geological study of aerial photographs, but who wish a rapid, yet simple, way to rectify slope and dip estimation.

In conclusion, it is well to repeat that the illustrations accompanying this paper are intended to depict the principles, and not actual conditions of photography and stereoscopy; also, the magnitude of variation in vertical exaggeration and distortion produced by each variable discussed may or may not be significant. Each variable must be subjected to an intensive quantitative test before its relative importance can be known.

REFERENCES CITED

Miller, Victor C., "Rapid Dip Estimation in Photo-Geological Reconnaissance." *Bulletin Amer. Assoc. Pet. Geol.*, Vol. 34, No. 8 (August, 1950).

Salzman, M. H., "Note on Stereoscopy." PHOTOGRAMMETRIC ENGINEERING, Vol. XVI, No. 3 (1950).

AN EQUATION FOR APPROXIMATING THE VERTICAL EXAGGERATION RATIO OF A STEREOSCOPIC VIEW

E. R. Goodale, Staff Photogrammetrist, Creole Petroleum Corporation, Caracas, Venezuela

AT THE VII International Congress of Photogrammetry, attention was called to the interesting subject of the stereoscopic estimation of dip-slope angles by the photogeologist. One of the important factors involved in estimating dip angles is the amount any given stereoscope exaggerates the vertical scale over the horizontal.

This vertical exaggeration of the stereoscopic view need only be determined approximately. However, up to the present time, the author has not seen any mathematical expression of the vertical exaggeration ratio, although several articles on the subject have appeared recently in PHOTOGRAMMETRIC ENGINEERING.

The purpose of this paper is to offer such an equation and, at the same time, to substantiate it with sufficient proofs to at least invite its further discussion.

The geometry in this paper is limited to that concerned with truly vertical air photographs; also, for simplicity and convenience, ground relationships are represented in the positive counterpart of the lens—negative plane.

For use of the equation, the only information needed is the focal length of the air-camera lens. All other terms can be measured on a stereo-pair of contact prints. The eye base and viewing distance will correspond to the individual observer and the stereoscope used, and can easily be measured.

DEFINITIONS

At this point some of the terms to be used in the discussion will be defined. No claims are made for these definitions of terms other than to clarify their meaning within the text of this paper. Photogrammetric terms of general usage have the same definitions as given in chapter XIX of the MANUAL OF PHOTOGRAMMETRY.

1) *Vertical Exaggeration*

Exaggerated impression of depth over image size when viewed stereoscopically.

2) *Photo Base*

The distance measured between the center points of two photographs of a stereo-pair, when the images of all common ground points which lie at the datum elevation are in coincidence.

3) *Photographic Datum Plane*

Theoretical plane at the elevation of any selected photographic image point (in this paper a low point of the stereo-view), from which image displacements are measured, and to which the measurement of the photo-base is referred.

4) *Object Ray*

Ray which has passed through the air camera lens and by which the photographic impression of a ground object has been made.

5) *Image Ray*

Ray reflected from the photographic image of a ground object to the eye or to the stereoscope eyepiece.

6) *Stereoscope Focal Distance*

Distance from the image plane to the stereoscope eyepiece, measured along course of image rays.

7) *Viewing Distance* (pertaining to stereoscopic viewing)

a) Without lens instrument: distance from the image plane to the eye, measured along course of image rays.

b) With lens instrument: same as Stereoscope Focal Distance.

8) *Eye Axes*

Lines of sight along which observation is directed, which usually converge at the point on which the eyes are focused.

9) *Convergency Angle*

Angle defined at the eye by two positions of the eye axis when the eye lens rotates to focus on objects at different distances. That part of the angle of depth perception which corresponds to one eye.

10) *Image Displacement*

Parallax difference of any image point in the stereo-view relative to the low (datum) point, which can be measured on two properly aligned photographs of a stereo-pair. May also refer to the fractional part of the parallax difference corresponding to one photograph subtended by the convergency angle of one eye.

11) *Image Plane*

Plane facing the eye which contains the images for stereo-viewing. It can be different for each eye.

12) *Plane of Stereoscopic Fusion*

Plane parallel to the Image Plane, on which the fused images *appear* to lie when viewed stereoscopically.

13) *Print Separation Distance*

a) With naked eye or simple magnifier lens-type stereoscope: Variable distance between common image points which lie in the photographic datum plane, measured with the photographs in alignment for stereoviewing. Equivalent to the Absolute Parallax of such points.

b) With mirror-type stereoscope: Variable distance corresponding to the additional amount of print separation from that at which the geometric conditions for no print separation are satisfied.

Stereo-Viewing with no Print Separation

Stereoscopic viewing with no separation is an important factor in the derivation of the equation. It is employed as a base for the geometric constructions and therefore warrants being commented upon.

It is possible to see stereoscopically with no effective print separation. For example, with a mirror-type instrument, the photographs of a stereo-pair can be placed so that the rays of common image points enter the eyepieces at the same angles as though they were in coincidence on the table below. The mirrors and prisms make it possible to view thus without hindrance of overlapping prints.

Also, it is possible to view two photographs stereoscopically with or without an instrument, gradually reducing the separation until common image points are almost over one another. Of course, the field of view will be reduced by the upper print's sliding over and covering the detail of the lower photo, until only a narrow band can be seen stereoscopically. In fact, stereo-vision is lost only by the complete covering up of common detail on the bottom print.

Two common examples of stereoscopy without separation are the vectograph and anaglyph. Their registry is made by separating the images of common points distances corresponding to their displacements, or parallax differences, referred to a convenient datum plane. The common images of points lying on the datum plane are in coincidence.

Approach to the Problem

If possible, it would be helpful to depict or simulate the condition of stereo-viewing for the purpose of analyzing the physical and psychological elements involved, in order to obtain a vertical exaggeration formula derived by a factual reconstruction of the phenomenon.

Such an approach has already been furnished by C. A. J. von Frijtag Drabbe.[5] This is a detailed treatise on the physical and "psychical" elements involved in monocular and binocular vision which, in v. Frijtag Drabbe's opinion, contributed to a proper understanding of stereoscopic vision. His statements are substantiated by a series of ingenious and simple experiments, which to this writer opened the door to a new and clearer perspective of the subject of vertical exaggeration.

Of particular importance is v. Frijtag Drabbe's[5] attempt to disprove the long-accepted theory of Wheatstone that the plane of stereoscopic fusion is located at the point of convergence of the eye axes when viewing stereoscopically (see Figure 1). Also, by logical reasoning, the Wheatstone theory is further refuted by the fact that it is possible to view stereoscopically with the eye axes parallel, or even diverging. If the Wheatstone theory were true, this would not be possible, since there would be no point of convergence and hence, no point of fusion.

C. A. J. von Frijtag Drabbe's last two experiments, first with matches and later with colored pencils, simulate the "condition of stereo-viewing," and, in the opinion of the author, prove that the plane where one appears to observe the "stereoscopic, three-dimensional reconstruction" lies between the image (pencil) plane (*ab*) and the plane passing through *f* (Figure 1), but closer to *ab* than to *f*.[5]

In making the experiment described by v. Frijtag Drabbe, with the red and blue pencils, the author was able to observe three images formed by the pencils; two single images on the outside, and a fused (red and blue) image in the middle. The two single images on the outside correspond in color to their respective pencils and appear to lie at the same distance from the eyes at which the pencils are held. The fused image, in the center, appears to lie in a more remote plane; but, at the same time, maintains its size and focus, as though it were still on the image plane.

In the above experiment, as the matches or pencils are removed from the eyes along the same eye axes, the fused image appears to precede them toward the point of convergence *f*, but never quite succeeds in reaching *f*. As the matches are made to approach *f*, their separation becomes less until the planes begin to merge; and, in v. Frijtag Drabbe's words, "that last moment we can hardly observe."

Experimenting still further along the lines described by v. Frijtag Drabbe, and holding the red and blue pencils in the lines of vision focused on a remote point of convergence (point *f* of Figure 1) the pencils may be separated still further, at the same time concentrating on the fused image. (See Figure 2.)

As the pencils are separated, the fused image appears to recede still further from the plane of the single images (*ab*).

Inversely, as the pencils are moved closer together, the fused image appears to approach the pencil, or image, plane; until, with the pencils almost touching, the fusion is lost due to the tendency to merge on the part of the images and planes.

Therefore, it can be assumed that, if the pencils could be made to coincide, the stereoscopic fusion would coincide with the position of the pencils.

These experiments prove that the position of the plane of stereoscopic fusion is dependent upon three factors, to wit: eye base; viewing distance; and image separation.

It follows, then, that the apparent change of the plane of fusion is that which gives the impression of relief.

It can now be stated that the problem consists in the ability to measure the distance between the various planes of stereoscopic fusion resulting from many image displacements and from varying amounts of print separation.

Is Stereo-Viewing a Physical or Mental Process?

Offering his experiments as a basis for further investigations, v. Frijtag Drabbe does not attempt to locate the planes of stereoscopic fusion mathematically, nor does he categorically opine as to whether the stereoscopic fusion is physical or mental.

Ignoring the proverb that a fool steps in where wise men fear to tread, the author risks advancing the theory that stereoscopic fusion takes place in the brain. From experiments, it is apparent that the fusion does not take place in the field of observation, but results from the separate projection of two images through the eye lenses to the retina, thence transmitted to the nerve cells of the brain, which in turn translate the two images into the stereoscopic illusion.

The author believes, moreover, that the fused image registered in the brain depends upon the geometry of the image rays in the same manner as the photographic impression depends upon the geometry of

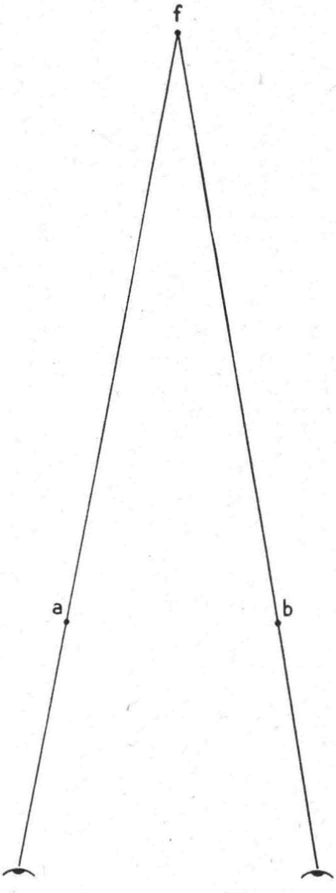

FIG. 1

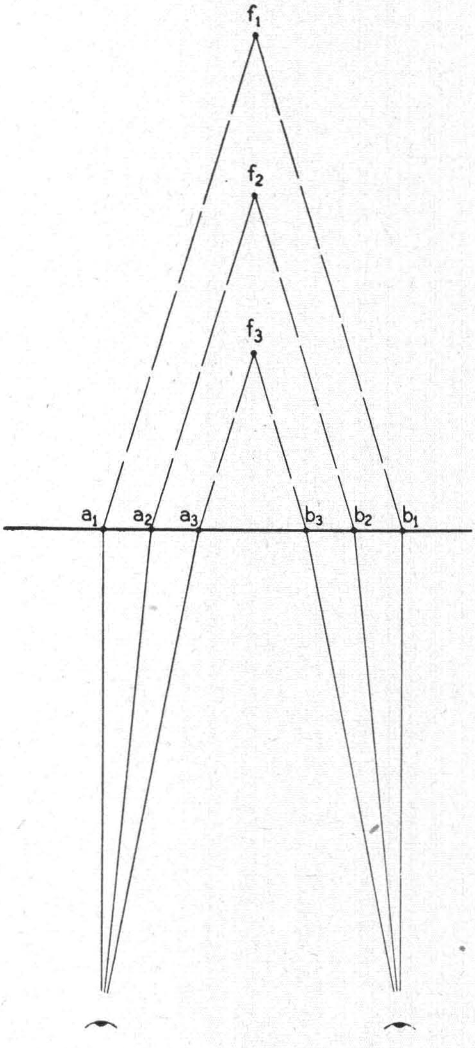

FIG. 2

the object rays entering the air camera, with which the photogrammetrist is already quite familiar.

EQUATION

The equation for determining the approximate vertical exaggeration ratio (E_v) is based upon the geometry of the object and image rays as presented in Figures 3, 4 and 5, and is stated as follows:

$$E_v = \frac{f_s(b_e + s)(b + d)}{f_a(b_e)(b_e + md)}. \qquad (1)$$

When:

- f_s = viewing distance, or stereoscope focal distance.
- f_a = focal length of air camera lens.
- b_e = eye base.
- s = print separation.
- b = photo-base.
- d = image displacement.
- md = image displacement multiplied by the magnifying power of the stereoscope.

When the print separation is equal to the eye base, i.e., when $s = b_e$, equation (1) can be simplified to:

$$E_v = \frac{2f_s(b + d)}{f_a(b_e + md)} \cdot \qquad (2)$$

EXPLANATION OF FIGURE 3

Figure 3 is a vertical section of a stereoscopic model, showing the spatial relationships of object and image rays which form the three-dimensional view, when the images of the low (datum) point are in coincidence.

It will be noted that the image displacements are purposely distributed unequally between the two photographs and that the stereoscope eyepieces are centered over the photographic images of the low (datum) point.*

From two positions of the air-camera lens, established by the coincidence of common images of the low (datum) point, object rays are projected to points u_r, c and u_l of the photographic image. Then, due to the magnifying power of the stereoscope, the displaced images u_r and u_l are enlarged

* By experimentation, it will be found that changing the position of the stereoscope eyepieces along a line parallel to the image plane does not make any appreciable change in the height of the intersections which determine the planes of fusion. It is noted, however, that the optimum viewing efficiency, with least distortion, is obtained when the stereoscope eyepieces are directly over their corresponding images, i.e., when the eye axes are approximately parallel, or when the print separation is equal to the eye base.

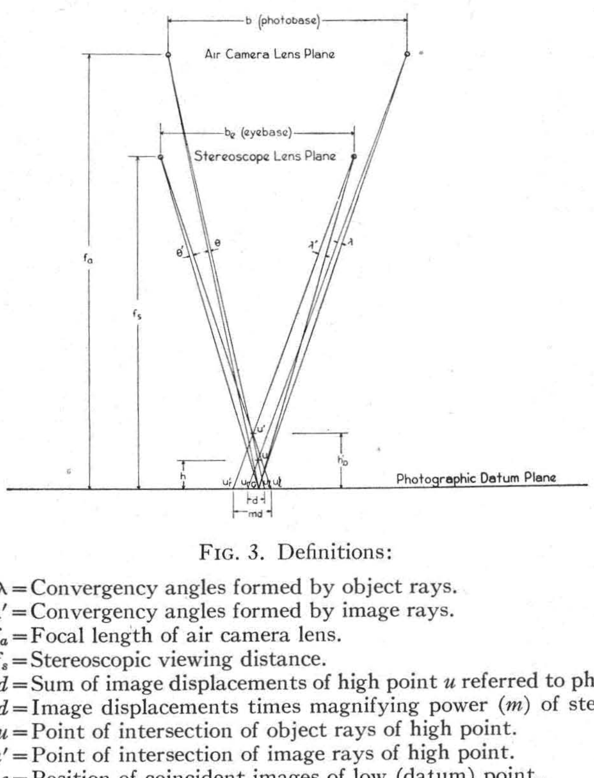

FIG. 3. Definitions:

Angles θ and λ = Convergency angles formed by object rays.
Angles θ' and λ' = Convergency angles formed by image rays.
f_a = Focal length of air camera lens.
f_s = Stereoscopic viewing distance.
d = Sum of image displacements of high point u referred to photographic datum.
md = Image displacements times magnifying power (m) of stereoscope.
u = Point of intersection of object rays of high point.
u' = Point of intersection of image rays of high point.
c = Position of coincident images of low (datum) point.
u_r = Position of image of point u on right-hand photograph by displacement from datum point c.
u_l = Position of image of point u on left-hand photograph by displacement from datum point c.
u_r' and u_l' = Apparent positions of u_r and u_l when viewed through magnifying (stereoscope) lenses.
h = True height of high point u at photographic datum scale.
h_o' = Stereoscopic height of high point u with no separation.

to appear at u_r' and u_l by which displacements d are enlarged to md. Thence, the image rays are projected to the stereoscope eyepieces, by which angles θ and λ, formed by the object rays, become angles θ' and λ', formed by the image rays converging to the stereoscope eyepieces. These are the convergency angles of definition No. 9.

The intersection of the object rays at u indicates the true height (h), over the photographic datum, of the point at elevation h.

The intersection of the image rays at u' indicates the apparent stereoscopic height (h_o'), over the photographic datum, of the point at elevation h with no print separation.

The standard formula for computing the approximate magnifying power of a simple lens stereoscope, [1,3,4] is repeated here:

$$m = \frac{10}{\text{focal length in inches}}; \quad \text{or,}$$
$$= \frac{250}{\text{focal length in mms.}}. \tag{3}$$

From the geometry of the object rays in figure 3, we have

$$h = \frac{f_a \cdot d\dagger}{b + d} \tag{4}$$

and

$$h_o' = \frac{f_s \cdot md\dagger}{b_e + md}. \tag{5}$$

EXPLANATION OF FIGURE 4

In Figure 4, there has been dropped the illustration of the object rays. Attention is confined to the projection of the image rays first repeating those of Figure 3 as a starting point (Figure 4-a); and later, attempting to develop their counterparts below the photographic datum plane with separation s equal to b_e (Figure 4-b).

The separation of the photographs for convenient stereo-viewing is accomplished by rotating the convergency angles, together with the image planes, around the stereoscope eyepieces and through the angles α and β respectively, until the images of the low (datum) point lie directly beneath the stereoscope eyepieces.

† Equations (4) and (5) can be more easily recognized from the simplified geometry of Figure 5.

Angles θ and λ have been rotated to lie at θ' and λ'.

Likewise, the image planes (corresponding to each eye separately), rotated from ab, assume new positions at ax and by.

Line kc has been constructed perpendicular to the image plane ab at point c, with no separation.

Lines ca' and cb' represent the continuations of kc, perpendicular to image planes ax and by.

The new intersections, which ca' and cb' produce with the rotated image rays, provide the following similarities in the geometry with and without separation:‡

1) Angle γ plus angle δ = angle ϕ.
2) Angle ι plus angle K = angle ω.§
3) a' and b' are the counterparts of c.
4) Plane $a'b'$ is the counterpart of plane ab.
5) Points u_s' and u_s'' are the counterparts of point u'.¶
6) h_s' is the counterpart of h_o'.
7) The plane passing through u_s' and u_s'' is the counterpart of the plane passing through u'.

It can be seen from Figure 4-b that the separating of the prints has the effect of creating a new reference plane for relief measurement ($a'b'$). This plane passes through the new intersections of the projected image rays of the low (datum) point with the construction lines ca' and cb'. From this plane, the stereoscopic relief, with separation s, is measured vertically to the plane passing through the new intersections of the projected image rays of the high point, and is indicated in the figure as h_s'.

In Figure 4-b, this new reference plane is called the "stereoscopic datum plane with separation s." It is removed from the photographic datum plane (ab) by the distance n_s, the value of which is found geometrically by locating plane $a'b'$, as explained in the previous paragraph.

‡ These similarities are based on the assumption that $\theta = \theta'$ and $\lambda = \lambda'$, equalities which are only approximate, due to the rotation movements.

§ Angles ϕ and ω will be recognized as "angles of convergence," defined on page 524 of the "MANUAL OF PHOTOGRAMMETRY."[1] As a matter of interest, $\phi - \omega = \theta + \lambda$.

¶ There are four possible constructions for locating points u_s' and u_s'', only one of which is shown.

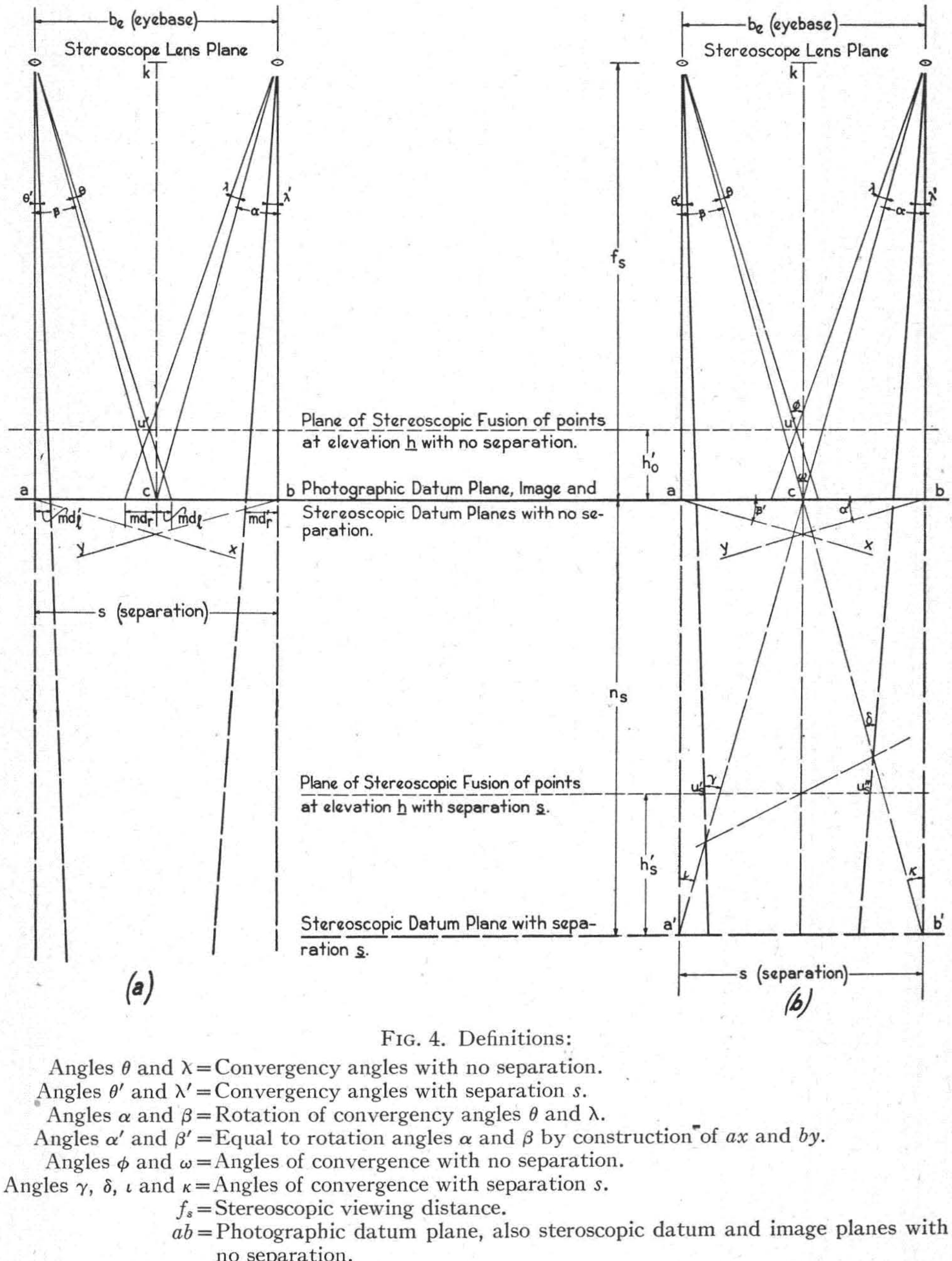

FIG. 4. Definitions:

Angles θ and λ = Convergency angles with no separation.
Angles θ' and λ' = Convergency angles with separation s.
Angles α and β = Rotation of convergency angles θ and λ.
Angles α' and β' = Equal to rotation angles α and β by construction of ax and by.
Angles ϕ and ω = Angles of convergence with no separation.
Angles γ, δ, ι and κ = Angles of convergence with separation s.
f_s = Stereoscopic viewing distance.
ab = Photographic datum plane, also steroscopic datum and image planes with no separation.
md_l and md_l' = Image displacements on left photograph, enlarged by magnifying power (m) of stereoscope.
md_r and md_r' = Image displacement on right photograph, enlarged by m.
kc = Construction line perpendicular to position of image plane with no print separation.
h_o' = Apparent stereoscopic height with no separation.
ax and by = New positions of image planes, rotated together with angles θ and λ.
ca' and cb' = Continuation of kc perpendicular to rotated image planes ax and by.
n_s = Distance between photographic and stereoscopic datum planes with separation s.
h_s' = Apparent stereoscopic height of point at elevation h with separation s.

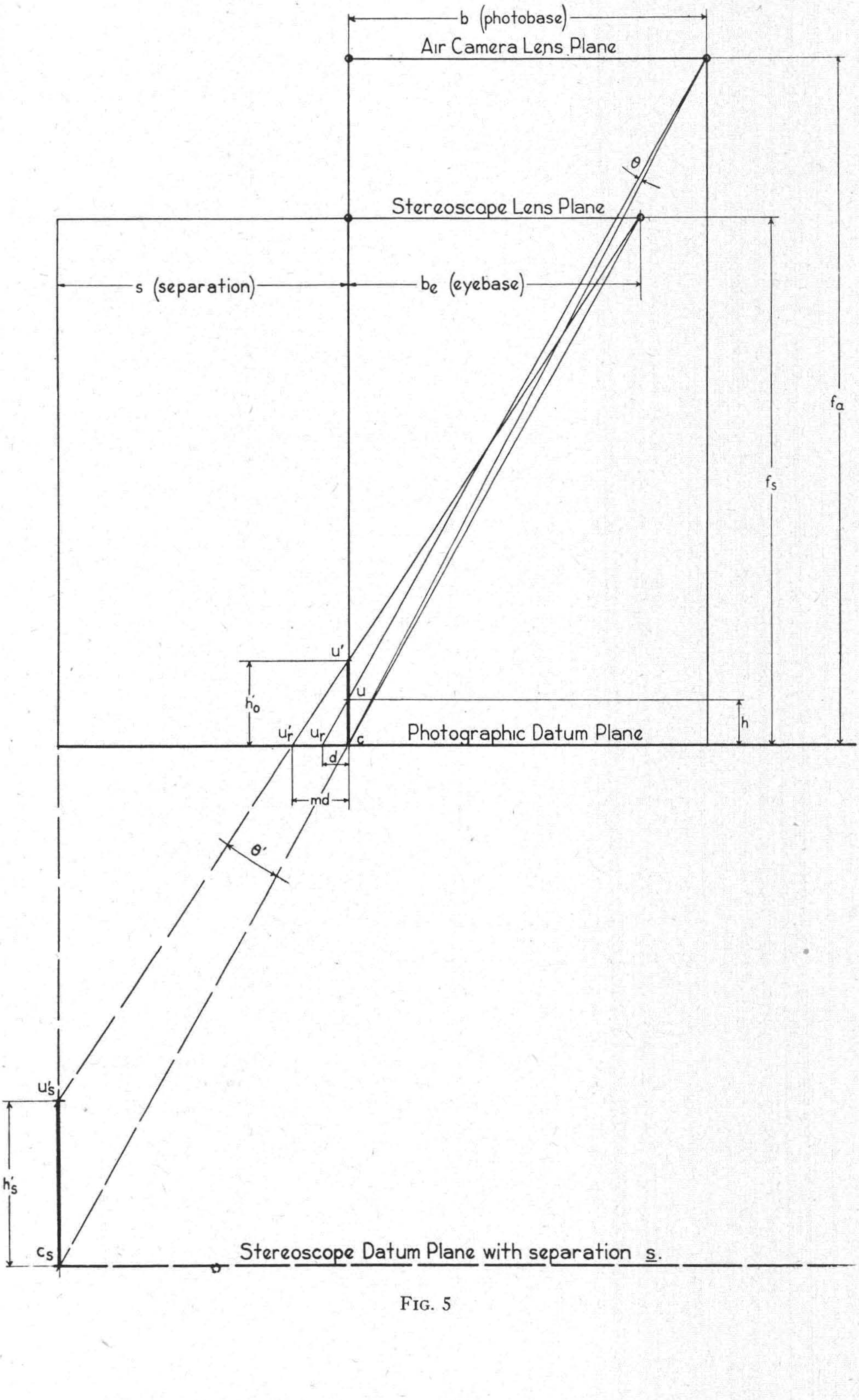
b (photobase)
Air Camera Lens Plane
θ
Stereoscope Lens Plane
s (separation)
b_e (eyebase)
f_a
f_s
u'
u
h'_o
u'_r
u_r
c
Photographic Datum Plane
h
d
md
θ'
u'_s
h'_s
c_s
Stereoscope Datum Plane with separation s.

FIG. 5

From the foregoing, two conclusions are evident:

1) The greater the print separation, the farther removed from the photographic datum plane will the new "stereoscopic datum plane" lie.

2) The more the prints are separated, the greater will be the stereoscopic relief (h_s').

n_s can also be derived algebraically from the previously mentioned experiments, by which n_s is seen to be proportional to f_s; to s; and to $1/b_e$. It follows, then, that n_s is proportional to $(f_s \cdot s)/b_e$; or,

$$n_s = K \frac{f_s \cdot s}{b_e} \tag{6}$$

when K is the proportionality constant.

The value of K may well be determined by further experimentation, by finding a direct way of measuring n_s. However, in using equation (6), is is assumed that K has a value of one (1), inasmuch as this agrees with the geometry of figure 4-b.

Should the value of K be proven experimentally to be unity, there would be a strong indication that the theory of rotating the convergency angles and image planes to allow for the print separation, as assumed in Figure 4, is correct. (See second paragraph under "Explanation of Figure 4.")

Assuming a value of one for K, equation (6) becomes

$$n_s = \frac{f_s \cdot s}{b_e}. \tag{7}$$

Returning to the geometry of Figure 4, it is apparent that

$$\frac{h_o'}{h_s'} = \frac{f_s}{f_s + n_s}\text{; therefore, } h_s' = \frac{h_o'(f_s + n_s)}{f_s}. \tag{8}$$

Substituting for h_o' from equation (5), and for n_s from equation (7), we have

$$h_s' = \frac{f_s \cdot md(b_e + s)}{b_e(b_e + md)}. \tag{9}$$

The value of h_s', which has just been determined by equation (9) is the key to finding the vertical exaggeration. The ratio of the apparent stereoscopic height (h_s') to the true height (h), divided by the magnifying power of the stereoscope (m), gives the amount of vertical exaggeration (E_v), and is expressed algebraically as follows:

$$E_v = \frac{h_s'}{mh}. \tag{10}$$

Restating equation (10) by substituting equations (4) and (9) for h and h_s', we have

$$E_v = \frac{f_s(b_e + s)(b + d)}{f_a(b_e)(b_e + md)},$$ which is equation (1) (q.e.d.).

Explanation of Figure 5

Figure 5 is also a vertical section of a stereoscopic model and expresses the geometry of Figures 3 and 4 in simplified form. Equations (1) and (2) can also be derived from this figure. The definitions of the terms will easily be recognized from those of Figures 3 and 4.

Conclusion

The vertical exaggeration equation, as presented here, has been checked visually by several observers of unequal experience in stereo-viewing, using several types of stereoscopes, with varying amounts of print separation and displacements.

An interesting example of possible ways of using the equation was called to the author's attention by Mr. W. R. Drake. Using the author's data, he plotted a graph in terms of the calculated vertical exaggeration related to the stereoscopic viewing distance. For the same air-camera lens, photo-base, eye base, displacement and print separation, the vertical exaggerations for the following stereoscopes were plotted: Zeiss pocket, Ryker folding, Abrams-model CF-8, Abrams-model CB-1, Fairchild Stereo-comparagraph and Fairchild F-71. The plotted positions fell in a straight line on the graph.

Plotting a number of such graphs should make possible selecting the proper stereoscope and establishing the adequate print separation to give the desired amount of vertical exaggeration.

Some of the conditions which affect the vertical exaggeration, and which are apparent from equation (1), are:

1) The longer the focal length of the air-camera lens (f_a), the less the E_v.
2) The longer the photo-base (b), the greater the E_v.
3) The longer the viewing distance (f_s), the greater the E_v.
4) The wider the eye base (b_e), the less the E_v.
5) The greater the magnifying power of

the stereoscope lenses (m), the less the E_v.

6) The greater the image displacements (d), the less the E_v.
7) The more print separations (s), the greater the E_v.

BIBLIOGRAPHY

1. MANUAL OF PHOTOGRAMMETRY, second edition, 1952, American Society of Photogrammetry. Chapters II, VI and XI.
2. Elliott, D. H., 1952, "Photogeologic Interpretation Using Photogrammetric Dip Calculations—Special Report No. 15," State of California, Department of Natural Resources, Division of Mines.
3. Hardy, A. C. and Perrin, F. H., 1932, "Principles of Optics," McGraw-Hill, Chapters XXII and XXV.
4. Talley, B. B., 1938, "Engineering Applications of Aerial and Terrestrial Photogrammetry," Pittman Publishing Company, Chapter IX.
5. Frijtag Drabbe, C. A. J. von, 1951–1952, "Some New Aspects in Stereoscopic Vision," *Photogrammetria*, Volume VIII-4, Special Congress Number, pages 168–179.

ACKNOWLEDGMENTS

For the help and encouragement given the author expresses thanks to the following members of the Creole Staff: Mr. W. R. Drake, who acted as mathematical consultant; Drs. George R. Heyl and G. D. Johnson, and Messrs. V. M. W. Petzall and P. Jacobsen for their help in improving the text.