SOME THEORETICAL AND PRACTICAL PROBLEMS IN PHOTOGRAMMETRIC BRIDGING*

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THE problem of extending vertical and horizontal control over inaccessible areas by means of photogrammetry has always been of great interest to many photogrammetrists and pioneers in this relatively young method of surveying. With the development of the first precise stereoplotting instruments, like the Zeiss Stereoplanigraph and the Wild Autograph, the experimental and theoretical work has been accelerated. During these experiments different errors have been encountered which influence the accuracy of bridging and limit its practical applications. The discovery of the causes of these errors and the derivation of the mathematical functions which the different errors follow, is still an actual but indeed not an easy task for the photogrammetrist. The development of very precise instruments, aerial lenses, and plate cameras during the last few years has made possible recognizing some of the causes of errors and discovering the laws which the errors follow in a bridging operation.

As in geodesy, there are in photogrammetry two groups of errors, systematic and accidental. The systematic errors falsify the observations of the same group in the same sense, and they cannot be detected from the internal agreement of the observations. These errors can only be detected in an experimental or theoretical way. The accidental errors are those which follow the well known law of Gauss.

The system of accumulation of errors which finally result in the deformation of the strip is complicated. A very systematic and thorough analysis of all the bridging operations is essential to an understanding and explanation of the deformations of a strip.

The publications of Gotthardt Bachmann, and Roelofs show that all of the strip deformations can be explained as the effect of purely accidental and systematic errors. Any introduction of new definitions like pseudo-accidental and pseudo-systematic errors is not necessary and only complicates the problem.

The bridging methods can be divided into two groups. In the first, which can be called the classical method, none of the absolute orientation elements of the aerial camera is known. The bridging depends completely upon the absolute orientation of the first model, the accuracy of the relative orientation throughout the strip, transferring of scale from model to model within the strip and the existing control. In the second group of bridging methods, at least one of the elements of absolute orientation is known and has been determined independently during the photo flight. From the viewpoint of theory of errors the main difference between these two groups is the effect of the convergence error.

In the classical method the convergence error and the earth curvature result in a bend of the strip in the XZ plane. As a result there is a very large error in elevation at the end of the strip and the last models are tipped an appreciable amount. This limits the length of the strip to a maximum of 30 to 35 models. For the bridging of long strips in areas with limited control, it is advantageous to record some of the elements of absolute orientation and to use this in bridging.

To explain and to understand the different summation of errors in aerotriangulation it is necessary to discuss some of the basic equations of the theory of errors of aerotriangulation (Figure 1).

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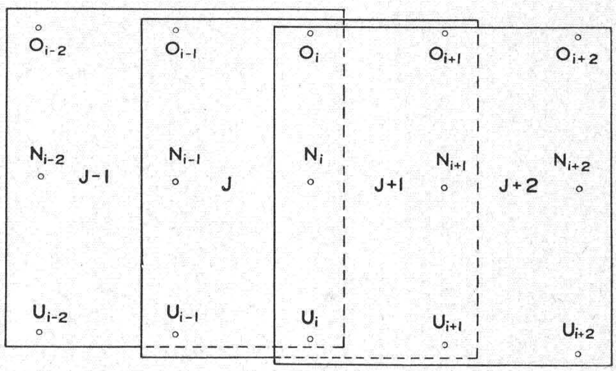

FIG. 1

BASIC EQUATIONS OF THE THEORY OF ERRORS OF AEROTRIANGULATION

In the following a short analysis* of the errors ΔX, ΔY, ΔH of the co-ordinates X, Y and H in the axis of the strip will be given. It is assumed that the models are relatively flat and the forward overlap is 60 per cent. Systematic errors only are considered. Figure 2 shows the influence of the convergence error $d\gamma$.

The first model will have a tip error of $d\Phi+\Delta\phi$ when $\Delta\phi$ is the error in tip because of errors in absolute orientation and $d\phi$ is instrumental error. The

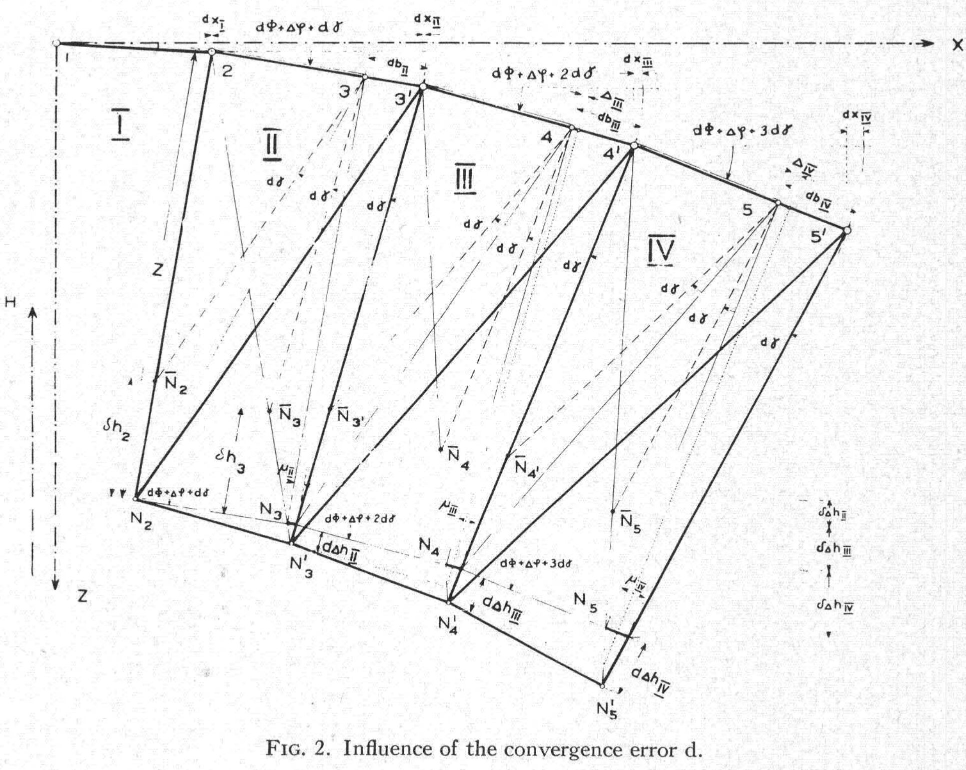

FIG. 2. Influence of the convergence error d.

* A detailed theory of errors of aerotriangulation is given in author's publication "Theory of Errors of Spatial Aero-triangulation" Zuerich—1952.

camera 2 will have a convergence error $d\gamma$ as a result of limited accuracy in relative orientation of model I. In model II there are the following errors. First the base b and therefore the entire model II has a tip error of $d\Phi+\Delta\phi+d\gamma$. As a result of the convergence error $d\gamma$ of camera 3, the nadir point N_2 will move to the position $\overline{N}_2$ and the nadir point N_3 to position $\overline{N}_3$. In the practical bridging the base is being changed in length until point $\overline{N}_2$ coincides with point N_2; i.e., so that it will have the same elevation as read in model I. This operation results in enlargement db_{II} of the base b, but on the other hand compensates mainly the influence of the convergence error on the distance between nadir points N_2 and N_3. The error in distance between nadir points N_2 and N_3 is finally only μ. This operation results also in an elevation error $d\Delta h_{II}$ of the nadir point N_3. Because of the error $d\Phi+\Delta\phi+d\gamma$ in the tip of the base b_{II}, the nadir point N_3 will also show an error $\delta\Delta h_{II}$ in elevation, and an error dx_{II} in distance.

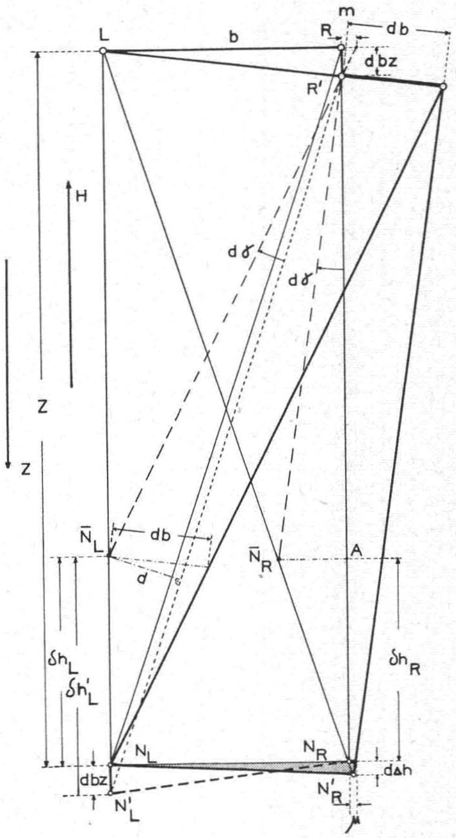

FIG. 3

Figure 3 shows the influence of the error dbz of the bz component of the base and the convergence error $d\gamma$. From this figure we get:

$$\delta h_L = \frac{b^2 + Z^2}{b}\cdot d\gamma - dbz \quad (1)$$

$$\delta h_R = \frac{Z^2}{b}\cdot d\gamma \quad (2)$$

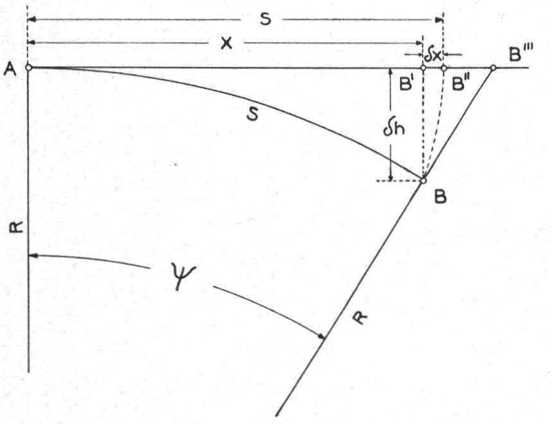

FIG. 4

From the above introductory remarks it can be seen that the error ΔH in elevation is built up from the following differential errors:

$d\Delta h$ results from the convergence error $d\gamma$, the dbz error and the operation of transferring of scale from Model J to Model $J+1$.

$\delta\Delta h$ results from the tip error of the base (Figure 4).

The influence of the earth curvature is: $\delta h = -(X^2/2R)$.

From Figures 2 and 3:

$$\begin{aligned} d\Delta h_{II} &= dbz - bd\gamma \\ d\Delta h_{III} &= 2(dbz - bd\gamma) \\ d\Delta h_{IV} &= 3(dbz - bd\gamma) \end{aligned} \tag{3}$$

Finally the error in model N is

$$d\Delta h_N = (N-1)(dbz - bd\gamma) \tag{4}$$

Also:

single summation

double summation

$$\begin{aligned} \delta\Delta h_I &= -\, b(d\Phi + \Delta\phi) \\ \delta\Delta h_{II} &= -\, b(d\Phi + \Delta\phi + d\gamma) \\ \delta\Delta h_{III} &= -\, b(d\Phi + \Delta\phi + 2d\gamma) \\ \delta\Delta h_{IV} &= -\, b(d\Phi + \Delta\phi + 3d\gamma) \\ &\;\vdots \\ \delta\Delta h_N &= -\, b[d\Phi + \Delta\phi + (N-1)d\gamma] \end{aligned}$$

$$\sum_{i=I}^{i=N} \delta\Delta h_i = \Delta h_N = -\, b\left[N(d\Phi + \Delta\phi) + N(N-1)\frac{d\gamma}{2}\right] \tag{6}$$

The total error ΔH will be:

$$\Delta H_N = d\Delta h_N + \sum_{i=I}^{i=N} \delta\Delta h_i + \frac{X_N^2}{2R} \tag{7}$$

Introducing $N\ X/b$ there is obtained:

$$\begin{aligned} \Delta H_i = b\cdot d\gamma - dbz + X_i\left[\frac{dbz}{b} - d\gamma - \Delta\phi + \frac{d\gamma}{2} - d\Phi\right] \\ + X_i^2\left[-\frac{d\gamma}{2b} - \frac{1}{2R}\right] \end{aligned} \tag{8}$$

or in general form:

$$\Delta H_i = C_0 + C_1X_i + C_2X_i^2 \tag{9}$$

THE ERROR ΔX

This error of the X co-ordinates of the nadir points results from the following differential errors.

μ results from the influence of convergence errors $d\gamma$ the dbz error and the operation of transferring of scale from Model J to model $J+1$.

$dx_{d\Delta\kappa}$ results from the swing error $d\Delta\kappa$.

dx results from the tip error of the base because of convergence error $d\gamma$ and errors of Φ and $\Delta\phi$.

$\delta X = X^2/6R^2$ is the influence of the earth's curvature.

From Figures 2 and 3 there is obtained:

$$\begin{aligned} \mu_{II} &= db_{II} - Zd\gamma \\ \mu_{III} &= db_{III} - Zd\gamma \\ &\vdots \\ \mu_N &= db_N - Zd\gamma \end{aligned} \tag{10}$$

$$\sum_{J=II}^{J=N} \mu_J = M_N = \sum_{J=II}^{J=N} db_J - (N-1)Z\cdot d\gamma$$

and:

single summation

double summation

$$\begin{aligned} db_{II} &= \delta h_L \frac{b}{Z} && = Zd\gamma + \frac{b^2 d\gamma}{Z} - dbz\frac{b}{Z} \\ db_{III} &= \left[\delta h_L + (bd\gamma - dbz)\right]\frac{b}{Z} && = Zd\gamma + 2\left(\frac{b^2 d\gamma}{Z} - dbz\frac{b}{Z}\right) \\ db_{IV} &= \left[\delta h_L + 2(bd\gamma - dbz)\right]\frac{b}{Z} && = Zd\gamma + 3\left(\frac{b^2 d\gamma}{Z} - dbz\frac{b}{Z}\right) \\ &\vdots && \vdots \\ db_N &= \left[\delta h_L + (N-2)(bd\gamma - dbz)\right]\frac{b}{Z} && = Zd\gamma + (N-1)\left(\frac{b^2 d\gamma}{Z} - dbz\frac{b}{Z}\right) \end{aligned} \tag{11}$$

$$\sum_{J=II}^{J=N} db_J = (N-1)Zd\gamma + (N-1)N\left(\frac{b^2 d\gamma}{2Z} - \frac{bdbz}{2Z}\right)$$

or

$$\sum_{J=II}^{J=N} \mu_J = M_N = N(N-1)\left(\frac{b^2 d\gamma}{2Z} - \frac{bdbz}{2Z}\right) \tag{12}$$

or general:

$$M_N = A' + B'X + C'X^2 \tag{13}$$

The above equations give this part of the error ΔX in X co-ordinates which is caused by the convergence error $d\gamma$, dbz error and by the operation of transferring of scale.

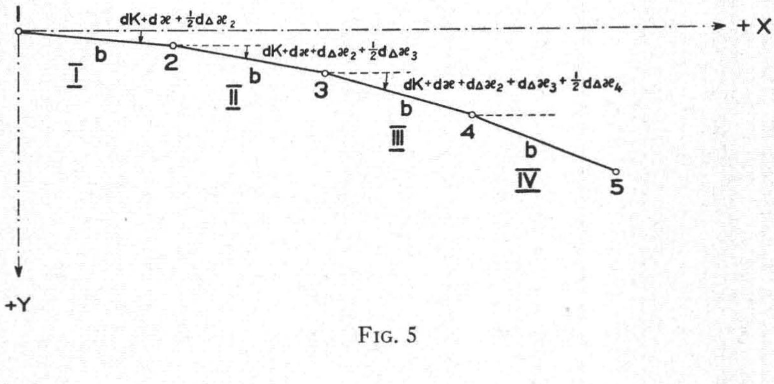

FIG. 5

The influence of the deformation of the strip in X-Z plane can be computed as follows:

single summation

double summation

$$\left.\begin{aligned} dx_{\mathrm{I}} &= \frac{b}{2}(\Delta\phi + d\Phi)^2 \\ dx_{\mathrm{II}} &= \frac{b}{2}(\Delta\phi + d\Phi + d\gamma)^2 \\ dx_{\mathrm{III}} &= \frac{b}{2}(\Delta\phi + d\Phi + 2d\gamma)^2 \\ &\vdots \\ dx_N &= \frac{b}{2}[\Delta\phi + d\Phi + (N-1)d\gamma]^2 \end{aligned}\right\} \tag{14}$$

$$\begin{aligned} \sum_{J=\mathrm{I}}^{J=N} dx_J = N\frac{b(\Delta\phi + d\Phi)^2}{2} + N(N-1)\frac{bd\gamma(\Delta\phi + d\Phi)}{2} \\ + [(N-1)(2N-1)N]\frac{db\gamma^2}{12} \end{aligned} \tag{15}$$

Similarly, the influence of the swing error can be computed (Figure 5):

$$\begin{aligned} \sum_{J=\mathrm{I}}^{J=N} dx_{d\Delta\chi,J} = N\frac{b(\Delta x + dK)^2}{2} + N(N-1)\frac{bd\Delta\chi(\Delta\chi + dK)}{2} \\ + [N\cdot(N-1)(2N-1)]\frac{bd\Delta\chi^2}{12} \end{aligned} \tag{16}$$

It can be easily seen that in this case there is also a double summation of errors.

The total error ΔX in model N will be

$$\Delta X_N = \sum_{J=\mathrm{II}}^{J=N} \mu_J + \sum_{J=\mathrm{I}}^{J=N} dx_J + \sum_{J=\mathrm{I}}^{J=N} dx_{j\,d\Delta\mathrm{K}} + \frac{X_N{}^3}{6R^2} \tag{17}$$

or

$$\begin{aligned} \Delta X_i = X_i\Big[-\frac{db\gamma}{2Z} + \frac{dbz}{2Z} - \frac{(\Delta\phi + d\Phi)^2}{2} \\ + \frac{d\gamma(\Delta\phi + d\Phi)}{2} - \frac{d\gamma^2}{12} + \frac{(\Delta\chi + dK)^2}{2} + \frac{d\Delta\chi(\Delta\chi + dK)}{2} - \frac{d\Delta\chi^2}{12}\Big] + \\ X_i^2\Big[\frac{d\gamma}{2Z} - \frac{dbz}{2bZ} - \frac{d\gamma(\Delta\phi + d\Phi)^2}{2b} + \frac{d\gamma^2}{4b} + \frac{d\Delta\chi(\Delta\chi + dK)}{2b} - \frac{d\Delta\chi^2}{4b}\Big] + \\ X_i^3\Big[-\frac{d\gamma^2}{6b} - \frac{d\Delta\chi^2}{6b^2} - \frac{1}{6R^2}\Big] \end{aligned} \tag{18}$$

and in general form:

$$\Delta X_i = A_0 + A_1X_i + A_2X_i^2 + A_3X_i^3 \tag{19}$$

THE ERROR ΔY

The following errors of orientation influence the error ΔY:

the general swing error dK
the swing error of the first model $d\kappa$
the differential swing error $d\Delta\kappa$
the differential tilt error $d\Delta\omega$
the error dby of the y component of the base (Figure 6)

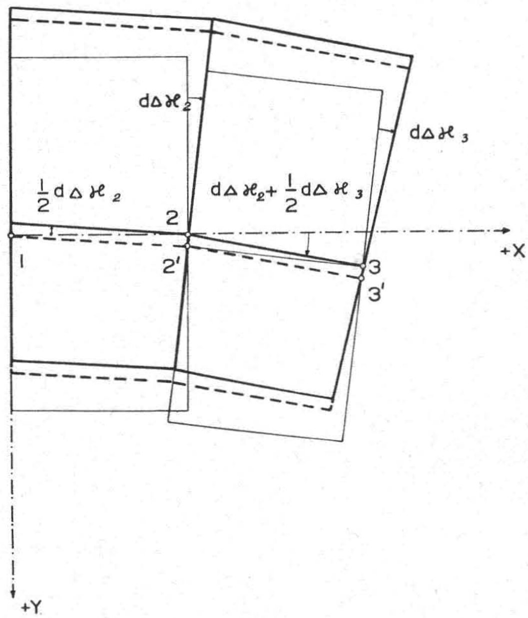

FIG. 6

The influence of the last two errors is practically eliminated when the coordinates of nadir point N in model $J+1$ are made to read the same as in model J.

From Figure 5 there is obtained:

single summation

double summation

$$\left\{\begin{aligned} d\Delta Y_{\mathrm{I}} &= b(\Delta\chi + dK) + \frac{b}{2}\, d\Delta\chi_2 \\ d\Delta Y_{\mathrm{II}} &= b(\Delta\chi + dK + d\Delta\chi_2) + \frac{b}{2}\, d\Delta\chi_3 \\ d\Delta Y_{\mathrm{III}} &= b(\Delta\chi + dK + d\Delta\chi_2 + d\Delta\chi_3) + \frac{b}{2} d\Delta\chi_4 \\ &\ \ \vdots \\ d\Delta Y_N &= b(\Delta\chi + dK + d\Delta\chi_2 + d\Delta\chi_3 + d\Delta\chi_4 + \cdots + d\Delta\chi_{n-1}) + \frac{b}{2}\, d\Delta\chi_n \end{aligned}\right. \tag{20}$$

Because practically:

$$d\Delta\chi_2 = d\Delta\chi_3 = d\Delta\chi_4 = \cdots d\Delta\chi_n = d\Delta\chi$$

and finally:

$$\sum_{J=I}^{J=N} d\Delta Y_J \quad \Delta Y_N = N\left[b(\Delta\chi + dK) + \frac{b}{2}\,d\Delta\chi\right] + N(N-1)\frac{bd\Delta\chi}{2} \tag{21}$$

and:

$$\Delta Y_i = D_0 + D_1X_i + D_2X_i^2 \tag{22}$$

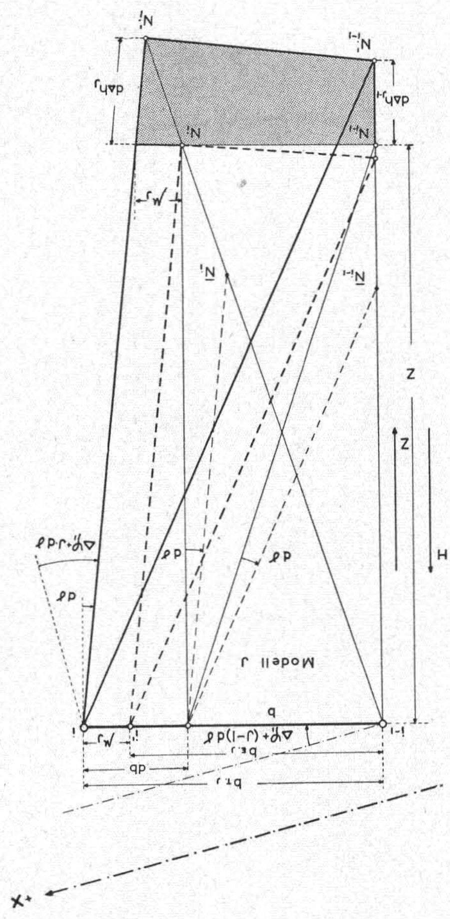

FIG. 7

When the differences between exposure stations are recorded with a statoscope, or the tip and tilt is recorded with a gyroscope, and those a priori established elements are used in bridging, the double summation does not apply to the errors in elevation.

THE RELATIVE SCALE ERROR *dm*

Figure 7 shows the influence of scale error. The light lines are showing the position of model J under the assumption that it has a correct base and is free from convergence error. The broken light lines are showing influence of the convergence error. The heavy lines represent the model after transferring of scale by using elevation of the nadir point N_{i-1}.

The heavy broken lines show the distorted model after the establishing of the correct scale by means of scaling to given control points.

The following definition of the relative scale error is introduced:

$$dm_J = \frac{\mu_J}{b_J} \tag{23}$$

From Figure 5

$$\mu_J = db_J - Zd\gamma \tag{24}$$

$$db_J = Zd\gamma + (J-1)\left(\frac{b^2d\gamma}{2Z} - \frac{bdbz}{2Z}\right) \tag{25}$$

and

$$\mu_i = (J - 1)\left(\frac{b^2 d\gamma - bdbz}{2Z}\right) \tag{24a}$$

finally

$$dm_J = \frac{\mu_J}{b_J} = \left(\frac{X}{b} - 1\right)\left(\frac{bd\gamma + dbz}{2Z}\right) \tag{26}$$

$$dm_J = B' + C'X_i \tag{27}$$

The influence of the relative scale error on the X co-ordinates can be derived as an integral of dm. Then:

$$\Delta X_{dm \cdot i} = \int_0^N dm = \int_0^{X_i} (B' + C'X_i)dX = A + BX_i + CX_i^2 \tag{28}$$

As was to be expected this equation and equation (12) which has been derived from the sum of the differential error have the same general form.

The above equations show that the relative scale error is independent from the error in elevation caused by the curvature of the strip in XZ plane which results from the convergence error and the earth's curvature.

Therefore the equation (27) applies also to this method of bridging when one of the orientation elements of the aerial camera in the moment of exposure is known.

THE EFFECT OF DOUBLE SUMMATION OF ACCIDENTAL ERROR

From the above derived equations it can be seen that in aerial triangulation there is a double summation of errors, and therefore the deformations of a strip are the result of the sums of errors and not of a single error. To show the different summations of errors in a strip during the bridging operation consideration for simplicity's sake has been restricted to systematic errors. But it is self evident that the single and double summation are also applicable to the accidental errors. Therefore the deformation of a strip is a result of single and double summations of systematic and accidental error. Unfortunately, the theory of errors in its present form cannot provide any information concerning this kind of double summation of accidental errors.

The effect of the double summation of accidental error can be shown on series of errors obtained, i.e., by direct observations of a priori known values. The differences between the true and observed values can be considered as accidental errors.

Naturally, it has to be made sure that no systematic error influences the observations.

Tables 1 and 2 and Figures 8 and 9 are showing different series of accidental errors and their single and double summations. From the above tables it can be seen that a *double summation of accidental error produces an effect similar to that caused by systematic errors*. This fact was not recognized for a long time and consequently it has been assumed that the strip is affected only by systematic errors. Accidental errors have been generally considered as unimportant and have been almost forgotten. It has to be pointed out that the double summation of accidental and systematic error complicates the research work on the theory of errors of bridging, but on the other hand, is very favorable for practical work because the interpolation polynoms used for the elimination of systematic errors

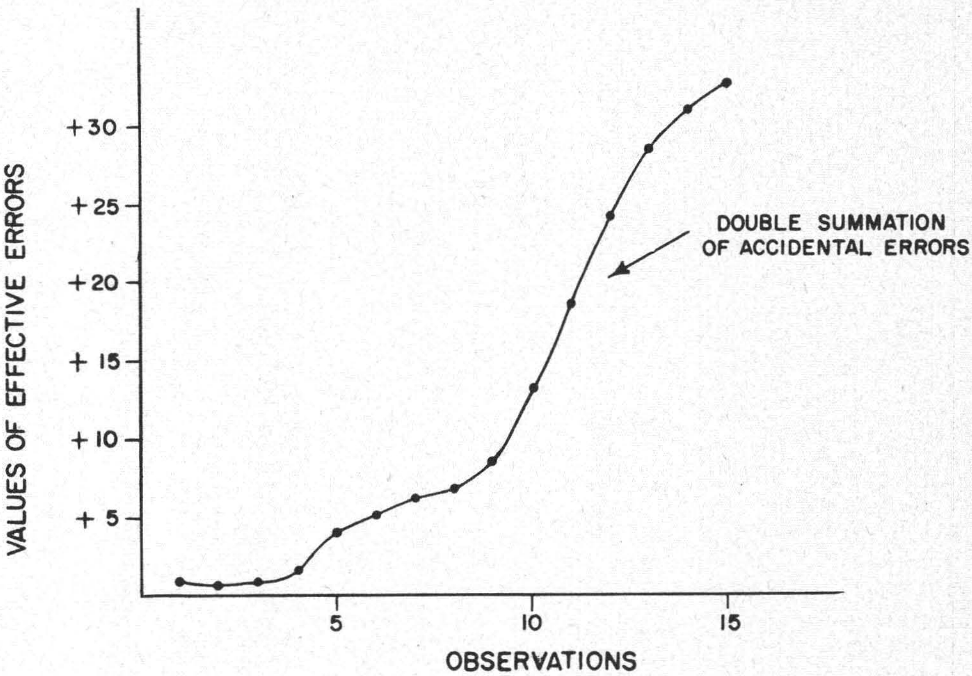

FIG. 8

are practically identical with those which eliminate the effect of double summation of accidental error.

GENERAL REMARKS ON ADJUSTMENT OF BRIDGING

Speaking of adjustment of aerial triangulations, it has to be pointed out that instead of an adjustment in a geographical sense there is an interpolation. Because the results obtained from a stereoplotting machine are corrected through interpolation, the term interpolation adjustment seems to be most correct.

TABLE 1

Errors	Single Summation	Double Summation
+ .7		
+ .1	+ .8	
−1.0	− .2	+ .6
−2.6	−2.8	− 2.2
+ .6	−2.2	− 4.4
+ .9	−1.3	− 5.7
+1.3	0	− 5.7
+ .1	+ .1	− 5.8
− .8	− .7	− 6.5
−2.3	−3.0	− 9.5
+2.2	− .8	−10.3
− .4	−1.2	−11.5
+ .9	− .3	−11.8
− .2	− .5	−12.3
− .3	− .8	−13.1
+1.0	+ .2	−12.9
− .9	− .7	−13.6
−1.5	−2.2	−15.8

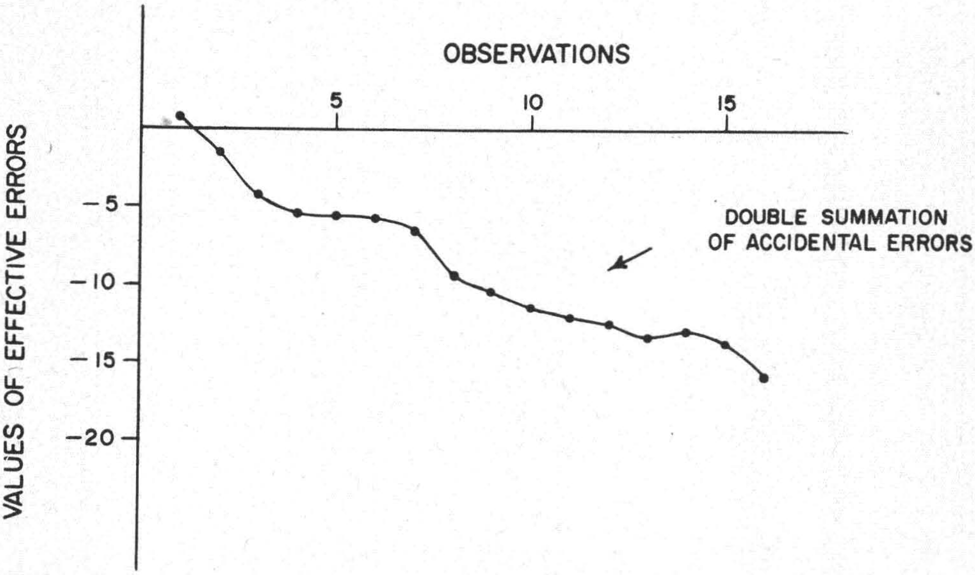

FIG. 9

To undertake an interpolation adjustment, the number of control points must at least equal the number of coefficients of the interpolation polynom. In graphical interpolation-adjustment, three groups of points are required: at the beginning, in the middle, and at the end of the strip. This method was published by the author in the November 1949 issue of PHOTOGRAMMETRIC ENGINEERING. It is therefore not necessary to explain it in detail. Let it be said only that the errors are interpolated from the longitudinal and lateral profiles of the surfaces of errors.

In the practical application, especially in the mapping of large inaccessible areas at small scales, the vertical and horizontal control are not sufficiently dense to insure three groups of control points in each strip. In this case the graphical interpolation-adjustment cannot be used in the classical form, but

TABLE 2

Errors	Single Summation	Double Summation
− .4		
+ .6	+ .2	
+ .6	+ .8	+ 1.0
−1.1	− .3	+ .7
+ .6	+ .3	+ 1.0
+ .3	+ .6	+ 1.6
+1.7	+2.3	+ 3.9
−2.1	+ .2	+ 4.1
+1.8	+2.0	+ 6.1
−1.3	+ .7	+ 6.8
+1.2	+1.9	+ 8.7
+2.6	+4.5	+13.2
+1.1	+5.5	+18.7
+ .1	+5.6	+24.3
−1.3	+4.3	+28.6
−1.6	+2.7	+31.3
−1.4	+1.3	+32.6

nevertheless a mathematically correct adjustment can be made if the layout of control survey is planned in the proper way.

To understand better the interpolation-adjustment the author introduces a new definition the "fictitious systematic error." As mentioned before, the deformations of a strip are the results of double-summations of accidental and systematic errors. The double summation of accidental errors affects the deformation of a strip in the same way as would a systematic error; therefore, it is possible to determine a value of a fictitious systematic error, which will give practically the same deformations of a strip as those caused by the combination of both systematic and accidental errors. In the interpolation-adjustment these fictitious systematic errors are used.

In the classical method of bridging the following interpolation polynoms are expressing the error ΔX, ΔY and ΔH.

$$\Delta X_i = A_0 + A_1X_i + A_2X_i^2 + A_3X_i^3 + A_4X_iY_i$$

$$\Delta Y_i = D_0 + D_1X_1 + D_2X_i^2 + D_3Y_i + D_4X_iY_i$$

$$\Delta H_i = C_0 + C_1X_i + C_2X_i^2 + C_3X_iY_i$$

When statoscope information has been used:

$$\Delta X_i = a_0 + a_1X_i + a_2X_i^2 + a_3X_iY_i$$

$$\Delta Y_i = d_0 + d_1X_i + d_2X_i^2 + d_3Y_i + d_4X_iY_i$$

$$\Delta H_i = c_0 + c_1X_i + c_2X_iY_i$$

Each of the above equations represents a surface which the author wishes to call a surface of error.

In short strips and bridging for large-scale mapping the classical method has an important advantage over the second group of methods because the accuracy of relative orientation is of a higher order than it is possible to obtain with existing devices used to establish the orientation elements of the camera at the moment of exposure. Therefore, the surfaces of errors and especially the surface of elevation's errors, are considerably more consistent and regular than when the second group of bridging methods is used. This fact is important because the interpolation polynoms will fit the true curve of errors and therefore there can be established more accurately the corrections to be applied to the co-ordinates of the pass points.

NEWS NOTE

A New Potentiometer

A new potentiometer providing a resistance range of 50 to 70,000 ohms with a standard linearity of ± 0.15 per cent has been announced by the Potentiometer Division of the Fairchild Controls Corporation, wholly owned subsidiary of the Fairchild Camera and Instrument Corporation.

Precision-built to close tolerances and guaranteed for long service and sustained accuracy, the new Type 747-E "pot" offers an innovation. A special clamp band provides an unrestricted tapping area allowing up to 19 taps and presents a simplified means of phasing units in a ganged assembly without disassembling the units.

The low noise level and high resolution of these units make them particularly desirable for computer assemblies, calibration controls, servo mechanisms and other similar applications.

The Type 747-E has a diameter of 2.100 inches and a cup width of .984 inches, and up to 6 units can be ganged on a single shaft. The units are furnished with welded taps and end leads. Low starting torque is only 1.0 oz.-in. per cup section.