6.00°!

Not much more can be expected from an approximation method.

With the true value of θ so easily derived, it is straight forward to go back to equations (10), (11), (12), and (14) to obtain the elements needed for a complete analysis of any object in the photographs, using now the applicable formulas for the S_i given earlier.

An important point to note in connection with this method (and the sample calculations) is that it utilizes no fancier computer than a standard engineer type 10″ slide rule. One criterion for evaluating usefulness or practicality which the author places on "field methods" is that they require no calculator more complex or bulky than a 10 inch slide rule. For exactly this reason, the Aerial Photo Slide Rule, designed by the author, (and described in Reference 2) has a complete polyphase duplex slide rule on one side.*

Example B

Take now the same problem for large depression angles, such as $\theta_T = 45°$. In this case, let us use a 12″, 9×9 camera cycling at one second intervals. Let the flying height be 500 feet, and the true ground speed 500 mph. Figure 6 shows this situation.

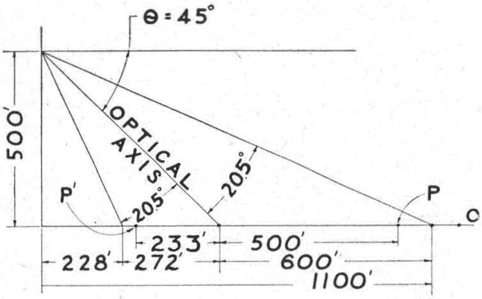

FIG. 6

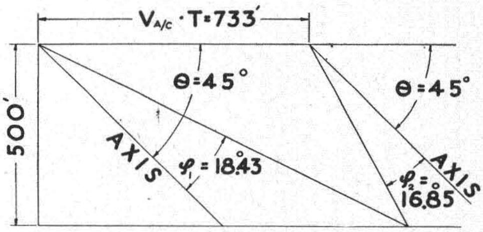

FIG. 7

It is clear that the aircraft advances 733 feet between successive photographs. The forward coverage per photograph is about 850 feet. Thus we cannot in advance guarantee that any particular ground object will be photographed twice. If it happens to be an object, as at O in Figure 6, lying just outside the coverage, O will be photographed in the next shot, but will be missed on the following one. An object at P for example, is captured in the background of the coverage angle shown, and will appear in the foreground in the next shot.

However, our analysis requires θ and H_1 which may be determined from two successive photographs, and which perforce must use some object or pair of points that occur in both photographs. Once θ and H are determined, the complete geometry of any photograph is at hand, and is applicable to other objects which may have been photographed but once.

Consider now the specific situation in Figure 7.

In this case, for the first photograph of P, taken at A, ϕ_1 is $-18.43°$. In the second photograph, taken at B, ϕ_2 is $+16.85°$. As before

$$\frac{I_2}{I_1} = \left[\frac{\cos -18.43°}{\cos 16.85°}\right]\left[\frac{\sin 61.85°}{\sin 26.57°}\right] = 1.954.$$

Assume now a value of $\theta_1 = 50°$.

* There has been much justified effort lately to raise the status (and recognition) of the photo-interpreter from that of a technician to that of a professional man. Surprisingly enough, most of this effort has come from outside the ranks of the photo-interpreter. Even more surprising—in fact fantastic—have been the counter efforts of some photo-interpreters to degrade, stunt, and permanently harm their cause by statements such as "the photo-interpreter has no need of trigonometric scales (or in fact any scales but the C and D). Trigonometric scales frighten and confuse the photo-interpreter." In this connection it should be remembered that every year around the first of October in every college in this fair land, seventeen and eighteen year old boys, from town and country alike, are initiated into the esoteric mysteries of the slide rule, and that many of these boys have yet to take their first shave. The fact that these boys are more or less randomly selected and that photo-interpreters and intelligence people are specially selected raises more questions than it answers.

We get

$$F(\theta_1) = -0.1135$$

and

$$F'(\theta_1) = -1.282$$

yielding

$$\Delta\theta = -\frac{F(\theta_1)}{F'(\theta_1)} = -0.0885 \text{ radians} = -5.07°.$$

Thus the first approximation to the 45° value of θ_T is $\theta = 44.93°$ showing again fast convergence to θ_T.

How about really large errors in estimating θ_T?

Take $\theta_1 = 30°$ (a 15° error), with the preceding data for ϕ_1 and ϕ_2.

Again

$$F(\theta_1) = \sin 46.85° - 1.97 \sin 11.57° = 0.3345$$

and

$$F'(\theta_1) = \cos 46.85° - 1.97 \cos 11.57° = 1.247$$

whence

$$\Delta\theta = -\left(\frac{0.3345}{-1.247}\right) = 0.2684 \text{ radians} = 15.36°.$$

Thus the first approximation to θ_T, is $\theta_1 + \Delta\theta = 45.36°$, which is within 0.36°!!

If this value of $\theta = 45.36°$ is used as the next approximation, the second approximation will be found to be $\theta = 45.026°$. Successive values of θ are 30°, 45.36°, 45.026°.

Actually, the first approximation, 45.36° is good enough for any photo interpretation purpose.

PROBLEM FOUR: ALTITUDE, DIVE ANGLE, AND SCALE DETERMINATION FROM FORWARD OBLIQUE PHOTOGRAPHY

Consider now the problem of the previous section extended to the case of the aircraft in a dive, a situation illustrated in Figure 8, where α is the dive angle. This problem is obviously and considerably more difficult than the preceding one, for there is the additional variable of α. A complete and exact solution can be achieved, and is given in schematic form in this paper; however, the method of solution is not one which is in the same class as the previous simple methods, and is not therefore, a good field method.

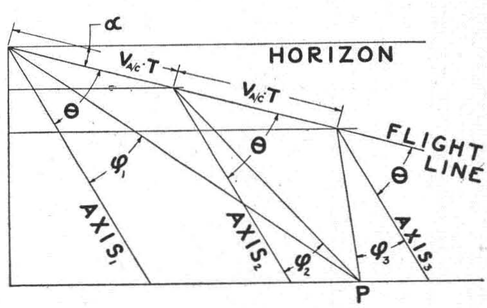

FIG. 8

Reference to Figure 8 (in which the fourth angle ϕ_4 is not shown because of complexity of the drawing) and previous similar analysis, enables us to write the series of equations for four successive photographs.

$$\frac{I_2}{I_1} = \frac{H_1 \cos \phi_1 \sin [(\theta + \alpha) + \phi_2]}{H_2 \cos \phi_2 \sin [(\theta + \alpha) + \phi_1]} \tag{21}$$

$$\frac{I_3}{I_1} = \frac{H_1 \cos \phi_1 \sin [(\theta + \alpha) + \phi_3]}{H_3 \cos \phi_3 \sin [(\theta + \alpha) + \phi_1]} \tag{22}$$

$$\frac{I_4}{I_1} = \frac{H_1 \cos \phi_1 \sin [(\theta + \alpha) + \phi_4]}{H_4 \cos \phi_4 \sin [(\theta + \alpha) + \phi_1]} \tag{23}$$

H_2, H_3, and H_4 are simple functions of H_1.

Let $p = V_{a/c}t$, the distance the aircraft travels between successive exposures. Then

$$\left.\begin{aligned} H_2 &= H_1 - \ \ p \sin \alpha \\ H_3 &= H_1 - 2p \sin \alpha \\ H_4 &= H_1 - 3p \sin \alpha \end{aligned}\right\}. \tag{24}$$

Thus equations (21), (22), and (23) are functions of three unknowns:

$$H_1, \alpha, \theta.$$

Everything else in these equations is determined from the photographs or from aircraft data (speed and camera interval). Equations (21), (22), and (23) can be written as

$$\left.\begin{aligned} \Theta(H_1, \alpha, \theta) &= 0 \\ \Phi(H_1, \alpha, \theta) &= 0 \\ \Psi(H_1, \alpha, \theta) &= 0 \end{aligned}\right\}. \tag{25}$$

Assume now that we have approximations H_{1_0}, α_0, θ_0 to the values of H_1, α, θ which satisfy equations (25) and that ΔH_1, $\Delta\alpha$, $\Delta\theta$ are the three desired corrections to the approximate values.

If then the three functions of equation (25) are expanded in Taylor's series for three variables about $(H_{1_0}, \alpha_0, \theta_0)$ and the higher order terms involving higher derivatives are dropped we have (as in the case of the single variable)

$$\left.\begin{aligned} &\Theta(H_{1_0} + \Delta H_1, \alpha_0 + \Delta\alpha, \theta_0 + \Delta\theta) \\ &\quad = \Theta(H_{1_0}, \alpha_0, \theta_0) + \Delta H_1\left(\frac{\partial\Theta}{\partial H_1}\right)_0 + \Delta\alpha\left(\frac{\partial\Theta}{\partial\alpha}\right)_0 + \Delta\theta\left(\frac{\partial\Theta}{\partial\theta}\right)_0 = 0 \\ &\Phi(H_{1_0} + \Delta H_1, \alpha_0 + \Delta\alpha, \theta_0 + \Delta\theta) \\ &\quad = \Phi(H_{1_0}, \alpha_0, \theta_0) + \Delta H_1\left(\frac{\partial\Phi}{\partial H_1}\right)_0 + \Delta\alpha\left(\frac{\partial\Phi}{\partial\alpha}\right)_0 + \Delta\theta\left(\frac{\partial\Phi}{\partial\theta}\right)_0 = 0 \\ &\Psi(H_{1_0} + \Delta H_1, \alpha_0 + \Delta\alpha, \theta_0 + \Delta\theta) \\ &\quad = \Psi(H_{1_0}, \alpha_0, \theta_0) + \Delta H_1\left(\frac{\partial\Psi}{\partial H_1}\right)_0 + \Delta\alpha\left(\frac{\partial\Psi}{\partial\alpha}\right)_0 + \Delta\theta\left(\frac{\partial\Psi}{\partial\theta}\right)_0 = 0 \end{aligned}\right\}. \tag{26}$$

In equations (26), the subscripts on the partial derivatives indicate that they are to be evaluated at $(H_{1_0}, \alpha_0, \theta_0)$.

Equations (26) are now three linear equations in the three unknowns ΔH_1, $\Delta\alpha$, $\Delta\theta$ and can be (more or less) readily solved by standard determinantal methods. At this point, the coefficients of the Δ's are numbers. (There is little point to give the actual values of the partial derivatives here, for they would only complicate the appearance of the equations.)

This then, is the complete solution for the problem.

An instructive solution to this problem is easily found if θ is known (as it may be for a particular aircraft-camera combination from previous solutions to the level flight problem).

In this case, let

$$\left.\begin{aligned} K &= \frac{I_2}{I_1}\,\frac{\cos\phi_2}{\cos\phi_1} \\ L &= \frac{I_3}{I_1}\,\frac{\cos\phi_3}{\cos\phi_1} \\ X &= \frac{H_1}{H_2} \\ Y &= \frac{H_1}{H_3} \\ \psi_i &= \theta + \phi_i \end{aligned}\right\} . \tag{27}$$

Substituting equations (27) in equations (21) and (22) yields

$$K \sin(\alpha + \psi_1) - X \sin(\alpha + \psi_2) = 0 \tag{28}$$

$$L \sin(\alpha + \psi_1) - Y \sin(\alpha + \psi_3) = 0. \tag{29}$$

Now substituting in equation (29) the value

$$Y = \frac{H_1}{H_3} = \frac{H_1}{H_1 - 2p \sin\alpha}$$

and solving for H_1 yields

$$H_1 = \frac{2pL \sin\alpha \sin(\alpha + \psi_1)}{L \sin(\alpha + \psi_1) - \sin(\alpha + \psi_3)} . \tag{30}$$

Substituting from equation (30) into equation (28) and replacing X by $H_1/H_1 - p \sin\alpha$ yields, after simplification,

$$KL \sin(\alpha + \psi_1) + K \sin(\alpha + \psi_3) - 2L \sin(\alpha + \psi_2) = 0. \tag{31}$$

This is a straightforward equation in α, $F(\alpha) = 0$ and is readily soluble by Newton's Method, as demonstrated previously.

This completes the solution, for having α, one finds H_1 from equation (30), and as before, the entire geometry of the situation is in hand.

There are other problems for which this type of scale-ratio analysis might prove useful. Some preliminary analysis of tilts in near vertical 6″ focal 9×9 cameras has so far demonstrated that when photographs are made at 60% overlap and image lines perpendicular to the flight paths are measured, we have

$$\frac{I_2}{I_1} = R_x = 1 + 0.01t^\circ \tag{32}$$

which says simply that under favorable circumstances (no relief, etc.) there is a 1% change in the scale ratio per degree of tilt. Now 1% of an image line 8 inches long is a sizeable and easily measured quantity. So it is clear that fairly small tilts yield measurable results. These preliminary thoughts are just that. Mr. Eldon D. Sewell is working on some problems which may combine this analysis with the height-differencing methods; it is hoped some fruitful results may soon be available for presentation in this journal.

IV. Conclusion and Acknowledgment

The four problems discussed previously cannot be regarded as a complete catalog of solutions. The author has demonstrated a method, and a different approach. This different approach, if there is a common denominator to the several problems considered, is characterized by gathering and using more data internal to the aircraft and its motions. What of the fellow in the field whose problem is slightly different and which does not fit neatly into one of the enumerated pigeon holes? If he can improvise his own methods, based, if necessary, on insights and suggestions found herein, he will be in good shape. If he insists on a made-to-order computing form for a non-routine problem, the author would be inclined to mistrust his results anyway. The foregoing is another way of saying what has been said loud, long, and often before—we need at least a few people in the field who can improvise methods for new problems on the spot. Routine problems are best handled routinely; but nonroutine problems can never be so handled.

Most ideas are the direct or indirect by-product of stimulating conversation; the thoughts in this paper are no exception. To Col. Richard Philbrick, Major James Henry, Eldon D. Sewell, Dr. Sam Burka, Captain Walter Levison, and Yale Katz, who, allowing themselves to be buttonholed, have shown interest when they could have legitimately shown boredom, much thanks. And since this paper was written at home, a special type of thanks to my wife.

References

1. Katz, Amrom H., "A Problem at Inch'on," Photogrammetric Engineering, Vol. XVII, No. 1, pp. 78–99, March 1951.
2. Katz, Amrom H., "Contribution to the Theory and Mechanics of Photo-Interpretation From Vertical and Oblique Photographs," Photogrammetric Engineering, Vol. XVI, No. 2, pp. 339–386, June 1950.
3. Bridgman, Percy W., "The Logic of Modern Physics," The Macmillan Co., New York, 1932.

NEWS NOTE

THE NEW FA-181 SENSITIVE ALTIMETER

This new Sensitive Altimeter has been designed particularly to withstand the severest service in the field. It meets the requirements of Military organizations. For durability, the Altimeter mechanism is supported by a shock mounting in a sturdy aluminum case with a latched metal lid. The instrument case contains a desiccant to absorb moisture and the instrument may be completely sealed to protect it during transport. The new Altimeter employs the W&T Mechanism featuring the self-balancing principle and custom calibration.

The W&T Palmer Altirule has been designed to compute elevations directly from observations when using the two-base method. Information on the FA-181 Altimeter, the W&T Palmer Altirule and modern Altimetry methods is available from the manufacturer, Wallace & Tiernan Products, Inc., 1 Main Street, Belleville 9, New Jersey.