PRINCIPLES OF DESIGN AND THE APPLICATIONS OF THE MM 101 SURVEYING CAMERA*

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TONIGHT I will speak on the principles of design and the applications of the Marine Metrical 101 mm surveying camera, the prototype of which is now under construction.

Before starting to read my paper, I want to pay tribute to a man—the late Lieutenant Colonel Guy Comer, of the United States Marine Corps. Colonel Comer died on September 5, 1953, while on active duty in Moscow. During his lifetime, his brilliant mind and energies were devoted to advancing the military science in behalf of the Marine Corps, which he loved so dearly. Among his many efforts was an earnest desire to advance the military science, by a broader application of metrical photography. It was in this connection that he initiated the design of the MM 101 surveying camera. In keeping with the highest tradition of the Marine Corps, Lieutenant Colonel Early, a close friend of Colonel Comer, is directing the continued development of this surveying camera. We, and Colonel Early, are resolved that this camera shall accomplish all that Colonel Comer expected.

Those who knew Colonel Comer will remember his warm personal friendship, his loyalty to his friends and the Marine Corps, his extreme honesty, his high sense of justice, and his keen progressive mind. I have lost a very dear friend, and the Marine Corps has lost a very fine officer. I cannot do less than dedicate this paper to Lieutenant Colonel Guy Comer of the United States Marine Corps.

I. Introduction

The MM 101 camera is a portable, multipurpose, tripod-mounted, goniometer type of instrument designed to obtain astronomic position data, astronomic azimuth data, and triangulation data quickly under the restricted circumstances of surveying in areas that are remote, unexplored or inaccessible by conventional means. The data obtained with this camera are expected to yield the following maximum errors:

Data	Maximum Error
Astronomic position	5 seconds
Astronomic azimuth	20 seconds
Terrestrial directions	20 seconds

Normally the zenith camera is employed to obtain astronomic position data. The zenith camera weighs 30 pounds and is about 14 inches in height. Its use is restricted to astronomic position data. The camera transit and the phototheodolite are generally used to obtain triangulation data. The camera transit weighs 56 pounds and is 16 inches high. The use of these cameras is restricted to triangulation data. No camera has been designed to obtain astronomic azimuth data. The MM 101 will weigh 22 pounds and be 9 inches in height. The MM 101 is designed to accomplish the combined functions of astronomic position and triangulation data normally obtained with two single-purpose cameras as well as the additional function of astronomic azimuth not obtainable with any camera. The capacity to obtain three types of data and the accuracy of the three types of data are a consequence of new design features and the integration of equally new methods of data reduction.

In fact, the MM 101 system is an optical mechanical solution to astronomic equations whose development and analysis preceded the first rough sketch. This is compatible with the role that mathematical analysis should play in the sound

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design of any photogrammetric instrument. The accuracy, multiplicity of function, simplicity of operation and portability of the MM 101 is symbolic of a new era in the metrical possibilities of terrestrial cameras and methods employed in photographic surveying from ground stations.

II. Description

The MM 101 consists of two basic elements (Figures 1, 2):

(1) *The camera*, and

(2) *The camera goniometer.*

A. CAMERA

1. Cone, Lens, and Focal Plane Element

The camera lens is a Kodak Ektar with a nominal focal length of 101 millimeters and an effective aperture of f/4.5. A cone angle of 42 degrees across the x axis and 28 degrees across the y axis provides a format size of 82×57 mm. The magazine takes 30 exposures of 70 mm. film which is regarded as sufficient for a local survey consisting of six frames for zenithal exposures for astronomic position and azimuth data and twenty-four frames for a

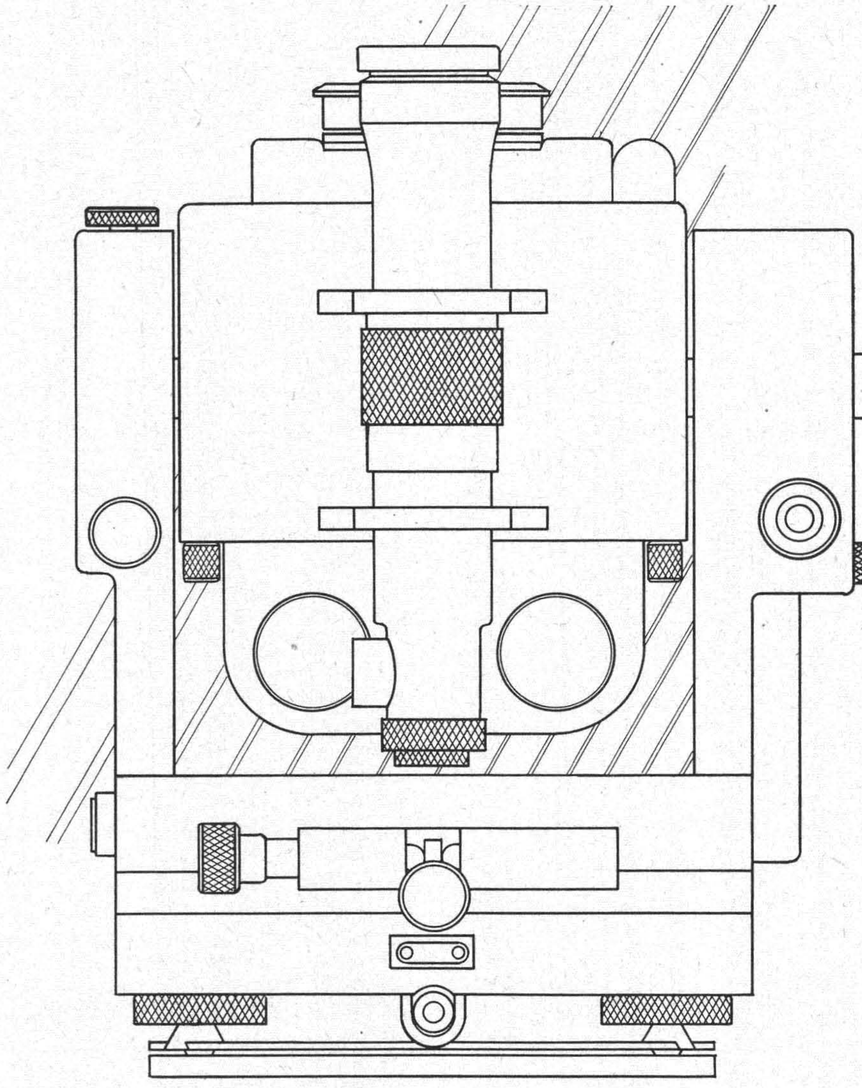

FIG. 1. Zenithal Orientation of MM 101.

round of terrestrial exposures at the terminals of a base line with a 12 degree sidelap on each exposure for a system of intersection points.

Considerable effort has gone into the metrical properties of the camera with regard to focal plane flatness, focal plane reproducibility, and film distortion control. For this purpose a plano-concave element is seated in the focal plane of the camera cone. A system of orthogonal lines with a 10 mm. interval is etched on the

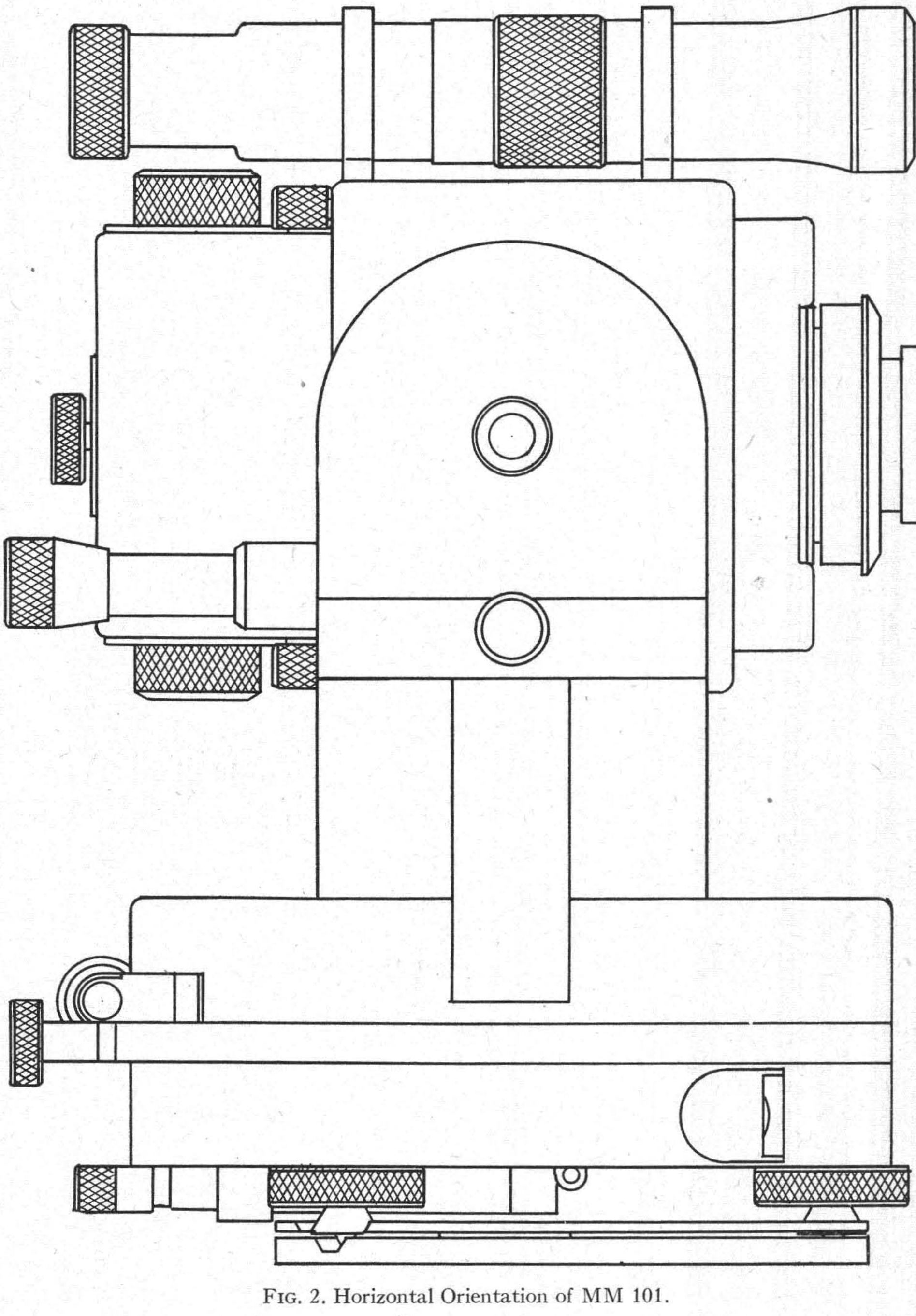

FIG. 2. Horizontal Orientation of MM 101.

planar surface of the element. The element is positioned in the focal plane so that the rear nodal point of the lens is the center of curvature of the concave surface, while the planar surface coincides with the focal plane. The geometric relations are illustrated in Figure 3.

The plano-concave element provides a planar surface and a control reference for the film without the negative refraction introduced by plano-parallel elements installed in some cameras for the same purpose. The freedom from negative refraction arises from the fact that rays from object space have normal incidence to the concave surface. Ray tracing through the combined system demonstrated that the image forming qualities for the lens system were improved inasmuch as the curvature of field is reduced approximately 50 per cent.

The planar surface of the element exhibited one fringe of helium light when subjected to a flatness test. This is nearly 10 times the requirement for the desired precision and 25 times the flatness specified in aerial cameras. Both surfaces of the element have a low reflectance coating. Measurement of the etched lines yielded widths of 7 microns ± one micron with a uniformity of interval ± 2 microns. No terrestrial or celestial detail is lost with lines of this width. Any accidental lack of flatness of the film is immediately detected in that the image of the lines increases in width. Precision in astronomic position would not be possible on film with a camera as small as the MM 101 without the planarity of the focal plane and the control over film distortion provided by the etched focal plane element. Glass plates lack operational efficiency in the loading of individual plates and add unduly to the weight and volume of the system. Special purpose cameras designed to accomplish the metrical functions under discussion are not adapted to film.

The intersection of the central x and y lines of the system is made to coincide with the principal point when the elements of the camera system are assembled in the laboratory. Two glass wheels are mounted tangent to the edge of the focal plane. Numbers etched on the outer circumferences are imaged with the exposure to define the exposure number. The focal plane element is seated in a metal frame so that the planar surface of the element defines a common surface with the frame. This prevents damage to the glass corners without reduction in the planarity of the focal plane. This is illustrated in Figure 4.

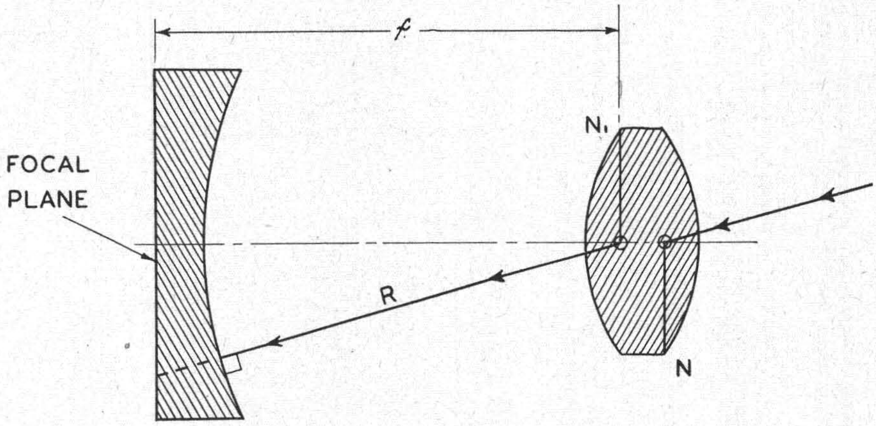

FIG. 3. Plano-Concave Focal Plane Element.

2. *Film magazine.*

A section coinciding with the xz plane is shown in Figure 5.

The basic components of the film magazine are:

(1) The platen.
(2) Feed and take-up spool.
(3) Film advance knob.
(4) Film contact knob.
(5) Exposure number knob.
(6) Film magazine housing.

The feed and take-up spool and platen are an integral unit that are translated along a normal to the focal plane within the

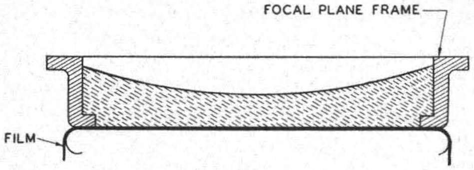

FIG. 4. Focal Plane Frame.

magazine housing by turning the film contact screw. This insures normal contact of the film without abrasion. The opposite ends of the platen are equipped with guide rollers and the spools with linking locking clutches to prevent abrasion of the film by lag or backlash of the film. This construction holds the film in contact with the platen under a slight tension at all times. The film contact screw is equipped with a release spring that prevents excessive pressure of the platen against the glass focal plane. The film advance knob has a safety clutch that prevents the film from being advanced in the film contact position. These combined features insure planar positioning of the film without abrasion of the emulsion which is a necessary condition to the successful use of glass in the focal plane.

3. *Orientation features.*

The camera body is equipped with two orientation features whose geometric principles are the basis for the precision in the astronomic azimuth and position: One feature consists of two nonadjustable 5 second level vials mounted normal to each other in the xy plane about the lens cell; while the other is the permanently attached autocollimating telescope mounted on the top center of the camera body with its optical axis normal to the horizontal axis of zenithal rotation. The two fixed bubbles are used for zenithal position exposures only. The two fixed bubbles are shown in Figure 6.

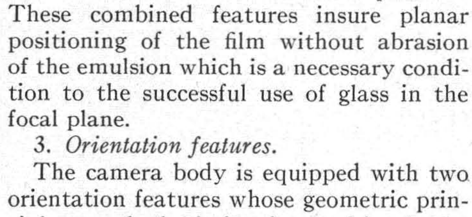

FIG. 5. Film Magazine.

a. Two fixed bubbles.—The two fixed-bubble feature embraces a geometric principle that is an innovation in zenithal orientation not employed heretofore in cameras designed for zenithal exposures. Conventional terrestrial and zenithal surveying cameras are mounted on vertical spindles which are made to define the direction of gravity by alternately working footscrews and adjustable level vials. The footscrews are turned and the level vials adjusted until the bubble remains centered in all azimuthal rotations of the spindle. Fundamentally an astronomic position observation consists of measuring directly or indirectly the direction angles of the gravitational vector at a point. The accuracy of

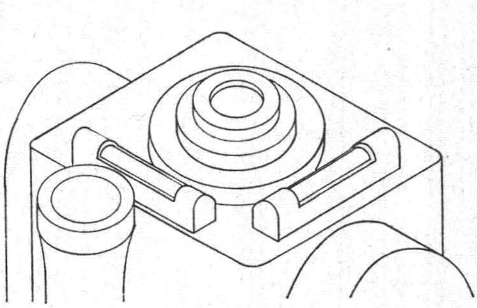

FIG. 6. Fixed Level Vials.

the position is dependent on the accuracy of the spindle rotation assembly and level vials. The position can be no more accurate than the spindle or level vials. In a perfectly functioning zenith camera the optical axis is collinear with the mechanical axis which is the spindle. Therefore when the mechanical spindle is adjusted with the aid of level vials and footscrews to define the vertical, the optical axis also defines the vertical. This is virtually impossible to achieve in practice, neither is it necessary. The spindle or the mechanical axis of rotation must define the direction of gravity while the optical axis has only to define a fixed angle with the direction of gravity, such that in an azimuthal rotation through 360 degrees the optical axis generates a cone. Two positions separated in azimuth by 180 degrees define opposite equal angles with the zenith point when the mechanical axis is vertical. The geometry of the conventional method is shown in Figure 7.

The reverse position is 180 degrees about zz from the direct. Assuming the vertical

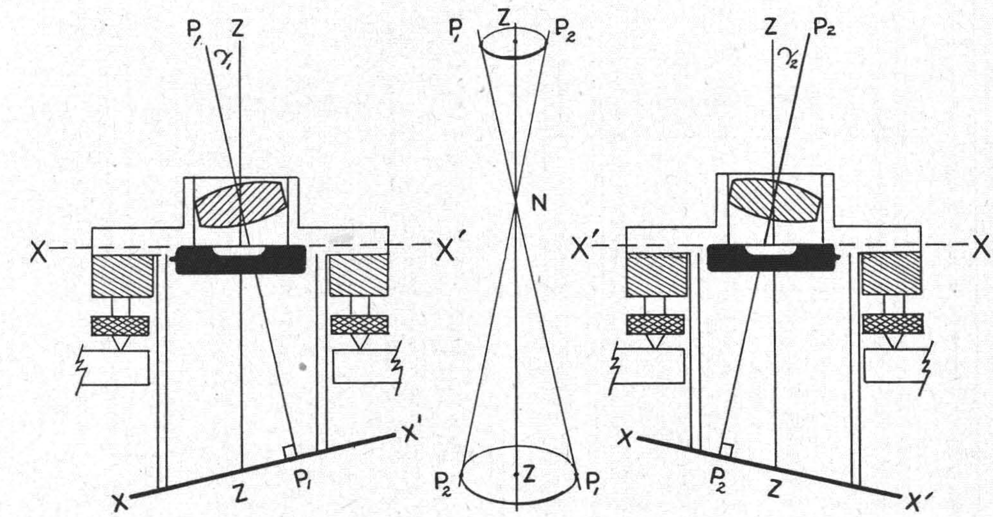

FIG. 7. Geometry of Conventional Method of Zenithal Observation.

bearing surface of the camera spindle to be parallel to zz, by virtue of the bubble being centered in all azimuthal orientations of the camera, the optical axis may be assumed to define a fixed angle γ for all azimuthal orientations of the camera. Two exposures made on the same film will provide two images of points on the celestial sphere at the principal point. The mean astronomic coordinates of these two images are equal to the astronomic coordinates of the zenith and the camera station since the principal point defines equal opposite angles with the zenith point in the direct and reverse positions.

The minimum geometry for defining the zenith may be rigorously duplicated with a camera equipped with two fixed or nonadjustable bubbles without a precise spindle or without a spindle. The level vials are mounted normal to each other on a camera supported on three footscrews attached to the camera body. If the focal plane occupies a fixed angle with a plane defined by any position of the level bubbles in the level vials, a normal to the film plane will define equal angles with the direction of gravity for all azimuthal orientations so long as the initial positions of the bubbles in the level vials are unaltered or recaptured for each azimuthal orientation. This concept is illustrated in Figure 8 in a plane through one bubble.

Assume the bubbles to be centered without adjustment in two positions 180 degrees apart, and at each of these positions exposures are made without advancing the film. The centered bubbles define the same tangent plane in both positions within the precision of the duplicate centering of the bubbles; the film has a fixed relation to the level vials in both positions. Therefore the principal point defines opposite equal angles with the zenith point. This is essentially the geometric principle of the two fixed bubbles seated on the lens end of the camera for zenithal exposures. The two fixed-bubble principle has several advantages over the adjustable-bubble, vertical spindle combination:

(1) Precise vertical spindles are costly, difficult to manufacture, difficult to assemble, and difficult to maintain in a state of precision.

(2) Much less skill is required to center two fixed bubbles twice for a short period than to center a bubble or several bubbles for all azimuthal orientations by alternately turning the footscrews and adjusting level vials.

(3) Two fixed bubbles result in greater position accuracy than two adjustable bubbles of the same sensitivity.

The latter advantage arises from two facts: The bubbles are centered deliberately for each exposure which in turn permits greater deliberation and concentration on the shutter actuation for each exposure, and the mechanical error is reduced to that of the bubble, whereas normally the mechanical error is a resultant of the bubble

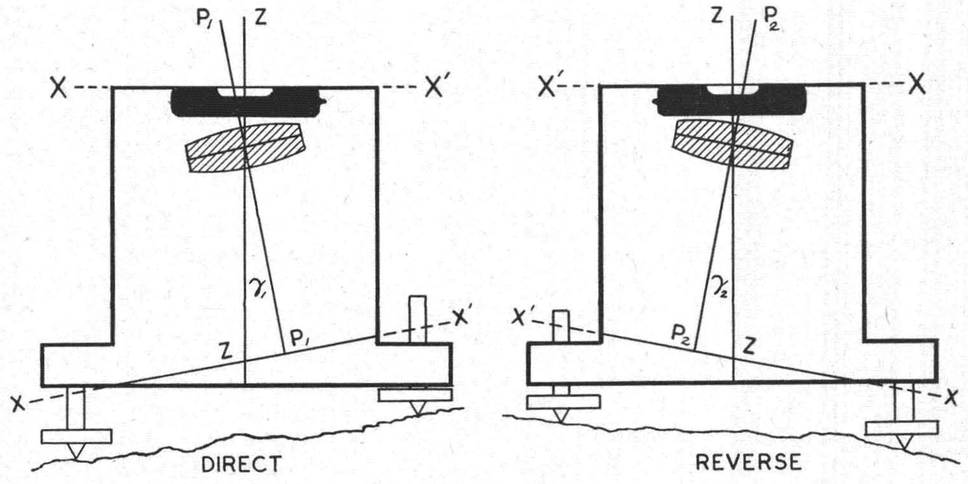

FIG. 8. Fixed Level Principle.

and the spindle. Level vials are selected for this purpose whose precision and sensitivity are compatible with the desired position accuracy and the metrical properties of the camera.

It is generally agreed that an operator can recenter a bubble with a precision of one-quarter division. On this basis two five-second level vials are considered to be compatible with the position requirements and the metrical characteristics of the MM 101 camera. Two such vials used in a test of the fixed-bubble principle were calibrated on a level trier. These vials were found to have a sensitivity of 5 seconds with a probable error of one-tenth second. Though the MM 101 is spindle mounted, the spindle imparts rotational ease to the camera—not precision. The spindle, therefore, has no relation to the precision of the zenithal orientation for position—only ease in rotating the camera to the reverse position. Finally since the bubbles are recentered for the direct and reverse position, the operator need not concern himself about decentering the bubble in moving from the direct to the reverse position.

b. Autocollimating telescope.—An optical diagram of the autocollimating telescope is shown in Figure 9.

The objective is an achromat corrected for *C* and *F* wavelengths. The telescope tube length, fixed by the camera body length, is 150 millimeters. The eyepiece is 36 millimeters long. The rather excessive focal length is imposed by the autocollimation element, positioned between the field lens and eye lens, consisting of an optical flat. The focal length of the eyepiece system is 25 millimeters. A magnification of

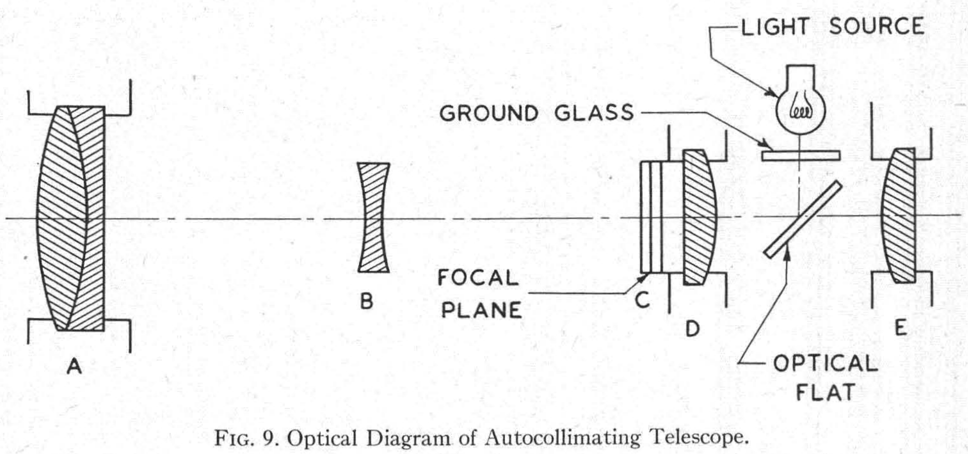

FIG. 9. Optical Diagram of Autocollimating Telescope.

twelve times is required for military purposes. The quotient of the tube length and eyepiece length is six. For this reason a double concave element, lens *B*, is designed to produce a focal length of 300 millimeters and at the same time provide means of internal focus. A 14 millimeter axial range of lens *B* permits an object distance as short as three feet. The longer telescopic focal length gives the desired magnification of twelve times. The reticule is placed at the contact surfaces of two plano-parallel pieces of glass sandwiched together. This design frees the reticule from troublesome specks of dust that inevitably collect in the focal plane of a sighting telescope. Any dust collecting on the back or front surface of the two elements will not be seen by the eye. A perfect square etched on the reticule permits reading stadia intercepts with the stadia rod in either a vertical or horizontal position.

The eyepiece is a Ramsden design selected because it is reasonably free from spherical aberrations, with a wide separation between the field lens and the eye lens. The wide separation between the elements permits the introduction of an optical flat whereby the return rays of light projected through the telescope to a three-cornered mirror may be viewed with the eye. A three-cornered mirror consists of three mutually orthogonal reflecting surfaces and has the property of returning a ray parallel to the path of incidence. This means that the azimuth mark will be seen only by the observer. Such an azimuth mark has advantages in military operations where it may be convenient to have an unattended light.

B. CAMERA GONIOMETER

The camera goniometer is a mechanical means of rotating the camera about azimuthal and zenithal axes. These rotations are referenced to glass circles graduated to single degrees. The azimuth circle is contained in the base, while the zenith circle is contained in the right standard. The optical mean of either circle is read in a common eyepiece projecting from the right standard. An accuracy of several seconds is achieved for any degree setting by the optical coincidence of diametrically opposite positions of either circle. The optical system consisting of the circles, microscope objectives, prisms, changeover prism, and reading eyepiece is shown in Figure 10. The circle graduations are magnified ten times. The apparent size of the reading is shown in Figure 11. The microscope objectives are achromatic, positioned to give a common image size for either circle. Image shift from azimuth to zenith circle, or vice versa, is achieved by turning a changeover knob also located on the right standard.

The purpose of the system employed is to provide precise, preset readings of the azimuth and tilt to some convenient degree or half-degree on the premise that the direction of any significant detail not falling on the principal point may be obtained accurately from measurements made on the exposure negative, provided the azimuth and tilt are precisely known. The location of the reading microscope provides maximum speed and comfort in reading or presetting azimuth and tilt angles. In most cases terrestrial exposures will be made with the optical axis lying in a horizontal plane. The zenith circle is then locked at 90 degrees and the azimuth angle preset for each exposure in accordance with the desired sidelap. Suppose a sidelap of 12 degrees is desired. The azimuth circle is then preset on zero, while bisecting the azimuth mark. Thereafter, the azimuth circle is preset by turning the slow motion azimuth screw while sighting the reading microscope in increasing increments of 30 degrees until 360 degrees of the horizon have been photographed. Photogrammetric calculations are reduced to a minimum when azimuth and tilt may be considered known. These are among the data recorded with each exposure.

The base plate is leveled with a tubular 20 second vial. The most stringent leveling requirements are those of zenithal orientation for position, which are embraced in the two fixed-level vial system. The terrestrial exposures do not require comparable precision and therefore the camera goniometer rests, and rotates about the vertical, on a horizontal ring supported by ball bearings. A token spindle is retained whose only function is to center the rotation. This type of spindle system has maximum base stability and rotational ease without vertical accuracy in the sense of arc seconds associated with an astronomic position. These are the minimum required properties. A section through the camera

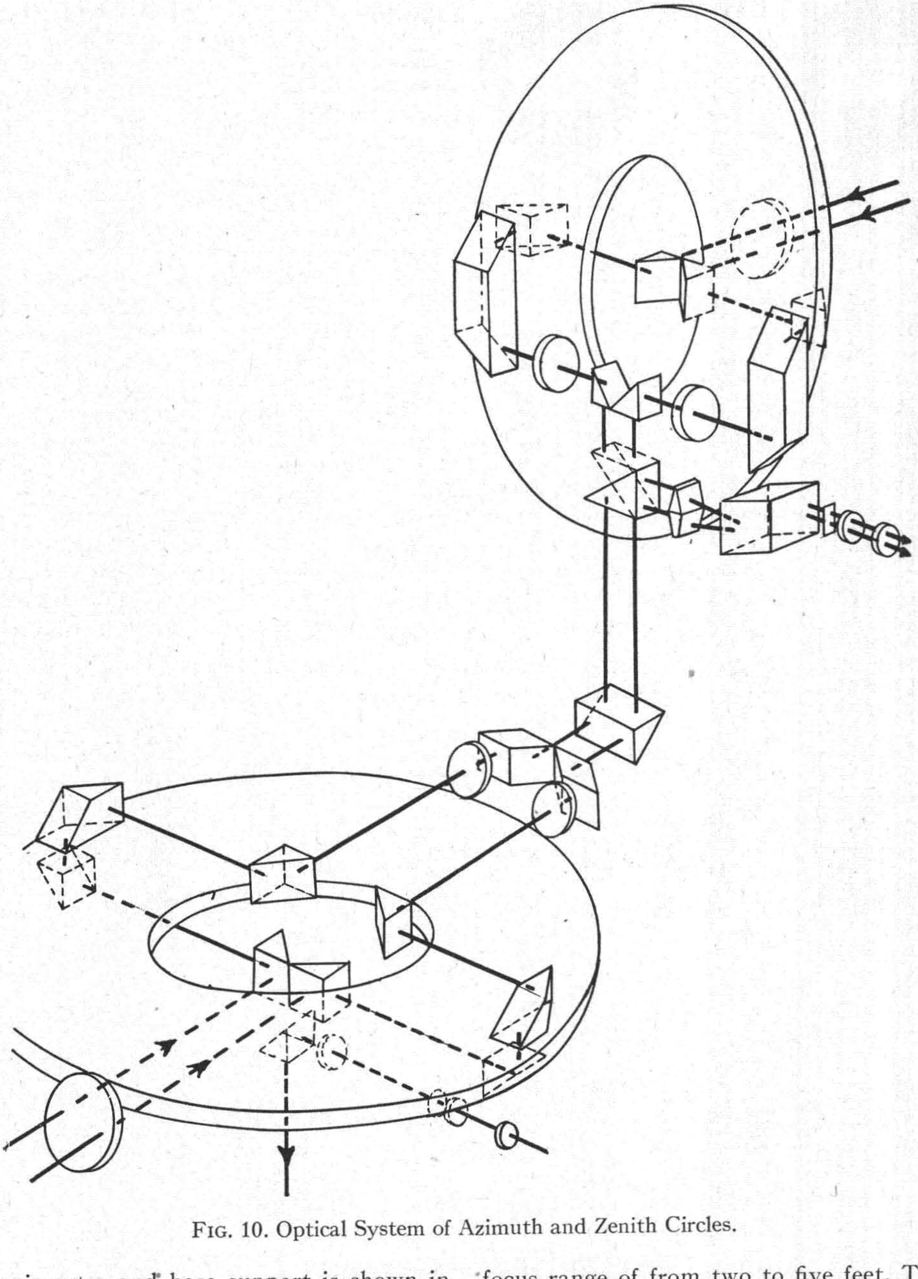

FIG. 10. Optical System of Azimuth and Zenith Circles.

goniometer and base support is shown in Figure 12.

The base of the goniometer is equipped with an optical plummet for rapid centering of the camera over a survey point. The optical plummet consists of a telescope, with a magnification of two times, lying in a horizontal plane. A 90 degree prism in front of the telescope objective directs the light downward. The eyepiece has a focus range of from two to five feet. The telescope may be adjusted about, and normal to, its axis for vertical autocollimation. The advantage of an optical plummet for centering an instrument in wind is obvious.

The circles and the autocollimating telescope reticules are illuminated by small plug-in light bulbs. A battery source is plugged into the base. Circular mirrors replace the light bulbs for day operation.

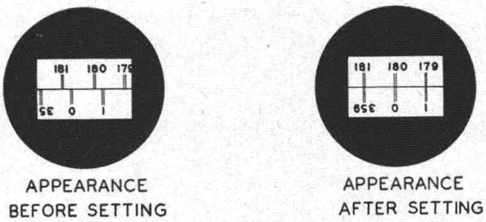

FIG. 11. Circle Reading Magnified by the Reading Microscope.

The distances from the light source to opposite positions on each circle are equal to insure equality of object surface illumination. The distances from the object surface to the focal plane of the reading microscope are also equal to insure equality in the focal plane illumination.

The fundamental objective in the camera goniometer design is a system in which the basic camera and the mechanical means of all orientations and operations are integrally related. Other cameras are either single purpose or improvised from existing related instruments. These terrestrial cameras suffer in appearance and function from the fact that the goniometers have not been especially designed for the specific camera or any other camera.

III. Field Procedure

The purpose surrounding the MM 101 camera system is twofold:

(1) the method and equipment for recording the metrical data necessary to establish a network of geodetic points at a stipulated accuracy in an area where no control exists; and

(2) the method and equipment that are adapted to rapid and simple observational procedure.

The supporting items of equipment necessary to complete the system for field procedure are some means of measuring a base line, such as a 100 foot tape; a radio receiving set, for time signals: and an altimeter, for one absolute elevation if an ab-

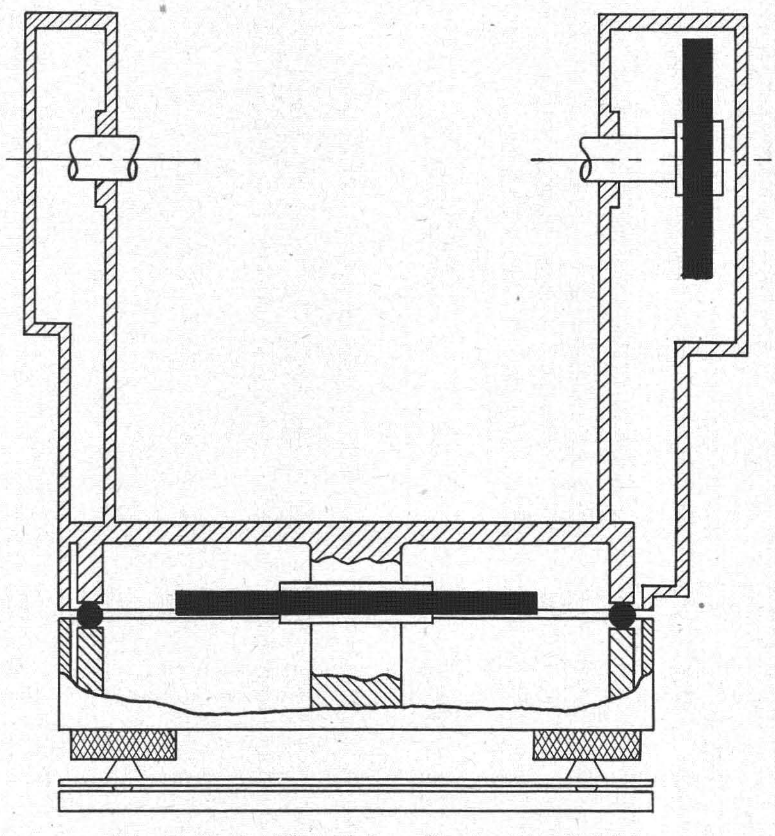

FIG. 12. Section Through Bore Showing Ring Spindle.

solute elevation is considered necessary. The basic operations are schematically keyed into Figure 13.

1. *Base line measurement:* The horizontal, or slope, distance is measured between A and B or any other two points and reduced to $\overline{AB}$.

2. *Vertical datum:* An altimeter reading or a series of altimeter readings are made at A.

3. *Triangulation:* A round of overlapping horizontal or near horizontal exposures are made at A and B.

4. *Astronomic position:* Three direct and reverse zenithal exposures referenced to audible radio time signals are made at A.

5. *Astronomic azimuth*: Three direct double zenithal exposures referenced to audible radio time signals are made at A referred to B.

In practice, any intersection point may be used as an astronomic position station, any two intersection points as an azimuth line, and any two intersection points may be used as the terminals of the base line. The notation and arrangement in Figure 13 are chosen for simplicity in outline only. The various observations may be accomplished in any convenient chronological order.

A. BASE LINE MEASUREMENT

Assume the base line to be measured by any conventional means whatsoever. In general, the base line must define a traversable route while the camera stations must define conjugate imagery of a common object space. The camera station base line provides the most direct data reduction and therefore defines the base line whenever possible. The base line may be measured between one camera station and one intersection point, or between two intersection points when the distance between the two camera stations is not accessible to measurement. The three cases are illustrated in Figure 14a, b, c. The distance AB is computed if it cannot be measured directly. Where the base line is defined by one camera station and one intersection station,

$$AB = \frac{\sin \theta_3}{\sin \theta_2} CA$$

$$\theta_3 = 180° - (\theta_1 + \theta_2)$$

and where the base line is defined by two intersection stations,

$$AB = CD \cdot K$$

$$K = \frac{1}{(m_1^2 + n_1^2 - 2m_1n_1 \cos \theta_1)^2}$$

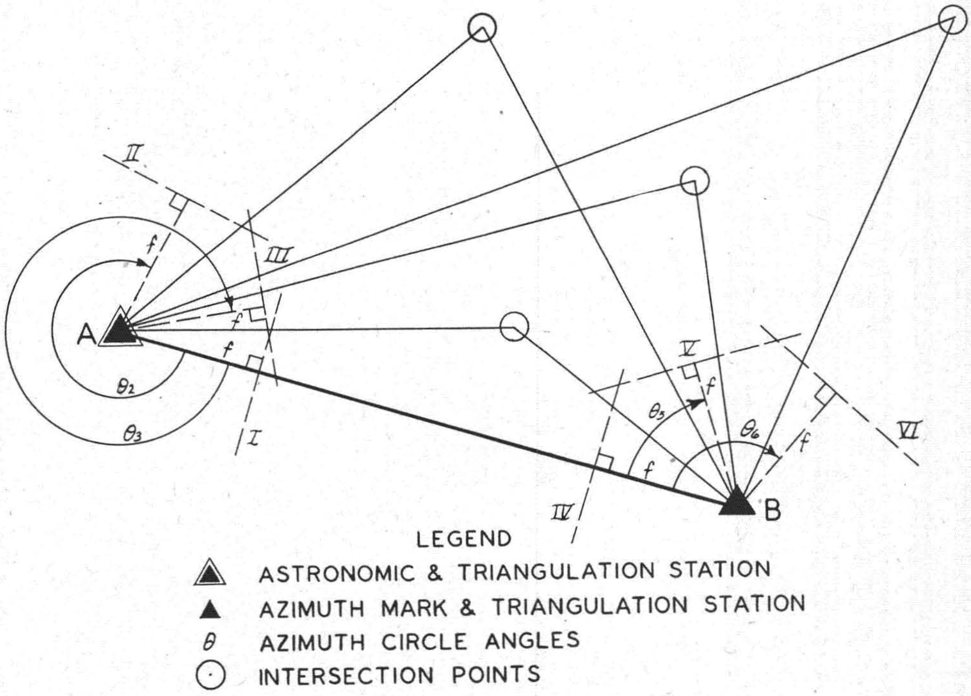

FIG. 13. Diagram of Basic Operations.

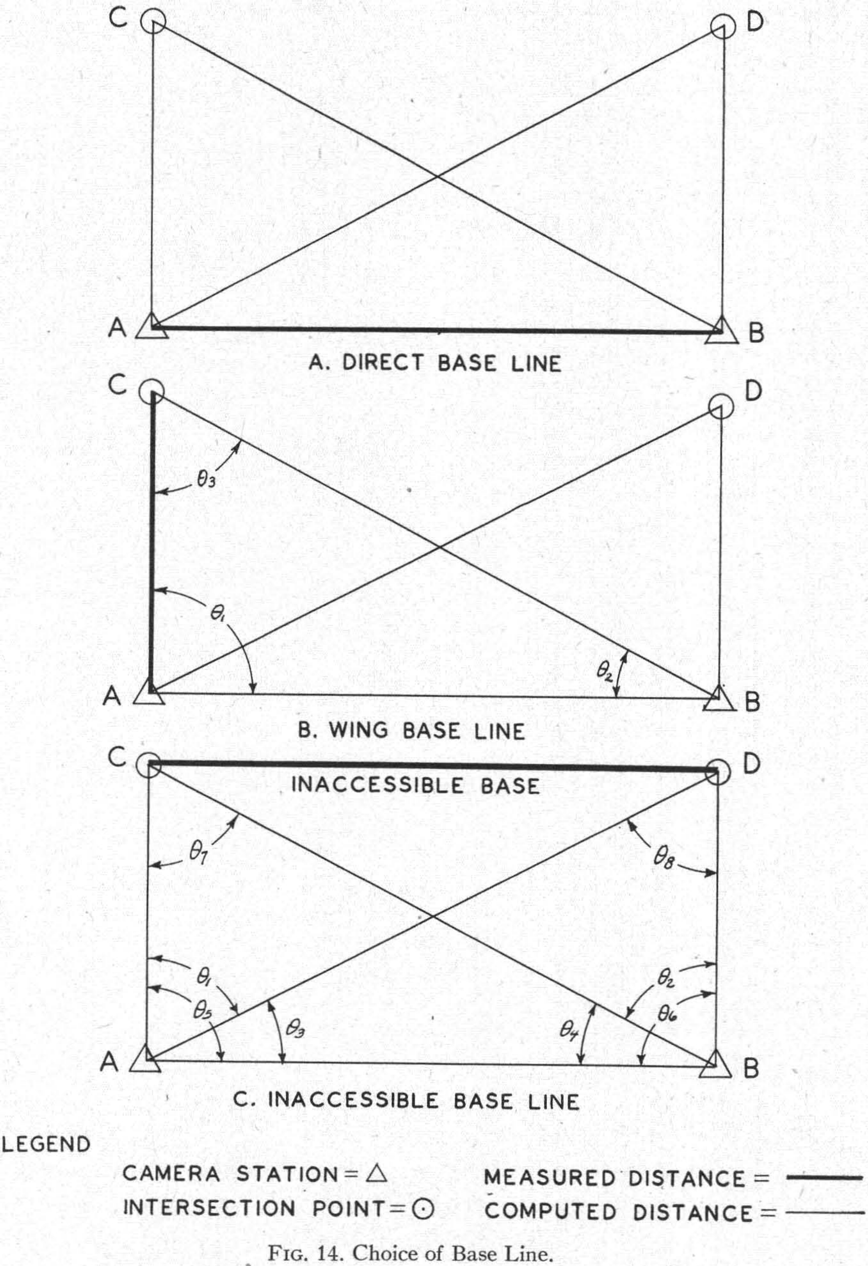

FIG. 14. Choice of Base Line.

or

$$K = \frac{1}{(m_2^2 + n_2^2 - 2m_2n_2 \cos \theta_2)}$$

$$m_1 = \frac{\sin \theta_4}{\sin \theta_7} \qquad n_1 = \frac{\sin \theta_6}{\sin \theta_8}$$

$$m_2 = \frac{\sin \theta_3}{\sin \theta_8} \qquad n_2 = \frac{\sin \theta_5}{\sin \theta_7}$$

$$\theta_7 = 180° - (\theta_4 + \theta_5)$$
$$\theta_8 = 180° - (\theta_3 + \theta_6).$$

Those values of θ at A or B are computed with photographic data.

B. VERTICAL DATUM

The absolute elevation of one point is measured with an altimeter. Even though this point may be a camera station, an altimetric measurement of any intersection point may be substituted. The purpose of the altimetric measurement is to refer the system of geodetic points to sea level datum. The differences in elevation between a point established by altimetry and all other points are determined with photographic data. The reference value may be assigned a value of zero in many surveys since the fundamental requirement is the differential elevations. Omission of a reference elevation has a negligible effect on the latitudes and longitudes computed from the space coordinates of intersection points obtained photogrammetrically. Therefore the errors of altimetry arising without reference to a base bench mark are more negligible. It is for this reason that an altimeter is used to establish the reference elevation if a reference elevation is considered necessary.

C. TRIANGULATION EXPOSURES

The camera is set up over *A*, leveled and the azimuth circle made to read zero while the telescope reticule bisects *B*. Though the camera may be inclined at any zenith angle, rarely will significant detail be omitted when the zenith circle reads 90 degrees or the camera optical axis is level. Imposing a 90 degree tilt on the camera simplifies the data reduction. In any case, the initial exposure is made with the azimuth circle zeroed on *B*. Assume a sidelap of 12 degrees is desired. Successive exposures are made thereafter at increasing azimuth circle increments of 30 degrees until the horizon has been photographed through 360 degrees. The zenith circle is left locked at 90 degrees unless some detail such as a mountain top or an array of high buildings would not be imaged at 90 degrees. Then the zenith circle is locked precisely to the nearest half-degree that will include significant detail in the exposure frame. An example of the recorded data is shown in Figure 15.

It may be noted in Figure 15 that the zenith angle of Exposure No. 2 is 95 degrees. This is assumed necessary to image the upper portion of a mountain peak or an array of high buildings. These operations are repeated at Station *B* so chosen as to provide conjugate images of the maximum number of points. The azimuth circle is zeroed on *A* at *B*.

D. ASTRONOMIC POSITION EXPOSURES

Three frames are double exposed on the zenith 180 degrees apart for astronomic position. The camera is rotated in azimuth so that the *y* axis approximates the observer's meridian and the azimuth circle is clamped. The camera is then rotated to the zenith and the zenith circle clamped at 180 degrees. The lens is set at full aperture and the shutter speed at *B*. The radio receiving set situated near the camera for hearing without effort is adjusted for sharp audible time signals while the fixed bubbles mounted on the lens end of the camera are becoming acclimatized.

After the bubbles are considered acclimatized, the bubbles of the two fixed vials are centered by working the footscrews. Initially the shutter is opened, without precise reference to time signals, for approximately one minute. During this time the operator mentally acquires the period of the time signal and selects a specific second of time about 20 seconds after the one-minute preliminary exposure.

In the 20-second interval before precise opening, the operator takes the slack out of the plunger of the shutter cable and mentally counts, making the count coincide precisely with the corresponding second. The shutter is then opened precisely on the selected second. The operator con-

FIG. 15. PHOTOGRAPHIC TRIANGULATION RECORD

Date: 28-5-53

Place: Utopia — Camera: MM 101
Observer: Nameless Nora — Film: 70 mm Super XX
Station: A — Reference Station: B

Exposure No.	Azimuth *A*	Zenith *A*	Aperture	Shutter Speed	Object
1	0°	90°	4.5	0.01^s	*B*
2	30°	95°	6.3	0.01^s	
...	...	...	...	...	...
12	330°	90°	5.6	0.01^s	

tinues to count, releasing the shutter precisely on the tenth second signal following the opening. This constitutes a direct exposure. The bubbles are examined for center position. Rarely will the bubbles be decentered. If they are, the exposure is cancelled. Assuming they are not, the camera is rotated in azimuth 180 degrees from the initial position without altering the zenith circle. The fixed bubbles are recentered by working the footscrews. The operator again takes the slack out of the cable, selects a specific second and mentally counts. Upon reaching the selected second, the shutter is precisely opened and precisely closed on the tenth second following the opening. This constitutes the reverse exposure and the double exposure of the first frame. At least two more frames are exposed in this manner.

A position exposure is shown in Figure 16. The purpose of the one-minute exposure is to increase the film speed by preexposure and to provide a legend for easy correlation of the 10-second trails with the corresponding radio times. The hour, minute, and second of both positions are recorded.

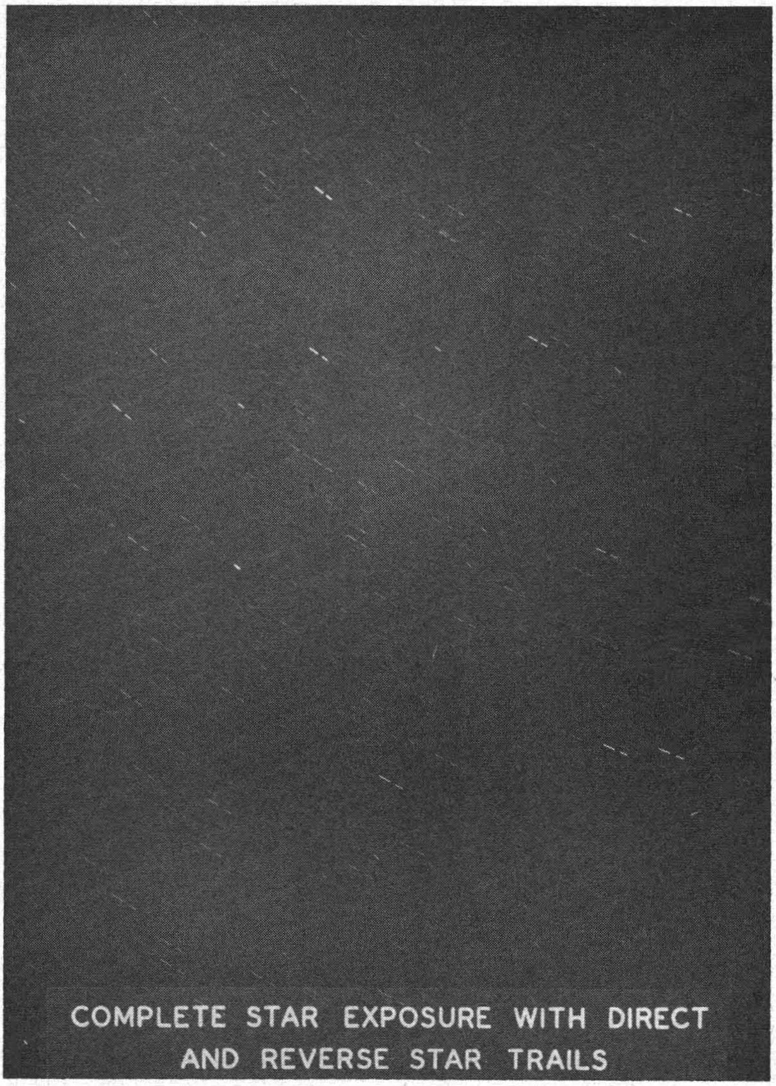

FIG. 16

It may be noted there has been no mention of stop watches, chronometers, or time comparators. The timing procedure removes the necessity for these items of equipment. A maximum error in longitude of one-half arc second with a 101 mm. test camera oriented on the zenith by the two fixed bubble principle is a gratifying testimonial to the accuracy obtáinable by the method of timing, the method of zenithal orientation, and the method of data reduction. The test camera mounted on a platform equipped with two fixed five-second bubbles is shown in Figure 17. A zenith exposure record is shown in Figure 18. The mean times of opening and closing are used in the data reduction.

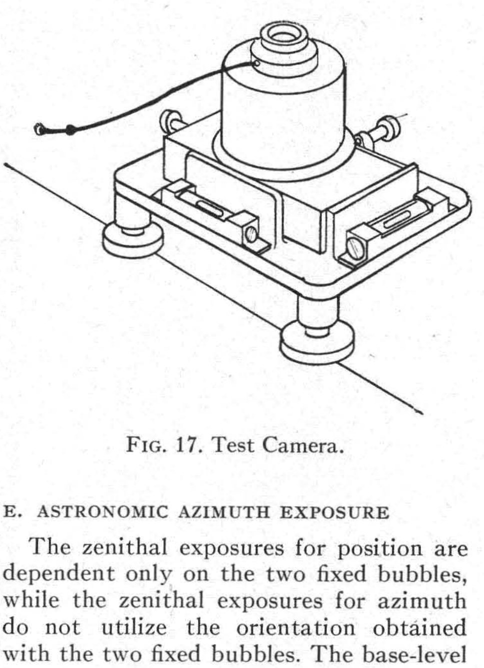

FIG. 17. Test Camera.

E. ASTRONOMIC AZIMUTH EXPOSURE

The zenithal exposures for position are dependent only on the two fixed bubbles, while the zenithal exposures for azimuth do not utilize the orientation obtained with the two fixed bubbles. The base-level bubble is adjusted, but the azimuth observation does not require that the zenith circle axis be horizontal, that 90 degrees on the zenith circle define the horizon, nor that 180 degrees define the zenith. Assume again that the camera occupies station *A*. The telescope is made to bisect station *B* with the azimuth circle clamped at zero degrees. The zenith angle of station *B* is recorded. The camera is then rotated to the zenith and exposures made at two arbitrary zenith angles at either side of 180 degrees, such as 175 and 185 degrees. Herein lie the assumptions:

(1) Even though the three values of the zenith angles are assumed to be not absolutely correct, the differences are correct within the accuracy of the circle system; and

(2) the azimuth mark and the two zenith points define an arc lying in a plane that is normal to the zenith circle axis.

This frees the azimuth observation from errors defined by a mechanical failure to recover the zenith.

In any case, two exposures are made in the direct position on the same frame. At the initial zenith angle the frame is pre-exposed for one minute, without reference to radio time signals, followed by a 10-second exposure, the opening and closing of which are precisely referenced to radio time signals. At the succeeding zenith angle a 10-second exposure is made precisely referenced to radio time signals. The exposure pattern follows that of astronomic position except that the pattern is restricted to the direct position of the camera. Two more frames are exposed in this manner. A zenithal exposure record for azimuth is shown in Figure 19.

This completes the field procedure.

FIG. 18. ZENITH EXPOSURE RECORD FOR POSITION

Date: GCT. 28-5-53

Place: Utopia	Radio: Hallicrafter S2-39
Observer: Nameless Nora	Signal: WWV
Camera: MM 101	Direct trial: $\underline{60^s}$ 20^s $\underline{10^s}$
Film: Super XX pan.	Reverse trail: $\underline{10^s}$

Exposure No. 1

Position	Open time	Close time	Mean time
Direct	$3^h\ 19^m\ 00^s$	$3^h\ 19^m\ 10^s$	$3^h\ 19^m\ 05^s$
Reverse	3 23 00	3 23 10	3 23 05

FIG. 19. ZENITH EXPOSURE FOR AZIMUTH RECORD

Date: GCT 28-5-53

Place: Utopia	Radio: Hallicrafter S2-39
Observer: Nameless Nora	Signal: WWV
Camera: MM 101	Direct trail (1) $\underline{60^s}$ ' $\underline{10^s}$ 20^s
Film: Super XX	Direct trail (2) $\underline{10^s}$
Camera Station: A	Azimuth M: B

Exposure No. 1

	Zenith Angle	Time Open	Time Close	Time Mean
Az. M.	92° 30′			
z	175° 00′	$4^h\ 32^m\ 00^s$	$4^h\ 32^m\ 10^s$	$4^h\ 32^m\ 05^s$
z	185° 00′	4 33 00	4 33 10	4 33 05

IV. DATA REDUCTION

The preliminary treatment of star data is the same for both position and azimuth. Three stars defining equal radii with the principal point are selected for each exposure on a given frame, or a total of six stars per frame. The following steps are preliminary to both azimuth and position determination:

(1) *Identification of stars.*
(2) *Reduction to apparent places for instant of exposure.*
(3) *Measurement of x and y coordinates referred to any arbitrary origin.* Both terminals of the star trails are measured. The mean value of the ends corrected for film distortion is used in the computations.
(4) *Calibration of exposures.*

Assume steps (1), (2), and (3) have been accomplished.

A. COMPUTATION OF CALIBRATION DATA

The small variation between the calibration data obtained in the laboratory and the calibration data existing at the instant of exposure is significant in astronomic position and azimuth computations. For this reason the star images used to compute the latitude, longitude, and azimuth are also used to determine the location of the principal point and the focal length satisfying the position of the star images. This, among other things, provides a value in which the effects of both refraction and lens distortion are made zero. The reduced coordinates and the angles subtended by pairs of images comprise the given data. The angles subtended by three images a, b, and c are computed with trigonometric functions of the right ascension and declination (α, δ):

$$\cos aLb = \sin \delta_a \sin \delta_b + \cos \delta_a \cos \delta_b \cos (\alpha_a - \alpha_b)$$
$$\cos bLc = \sin \delta_b \sin \delta_c + \cos \delta_b \cos \delta_c \cos (\alpha_b - \alpha_c)$$
$$\cos cLa = \sin \delta_c \sin \delta_a + \cos \delta_c \cos \delta_a \cos (\alpha_c - \alpha_a)$$

With these data the following equations are formed:

$$a_1\Delta x + b_1\Delta y + c_1\Delta z = q_1$$
$$a_2\Delta x + b_2\Delta y + c_2\Delta z = q_2$$
$$a_3\Delta x + b_3\Delta y + c_3\Delta z = q_3$$

where

$$a_1 = xam_1 + xbn_1 \qquad b_1 = yam_1 + ybn_1$$
$$a_2 = xbm_2 + xcn_2 \qquad b_2 = ybm_2 + ycn_2$$
$$a_3 = xcm_3 + xan_3 \qquad b_3 = ycm_3 + yan_3$$

$$c_1 = f'(m_1 + n_1) \qquad q_1 = LaLb \sin aLb'\Delta aLb$$
$$c_2 = f'(m_2 + n_2) \qquad q_2 = LbLc \sin bLc'\Delta bLc$$
$$c_3 = f'(m_3 + n_3) \qquad q_3 = LcLa \sin cLa'\Delta cLa$$

$$m_1 = 1 - \cos aLb' \frac{Lb}{La} \qquad n_1 = 1 - \cos aLb' \frac{La}{Lb}$$
$$m_2 = 1 - \cos bLc' \frac{Lc}{Lb} \qquad n_2 = 1 - \cos bLc' \frac{Lb}{Lc}$$
$$m_3 = 1 - \cos cLa' \frac{La}{Lc} \qquad n_3 = 1 - \cos cLa' \frac{Lc}{La}$$

$$La = (xa^2 + ya^2 + f'^2)^{1/2}$$
$$Lb = (xb^2 + yb^2 + f'^2)^{1/2}$$
$$Lc = (xc^2 + yc^2 + f'^2)^{1/2}$$

$$\cos aLb' = \frac{xaxb + yayb + f'^2}{LaLb}$$

$$\cos bLc' = \frac{xbxc + ybyc + f'^2}{LbLc}$$

$$\cos cLa' = \frac{xcxa + ycya + f'^2}{LcLa}$$

$$\Delta aLb = aLb - aLb'$$
$$\Delta bLc = bLc - bLc'$$
$$\Delta cLa = cLa - cLa'$$

and f' is the first approximation of f.

The equations are solved simultaneously for Δx_1, Δy_1, and Δz_1, which are applied as corrections to the x and y coordinates and f'. With the corrected values, new coefficients and constant terms are formed and a second set of simultaneous equations are solved for Δx_2, Δy_2 and Δz_2. The forming of revised coefficients and constant terms coupled with a simultaneous solution is repeated until the differential unknowns and constant terms vanish.

Then for any image a,

$$xa_c = xa + \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n$$
$$ya_c = ya + \Delta y_1 + \Delta y_2 + \cdots + \Delta y_n$$
$$f = f' + \Delta z_1 + \Delta z_2 + \cdots + \Delta z_n$$

Over three approximations are seldom required.

The sum of these corrections refer the coordinates to the principal point location and focal length that satisfy the star images selected.

B. LATITUDE AND LONGITUDE DETERMINATION

The following equations are formed for each position of the camera:

$$\cos \alpha_{sa} \cos \alpha_z + \cos \beta_{sa} \cos \beta_z + \cos \gamma_{sa} \cos \gamma_z = \cos \gamma_a$$
$$\cos \alpha_{sb} \cos \alpha_z + \cos \beta_{sb} \cos \beta_z + \cos \gamma_{sb} \cos \gamma_z = \cos \gamma_b$$
$$\cos \alpha_{sc} \cos \alpha_z + \cos \beta_{sc} \cos \beta_z + \cos \gamma_{sc} \cos \gamma_z = \cos \gamma_c$$

where

$$\cos \alpha_{sa} = \sin GHA_a \cos \delta_a \qquad \cos \beta_{sa} = \cos GHA_a \cos \delta_a$$
$$\cos \alpha_{sb} = \sin GHA_b \cos \delta_b \qquad \cos \beta_{sb} = \cos GHA_b \cos \delta_b$$
$$\cos \alpha_{sc} = \sin GHA_c \cos \delta_c \qquad \cos \beta_{sc} = \cos GHA_c \cos \delta_c$$

$$\cos \gamma_{sa} = \sin \delta_a \qquad \cos \gamma_a = \frac{f}{(xa^2 + ya^2 + f^2)^{1/2}}$$

$$\cos \gamma_{sb} = \sin \delta_b \qquad \cos \gamma_b = \frac{f}{(xb^2 + yb^2 + f^2)^{1/2}}$$

$$\cos \gamma_{sc} = \sin \delta_c \qquad \cos \gamma_c = \frac{f}{(xc^2 + yc^2 + f^2)^{1/2}}$$

$GHA = GST - \alpha$ (time)

$GST = GCT + GST$ 0^h+corr. (of mean time of opening and closing)

The camera data used in these computations are calibrated values.

The three equations are solved simultaneously for $\cos \alpha_z$, $\cos \beta_z$ and $\cos \gamma_z$.

Then

$$\tan \lambda = \frac{\cos \alpha_z}{\cos \beta_z}$$

$$\sin \phi = \cos \gamma_z.$$

Assume these equations have been solved for the direct and reverse positions. The mean values are the astronomic coordinates of the camera station if the arc distance separating the direct and reverse position of the principal point does not exceed several degrees.

$$\phi_z = \frac{\phi_D + \phi_R}{2}$$

$$\lambda_z = \frac{\lambda_D + \lambda_R}{2}.$$

The differences in latitude and longitude on two separate determinations was less than a second of arc.

The mean longitude is not the longitude of the zenith point if the arc separation exceeds several degrees unless the direct and reverse position define a line coinciding with the meridian or the prime vertical. The error of the mean longitude is due to the convergence of the meridians with latitude, and varies as the cosines of the latitudes of the direct and reverse positions of the principal point vary. Assume the direction cosines of the principal point have been determined in the direct and reverse position. The cosine of the arc separation is computed with these functions:

$$\cos \gamma_{z_1 z_2} = \cos \alpha_{z_1} \cos \alpha_{z_2} + \cos \beta_{z_1} \cos \beta_{z_2} + \cos \gamma_{z_1} \cos \gamma_{z_2} \qquad (1)$$

where subscripts 1 and 2 denote direct and reverse. Since the two positions define equal zenith angles with the zenith point,

$$\gamma_{zz_1} = \gamma_{z_2 z} = \frac{\gamma_{z_1 z_2}}{2}.$$

By the law of sines

$$\sin Az_1 = \frac{\cos \phi_2 \sin (\lambda_2 - \lambda_1)}{\sin \gamma_{z_1 z_2}}$$

$$\sin Az_2 = \frac{\cos \phi_1 \sin (\lambda_2 - \lambda_1)}{\sin \gamma_{z_1 z_2}}.$$

These equations are illustrated in Figure 20.

Then by the law of cosines,

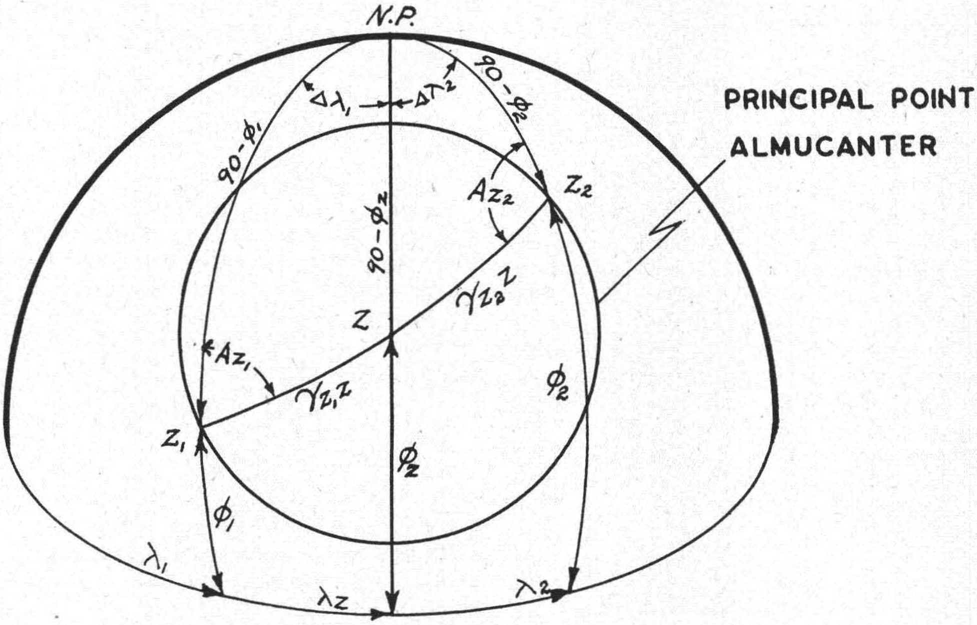

FIG. 20. $\lambda_1+\lambda_2/2\neq\lambda_z$ When $\gamma_{z_1z_2}$ is Excessive.

$$\left.\begin{aligned}\sin\phi_z &= \sin\phi_1\cos\frac{\gamma_{z_1z_2}}{2} + \cos\phi_1\sin\frac{\gamma_{z_1z_2}}{2}\cos Az_1\\ \sin\phi_z &= \sin\phi_2\cos\frac{\gamma_{z_1z_2}}{2} + \cos\phi_2\sin\frac{\gamma_{z_1z_2}}{2}\cos Az_2\end{aligned}\right\}\quad(2)$$

and

$$\left.\begin{aligned}\sin\Delta\lambda_1 &= \sin\frac{\gamma_{z_1z_2}}{2}\,\frac{\sin Az_1}{\cos\phi_2}\\ \sin\Delta\lambda_2 &= \sin\frac{\gamma_{z_1z_2}}{2}\,\frac{\sin Az_2}{\cos\phi_z}\end{aligned}\right\}.\quad(3)$$

Substituting for $\sin Az_1$ and $\sin Az_2$

$$\sin\Delta\lambda_1 = \cos\phi_2\left[\frac{\sin(\lambda_2-\lambda_1)}{\sin\gamma_{z_1z_2}}\,\frac{\sin\gamma_{z_1z_2}}{\cos\phi_z\cdot 2}\right]$$

$$\sin\Delta\lambda_1 = \cos\phi_1\left[\frac{\sin(\lambda_2-\lambda_1)}{\sin\gamma_{z_1z_2}}\,\frac{\sin\gamma_{z_1z_2}}{\cos\phi_z\cdot 2}\right].$$

Inasmuch as all terms in brackets are the same, $\sin\Delta\lambda$ varies as the cosine of the opposite latitude varies.

Therefore if $\cos\phi_2\neq\cos\phi_1$

$$\sin\Delta\lambda_1\neq\sin\Delta\lambda_2.$$

Then, since

$$\lambda_1+\Delta\lambda_1=\lambda_2-\Delta\lambda_2=\lambda_z$$

$$\frac{\lambda_1+\lambda_2}{2}\neq\lambda_z \text{ as we wanted to prove.}$$

When the zenith angle exceeds several degrees, equations (1), (2), and (3) are used to compute the astronomic coordinates of the zenith point.

C. AZIMUTH DETERMINATION

It is assumed that the direction cosines of the zenith point are known from the reduction of the direct and reverse astronomic position exposures. It is further assumed that:

1. *The optical axis of the sighting telescope is adjusted to be normal to the horizontal axis of the camera goniometer.*
2. *The camera coordinates of the image of an object bisected by the sighting telescope center cross are known* (x_z', y_z').
3. *The zenith circle defines the angle of rotation of the sighting telescope about the horizontal axis.*

The first assumption is accomplished by adjusting the telescope reticule. The second is a constant easily accomplished by making an exposure on spectroscopic plates, of a sharply defined distant object bisected with the telescope reticule. The coordinates x_z', y_z' of the center cross are measured directly. The third is accomplished during instrument assembly.

The astronomic coordinates of z_1' and z_2', illustrated in Figure 21, are computed with position equations where z_1' and z_2' are the positions of the center cross corresponding to the two azimuth exposures either side of the zenith:

$$\left.\begin{aligned}
&\cos\alpha_{sa_1}\cos\alpha_{z_1}' + \cos\beta_{sa_1}\cos\beta_{z_1}' + \cos\gamma_{sa}\cos\gamma_{z_1}' = \cos\gamma_{a_1}'\\
&\cos\alpha_{sb_1}\cos\alpha_{z_1}' + \cos\beta_{sb_1}\cos\beta_{z_1}' + \cos\gamma_{sb}\cos\gamma_{z_1}' = \cos\gamma_{b_1}'\\
&\cos\alpha_{sc_1}\cos\alpha_{z_1}' + \cos\beta_{sc_1}\cos\beta_{z_1}' + \cos\gamma_{sc}\cos\gamma_{z_1}' = \cos\gamma_{c_1}'
\end{aligned}\right\} z_1'$$

$$\left.\begin{aligned}
&\cos\alpha_{sa}\cos\alpha_{z_2}' + \cos\beta_{sa_2}\cos\beta_{z_2}' + \cos\gamma_{sa_2}\cos\gamma_{z_2}' = \cos\gamma_{a_2}'\\
&\cos\alpha_{sb_2}\cos\alpha_{z_2}' + \cos\beta_{sb_2}\cos\beta_{z_2}' + \cos\gamma_{sb_2}\cos\gamma_{z_2}' = \cos\gamma_{b_2}'\\
&\cos\alpha_{sc_2}\cos\alpha_{z_2}' + \cos\beta_{sc_2}\cos\beta_{z_2}' + \cos\gamma_{sc_2}\cos\gamma_{z_2}' = \cos\gamma_{c_2}'
\end{aligned}\right\} z_2'$$

where the coefficients and unknowns have the same meaning as before and

$$\cos\gamma_{a_1}' = \frac{x_z'xa_1 + yz'ya_1 + f^2}{L_z'La_1},$$

$$\cos\gamma_{a_2}' = \frac{x\,'xa_2 + y_z'ya_2 + f^2}{L_z'La_2}$$

$$\cos\gamma_{b_1}' = \frac{x_z'xb_1 + y_z'yb_1 + f^2}{L_z'Lb_1},$$

$$\cos\gamma_{b_2}' = \frac{x_z'xb_2 + y_z'yb_2 + f^2}{L_z'Lb_2}$$

$$\cos\gamma_{c_1}' = \frac{x_z'xc_1 + y_z'yc_1 + f^2}{L_z'Lc},$$

$$\cos\gamma_{c_2}' = \frac{x_z'xc_2 + y_z'yc_2 + f^2}{L_z'Lc_2}$$

The angles subtended by γ_{zz_1}' and γ_{zz_2}' are required.

$$\cos\alpha_z\cos\alpha_{z_1}' + \cos\beta_z\cos\beta_{z_1}' + \cos\gamma_z\cos\gamma_{z_1}' = \cos\gamma_{zz_1}'$$

$$\cos\alpha_z\cos\alpha_{z_2}' + \cos\beta_z\cos\beta_{z_2}' + \cos\gamma_z\cos\gamma_{z_2}' = \cos\gamma_{zz_2}'.$$

Then the angles enclosed by spherical triangle $zz_1'\,z_2'$ are computed.

$$\cos zz_1'z_2' = \frac{\cos\gamma_{zz_2'} - \cos\gamma_{zz_1'}\cos\gamma_{z_1'z_2'}}{\sin\gamma_{zz_1'}\sin\gamma_{z_1'z_2'}}$$

$$\cos zz_2'z_2' = \frac{\cos\gamma_{zz_1'} - \cos\gamma_{zz2'}\cos\gamma_{z_1'z_2'}}{\sin\gamma_{zz_2'}\sin\gamma_{z_1'z_2'}}.$$

Angle $\gamma_{z_1'z_2'}'$ is the difference in the two zenith angles, $(185°-175°) = \gamma_{z_1'z_2'}'$, observed directly. The index error of the zenith circle at the zenith and the inclination of the goniometer zenith axis are computed with Napier's analogies.

Inclination of the zenith circle axis $\gamma x'$

$$\sin\gamma x' = \sin\gamma_{zz_1}'\sin zz_1'\,z_2' = \sin\gamma_{zz_2}'\sin zz_2'z_1'$$

and the index error

$$e_z = \frac{z_1'z' - z'z_2'}{2}$$

$$\sin z_1'z' = \cot zz_1'z_2'\tan\gamma_x'$$

$$\sin z'z_2' = \cot zz_2'z_1'\tan\gamma_x'$$

with the inclination $\gamma x'$ and the corrected zenith angle of the azimuth mark γ_{Az}' the resultant zenith angle of the zenith circle axis γ_z' is computed.

$$\gamma_{Az'} = \gamma_{Az_1'} + z_1'z' = \gamma_{Az_2'} - z'z_2'$$

$$\tan\gamma_z' = \frac{\tan\gamma x'}{\sin\gamma_{Az'}}$$

Then

$$\sin\Delta A_1 = \frac{\sin\gamma_{Az_1'}\sin\gamma_z'}{\sin\gamma_{zz_1'}}$$

$$\sin\Delta A_2 = \frac{\sin\gamma_{Az_2}'\sin\gamma_z'}{\sin\gamma_{zz_2'}}$$

$$\sin Az_1' = \frac{\sin\lambda_z\cos\alpha_{z_1'} - \cos\lambda_z\cos\beta_{z_1'}}{\sin\gamma_{zz_1'}}$$

$$\sin Az_2' = \frac{\sin\lambda_z\cos\alpha_{z_2'} - \cos\lambda_z\cos\beta_{z_2'}}{\sin\gamma_{zz_2'}}$$

$$Az = Az_1' - \Delta A_1 = Az_2' - \Delta A_2.$$

D. DETERMINATION OF GEODETIC COORDINATES

The primary purpose of the survey is to establish a system of geodetic control points that may be used as control for a map to be compiled from the intermediate detail recorded on the terrestrial exposures. The geodetic control points are determined in two stages: The first consists of computing the space coordinates of selected conjugate images referred to a datum plane tangent to, and oriented on the meridian of the astronomic station by space intersection computations; the second consists of converting the space coordinates to latitudes, longitudes and elevations referred to sea level datum. Reference is made to Figure 22. Assume the geodetic coordinates of any point, P, are to be established. The astronomic coordinates (ϕ_z, λ_z) of point A have been determined from camera and star data obtained from direct and reverse zenithal position exposures made at A. The azimuth (Az_1) of a base line AB or any other line has been

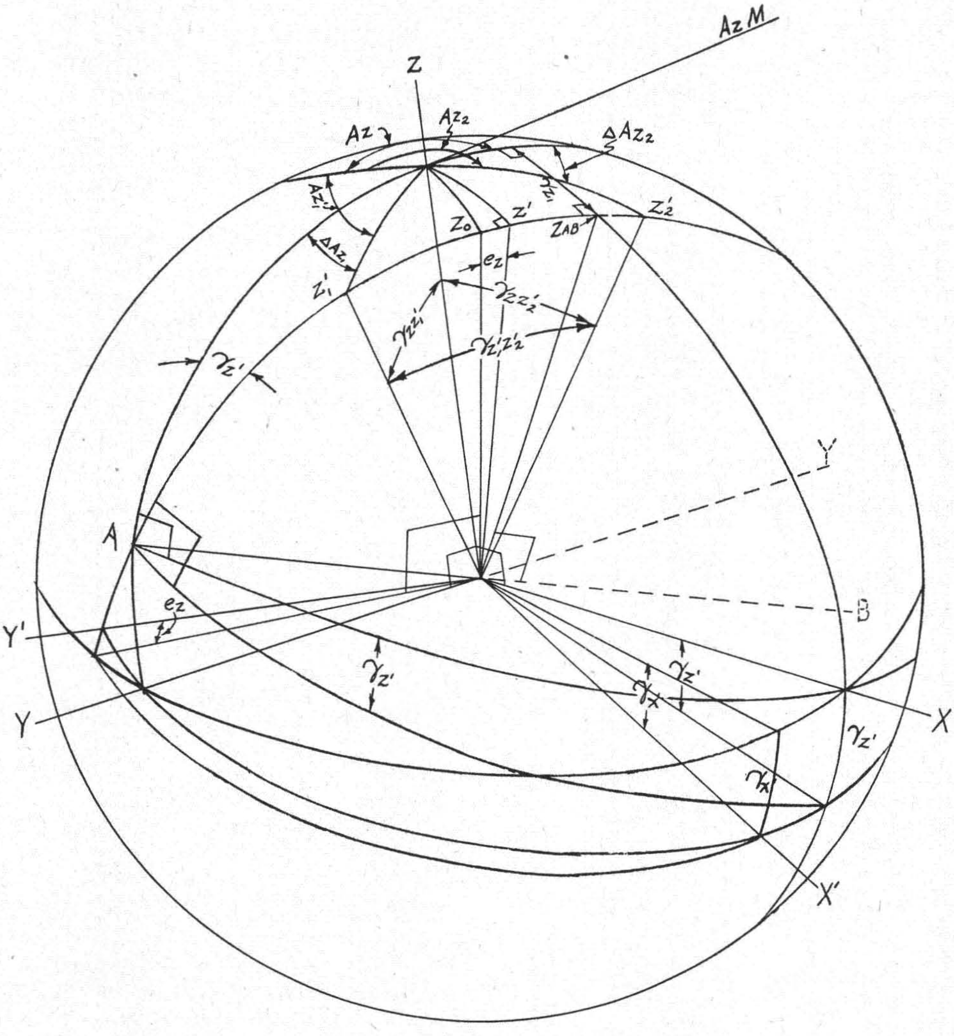

FIG. 21. Geometry of Azimuth Reduction.

determined from direct zenithal azimuth exposures also made at A. The azimuth of line BA is deduced from Az_1.

$$Az_2 = Az_1 \pm 180°.$$

Correction for convergence of the meridians is unnecessary since the data reductions are based momentarily on a local tangent plane. The azimuth circle was zeroed on the opposite terminal at each station of the base line during the terrestrial exposure. The azimuth circle angle (θ) of the camera principal plane referred to the base line, and the zenith angle (t), which corresponds to tilt, was read and recorded for each exposure. The base line AB has been measured by any means whatsoever. The elevation of A is measured with an altimeter or may be assumed to be zero. The conjugate images of point P are measured in x and y on the exposures made at A and B. These coordinates are corrected for film distortion and lens distortion. The corrected coordinates are combined with functions of tilt and swing to compute the horizontal angle η' referred to the principal plane and the vertical angle projected to the principal plane ξ'.

$$\tan \eta_1' = \frac{x \cos s - y \sin s}{(x \sin s + y \cos s) \cos t - f \sin t} = \frac{x'}{f'}$$

$$\tan \xi_1' = \frac{(x \sin s + y \cos s) \sin t + f \cos t}{(x \sin s + y \cos s) \cos t - f \sin t} = \frac{y'}{f'}.$$

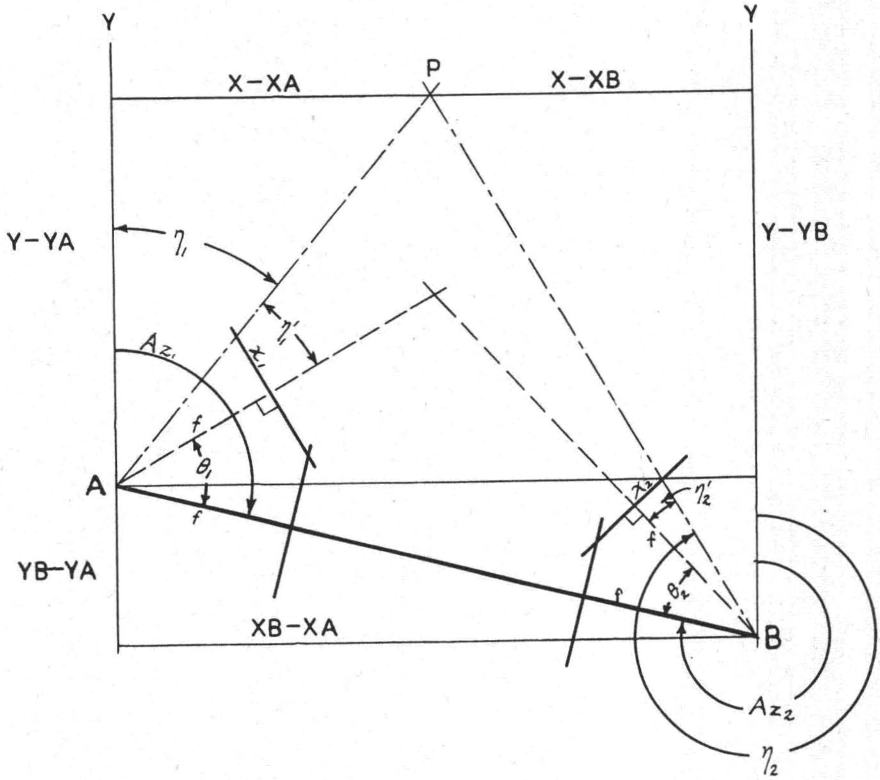

FIG. 22. Tangent Plane Coordinates of an Intersection Point.

These equations are illustrated in Figure 23.

Whenever possible, the exposure will be made with a tilt of 90° and the camera will, by adjustment, have a swing of zero.

When

$$s = 0 \qquad \sin s = 0 \qquad \cos s = 1$$
$$t = 90° \qquad \sin t = 1 \qquad \cos t = 0$$

in which case

$$\tan \eta' = \frac{x}{f}$$

$$\tan \xi' = \frac{y}{f}.$$

It may be seen from Figure 22 that

$$\eta_1 = Az_1 + \theta_1 + \eta_1'$$
$$\eta_2 = Az_2 + \theta_2 + \eta_2'$$

when clockwise angles are considered positive, and counter-clockwise angles are considered negative.

The principal plane projections of the vertical angles are projected to the meridian

$$\tan \xi = \frac{y'}{f' \cos (\eta - \eta')} = \frac{y'}{f' \cos (Az + \theta)}.$$

The space coordinates of any point P are obtained with the space coordinates of A and B and the conjugate tangent functions of η and ξ. The plane coordinates of A are assumed to be zero, and

$$XB = \sin Az_1 AB$$
$$YB = \cos Az_1 AB$$
$$ZB = \tan \xi_B' AB + ZA - dZ_B$$
$$= \tan \xi_A' AB + ZA - dZ_A.$$

ZA is the elevation of the camera at A above sea level datum, $\tan \xi_B'$ is obtained from the exposure at A zeroed on B and $\tan \xi_A'$ is the corresponding value obtained from the exposure at B zeroed on A. dZ is the subtractive correction due to curvature of a refracted ray. This is computed with $.00335\ M^2$ where M is the distance from the station to the point in thousand foot units. The z values are not corrected for earth curvature since datum is a local tangent plane. From the relation

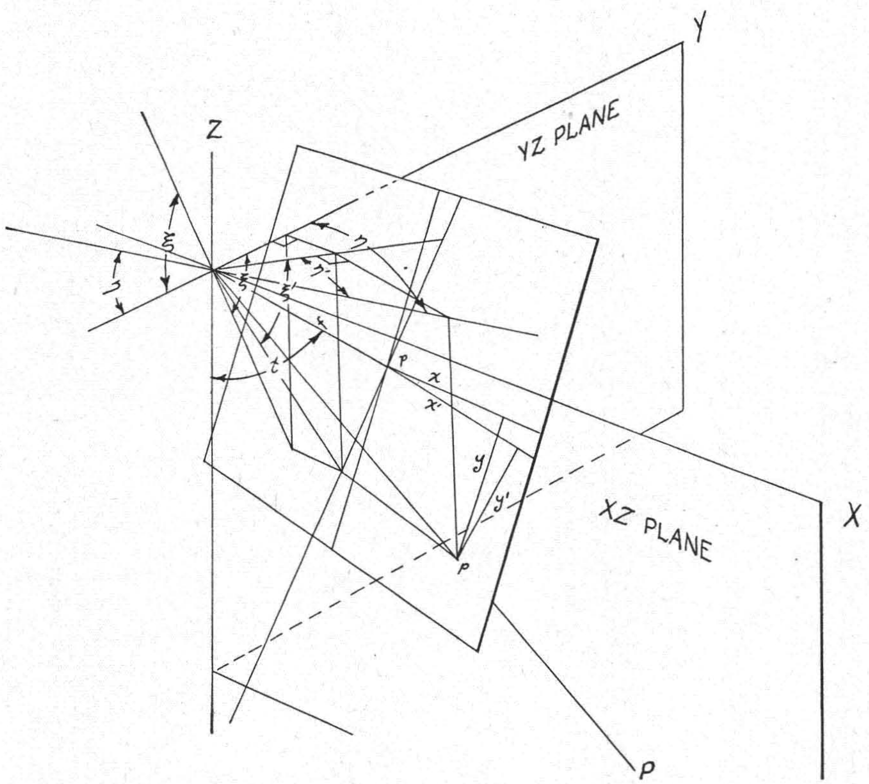

FIG. 23. Relation of η and ξ to Oblique Exposure.

$$\frac{X - XA}{Y - YA} = \tan \eta_1 \qquad \frac{X - XB}{Y - YB} = \tan \eta_2$$

$$\frac{Z - ZA}{Y - YA} = \tan \xi_1 \qquad \frac{Z - ZB}{Y - YB} = \tan \xi_2$$

when $XA = YA = 0$

$$Y = \frac{(XB - \tan \eta_2 YB)}{\tan \eta_1 - \tan \eta_2}$$

$$X = \frac{\tan \eta_1 (XB - \tan \eta_2 YB)}{\tan \eta_1 - \tan \eta_2} = Y \tan \eta_1$$

$$Z = \tan \xi_1 (Y) + ZA - dZ_1$$

$$= \tan \xi_2 (Y - YB) + ZB - dZ_2$$

These are the space coordinates of a point referred to a local tangent plane.

With space coordinates of any point P referred to A, and the astronomic coordinates of A, the astronomic or geodetic coordinates of any point P are determined.

$$Z_z' = \frac{a(1 - e^2) \sin \phi_z}{(1 - e^2 \sin^2 \phi_z)^{1/2}}$$

$$Y_z' = \frac{a \cos \phi_z}{(1 - e^2 \sin^2 \phi_z)^{1/2}}$$

where

$$e^2 = 1 - \frac{b^2}{a^2}$$

a = earth's semi-major axis = 6,378,388 meters

b = earth's semi-minor axis = 6,356,909 meters.

These are the space coordinates of A referred to the earth's equator and polar axis. Then the space coordinates of any point P referred to the earth's equator and the polar axis are

$$Y' = Z \cos \phi_z + Y \sin \phi_z + Y_z'$$

$$Z' = Z \sin \phi_z + Y \cos \phi_z + Z_z'.$$

The latitude, longitude and elevation E of any point P follows:

$$\tan (\lambda_z - \lambda) = \frac{X}{Y'}$$

$$\lambda = \lambda_z - (\lambda_z - \lambda)$$

$$\tan \phi = \frac{Z'}{Y'} \frac{\cos (\lambda_z - \lambda)}{(1 - e^2)}$$

$$E = Z' \csc \phi - \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{1/2}}.$$

The coordinates in this form may be converted to any chosen map projection.