*The Principles of Numerical Corrections in Aerial Photogrammetry**

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INTRODUCTION

EVERYONE, who is working with photogrammetry in practice knows how difficult it can be to adjust a model in an instrument up to the requirements of a certain standard, for instance, in accordance with the method of least squares. Even if numerical corrections of the elements of the relative orientation are computed from measured y-parallaxes and the corrections are applied to the instrument, the residual y-parallaxes of the model after the adjustment do not normally agree with the theoretical requirements. This unpleasant fact is, of course, mainly due to the lack of precision of the instrument. It is simply not possible to make an instrument that always reacts with mathematical precision. Consequently, if one is to use the requirements of the method of least squares as a standard for the adjustment of the model, every model set-up in the instrument will become more or less approximately oriented. In other words, the mathematical condition of the relative orientation normally cannot be physically satisfied. This means that normally there will remain unadjusted discrepancies in the relative orientation that will have systematic influence in functions of the elements of the relative orientation, as for instance the model coordinates and elevations.

A method to avoid these drawbacks of the photogrammetric procedure has been indicated in [2], but does not seem to be used in practice anywhere. The principle is very simple and means that *the results of a numerical adjustment of the relative orientation from measured y-parallaxes are applied in a numerical way to the actual functions of the relative orientation.* (See [2] pages 53–54.) The errors of the measured y-parallaxes are assumed to be of accidental character. The determination and treatment of some important systematic errors of the y-parallaxes will be discussed later in this paper.

In [2], the direct relation between the measured y-parallaxes and the corrections of elevations of the model has been given. (See also [12].) In a similar way the relation between the measured y-parallaxes and the corrections of the x- and y-coordinates of the model can be derived. From a combination of the formulae in [1] and [2] the preliminary results of an aerial triangulation can be corrected with respect to the measured y-parallaxes in the individual models.

An important result of this procedure is that numerical data of the highest possible precision can be obtained in a numerical way concerning, for instance, model coordinates, in spite of the fact that the individual models are not strictly adjusted in the instrument. Furthermore, the error propagation can be studied in a theoretically correct way in accordance with the method of least squares. Especially for the aerial triangulation, these consequences seem to be

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of great importance since the time-consuming work with the relative orientation may be considerably facilitated. Also for single models, where a limited number of points are of special interest, for instance, in connection with cadastral mapping, the mentioned principles may be of interest. As will be indicated later in this paper, also systematic errors can be corrected as far as the sources of the systematic errors are known. A new method for the determination of the most probable important sources of systematic errors has been given in [13].

A brief summary of the application of the principles will be given.

1. THE ADJUSTMENT OF THE RELATIVE ORIENTATION WITH THE AID OF MEASURED y-PARALLAXES, ACCORDING TO THE METHOD OF LEAST SQUARES

The general principles of such an adjustment problem are well known from geodetic literature; see for instance [14]. Also the principles of the error propagation in connection with an adjustment procedure, and in functions of the adjusted quantities, are clearly stated in the same publication.

A general treatment of the adjustment problems in photogrammetry in accordance with these well known principles is normally too time-consuming for practical use.* Under certain conditions however (i.e., a certain number of orientation points in certain locations within the model, approximately vertical photography and comparatively small elevation differences of the ground) the five normal equations of the adjustment problem can be solved in a general way. The complete general solution of the normal equations for the normal case of six orientation points, according to these conditions has been performed in [2]. In [3] the same problem has been treated for the case of nine symmetrically located orientation points. In [5] the corresponding problem for 15 symmetrically located orientation points has been treated, primarily in order to determine a more reliable value of the standard error of the y-parallax measurements. It is also possible to treat the adjustment problem in a similar way if there are 25 or more symmetrically located orientation points.

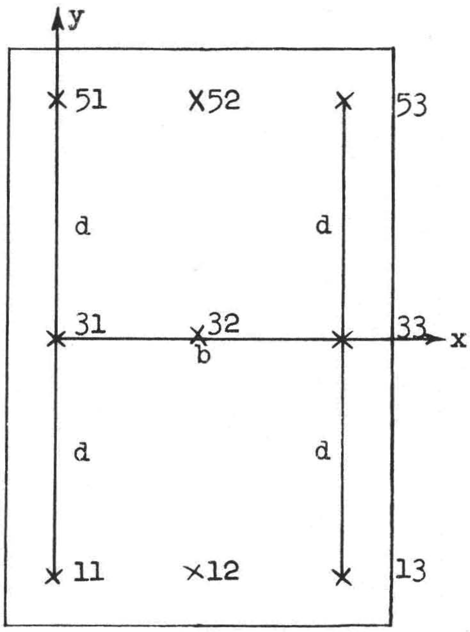

FIG. 1. The location and notation of 9 orientation points within the model, where y-parallaxes are measured.

As has been demonstrated in [4], and in other publications, the adjustment of the relative orientation of convergent photographs can be treated in the same manner.

For the following examples the formulas from [2] will be used.

* The solution of normal equations under arbitrary conditions can be performed conveniently with electronic computers. It may be advantageous to solve such normal equations *in advance* for a great number of cases that may be assumed in practice. If the results of such computations are collected in tables, the adjustment data may be determined from interpolation for most cases. The weight-and-correlation numbers are always of particular interest for the error propagation from the fundamental measurements. The theory of "the critical surfaces" can be studied conveniently with the aid of the weight-and-correlation numbers of the relative orientation.

The orientation points are chosen in accordance with Figure 1 and the elements of the orientation are defined in accordance with Figure 2.

The relation between the measured y-parallaxes in the six usual orientation

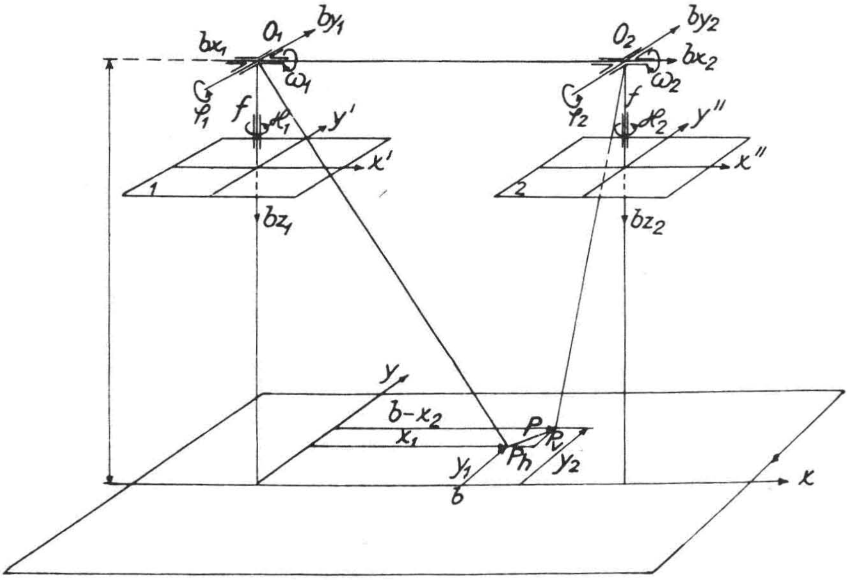

FIG. 2. The definition of the Elements of the External Orientation for the formulas in this paper. x- and y-parallaxes are notated p_h and p_v respectively.

points 11, 13, 31, 33, 51 and 53 and the elements of the relative orientation is for dependent pairs of pictures and comparatively flat ground [2]:

$$dby_2 = -p_{31}\left(\frac{1}{3}+\frac{h^2}{2d^2}\right) - p_{33}\left(\frac{2}{3}+\frac{h^2}{2d^2}\right) + (p_{11}+p_{51})\left(\frac{1}{6}+\frac{h^2}{4d^2}\right) - (p_{13}+p_{53})\left(\frac{1}{6}-\frac{h^2}{4d^2}\right) \tag{1}$$

$$d\kappa_2 = \frac{1}{3b}(p_{11} - p_{13} + p_{31} - p_{33} + p_{51} - p_{53}) \tag{2}$$

$$dbz_2 = \frac{h}{2d}(p_{53} - p_{13}) \tag{3}$$

$$d\phi_2 = \frac{h}{2db}(p_{51} - p_{53} - p_{11} + p_{13}) \tag{4}$$

$$d\omega_2 = \frac{h}{4d^2}(-2p_{31} - 2p_{33} + p_{11} + p_{13} + p_{51} + p_{53}). \tag{5}$$

The measured y-parallaxes are denoted p. The square sum of the residual y-parallaxes in the orientation points after the adjustment is obtained from the expression:

$$[vv] = \frac{1}{12}(-2p_{31} + 2p_{33} + p_{11} - p_{13} + p_{51} - p_{53})^2 \tag{6}$$

and the standard error of the observations as

$$\mu = \sqrt{[vv]}. \tag{7}$$

The *weight-and-correlation numbers*,* which facilitate the study of the error propagation in functions of the adjusted quantities according to their definition, can easiest be obtained directly from the expressions (1)–(5); see [14]. The weight numbers are defined as the square sums of the coefficients of the y-parallaxes of each expression (1)–(5). The correlation numbers are defined as the product sums of the coefficients of corresponding y-parallaxes of each pair of the expressions (1)–(5). Compare the method used in [3] and [2].

In [3] the corresponding expressions from y-parallaxes in the nine points 11, 12, 13, 31, 32, 33, 51, 52 and 53 have been derived. Of particular interest is the expression for $[vv]$ which can be used for the determination of the standard error of the observations.

Correcting a printing error in [3], we have:

$$\begin{aligned}[vv] = \frac{1}{12}(-\ 2p_{31} + 2p_{33} + p_{11} - p_{13} + p_{51} - p_{53})^2 \\ + \frac{1}{6}(p_{11} + p_{13} - 2p_{12})^2 + (p_{31} + p_{33} - 2p_{32})^2 + (p_{51} + p_{53} - 2p_{52})^2\}.\end{aligned} \tag{8}$$

The standard error of the measurements is then obtained as

$$\mu = \tfrac{1}{2}\sqrt{[vv]}. \tag{9}$$

Another printing error in [3] should also be corrected. As is immediately found from the normal equations, the expression dbz_2 should be:

$$dbz_2 = \frac{h}{12d}(-\ p_{51} + 5p_{53} + p_{11} - 5p_{13} + 2p_{52} - 2p_{12}). \tag{10}$$

In [5] similar expressions for y-parallax measurements in 15 points can be found.

Finally, in [4], the corresponding expressions for convergent photography have been derived. It is possible to derive similar expressions for other arrangements of the orientation points and for arbitrary directions of the camera axes. The fundamental differential formulas have, of course, to be applied to the actual arrangement of the axes of the plotting instrument. A new method for the derivation of the fundamental projection formulas and the corresponding differential formulas has been given in [6]. The application of simple matrix calculation procedures has proved to be of great value for the derivation of the fundamental projection formulas for arbitrary arrangements of the axes of the plotting instrument.

2. CORRECTION FORMULAS FOR SINGLE MODELS

a. *Elevations.* The relation between the y-parallaxes in the orientation points and the corrections of the elevations of the model has been given in [2]. The procedure is very simple. We have the well known relation between the elevations and the corrections of the elements of the relative orientation, for instance, for dependent pairs of pictures:

$$dh = \frac{yh}{b}d\kappa_2 - \frac{h^2 + (x - b)^2}{b}d\phi_2 + \frac{x - b}{b}dbz_2 + \frac{(x - b)y}{b}d\omega_2. \tag{11}$$

* These notations are in accordance with Tienstra and Roelofs. Other notations are square and rectangular weight coefficients.

Since we can express the corrections of the elements of the relative orientation as direct functions of the measured y-parallaxes, we obviously, from (1)–(5) and (11), can obtain the corrections of the elevations as direct functions of the measured y-parallaxes. Obviously an additional correction of the absolute orientation of the model can be necessary after the corrections for the relative orientation. This correction can be performed in a numerical way using the well known formulas for the rotations and translation of the model. See, for instance [3].

If the elevation control points are only three and they are located in the model points 11, 51 and 33 (see Figure 1) the corrections of the elevations of the model can be expressed as

$$dh = \frac{x^2 - bx}{2b^2d}(p_{11} + p_{53} - p_{51} - p_{13}) + \frac{xyh}{4bd^2}(-2p_{31} - 2p_{33} + p_{11} + p_{13} + p_{51} + p_{53}). \tag{12}$$

This expression has been derived in [12] under the assumption that the elevation errors in the control points due to the relative orientation have been entirely compensated by the absolute orientation of the model. In [7] similar principles have been used. The standard error of the elevations due to the relative orientation only and after the corrections, indicated by (12) can easily be found. The determination of the total standard error of the elevations of the model requires that the standard errors of the elevation measurements in the control points and in the model points be taken into account as well.

If more than three control points are given, the procedure indicated in [3] can be used. The influence of the relative orientation can be expressed by the weights of the model points with respect to the relative orientation according to (11).

b. *The plane coordinates*. The corrections of the x-coordinates of the model with respect to the corrections of the elements of the relative orientation can, according to [8] for instance, be expressed as:

$$dx = -\frac{xy}{b}d\kappa_2 + \frac{x\{h^2 + (x-b)^2\}}{bh}d\phi_2 - \frac{(x-b)xy}{bh}d\omega_2 - \frac{(x-b)x}{bh}dbz_2 \tag{13}$$

The corrections of the y-coordinates of the model with respect to the corrections of the elements of the relative orientation can, using the principle of symmetry, as indicated in [8] for instance, be expressed as:

$$dy = \frac{(x-b)}{2}d\kappa_2 + \frac{(x-b)y}{2h}d\phi_2 - \left(1 + \frac{y^2}{h^2}\right)\frac{h}{2}d\omega_2 - \frac{y}{2h}dbz_2 + \frac{dby_2}{2} - \frac{y}{h}dh. \tag{14}$$

The corrections of the plane coordinates can obviously be easily expressed as direct functions of the measured y-parallaxes with the aid of (1)–(5), (12)–(14).

After the corrections of the plane coordinates, a change of the scale of the model may be necessary.

The correction procedure is simple if only two control points are used. If more control points are present, another adjustment is required. The principles given in [3] can be used and the influence of the relative orientation can be expressed as weights with respect to the expressions (13) and (14). For an arbitrary number of control points in arbitrary positions, the formulas become rather complicated.

The standard errors of the corrections (13) and (14) can easily be determined with the error propagation formulas.

3. AERIAL TRIANGULATION

From the fundamental formulas for the correction of the elements of orientation along a triangulation strip as derived by Bachman in [1], we obtain in the coordinate system of Figure 2:

$$\Delta\kappa_n = \sum_{i=1}^{i=n} d\kappa_i \tag{15}$$

$$\Delta\phi_n = \sum_{i=1}^{i=n} d\phi_i \tag{16}$$

$$\Delta\omega_n = \sum_{i=1}^{i=n} d\omega_i \tag{17}$$

$$\Delta by_n = b\sum_{i=1}^{i=n}(n-i)d\kappa_i + \sum_{i=1}^{i=n} dby_i \tag{18}$$

$$\Delta bz_n = -b\sum_{i=1}^{i=n}(n-i)d\phi_i + \sum_{i=1}^{i=n} dbz_i \tag{19}$$

$$\begin{aligned}\Delta X_n = & -b\sum_{i=1}^{i=n}\left\{\frac{h}{b}\left(1+\frac{b^2}{h^2}\right)+\frac{b}{h}(n-i)\right\}d\phi_i \\ & -\frac{b}{h}\sum_{i=1}^{i=n}(2n-2i+1)dbz_i - \frac{b}{h}\sum_{i=1}^{i=n}(n-i+1)dH_{i-1}\end{aligned} \tag{20}$$

The six elements of orientation of each individual picture along the strip can thus be corrected with respect to the corrections of the individual models, obtained from the measured residual y-parallaxes in the orientation points of each model after a preliminary relative orientation. In these formulas the picture No. 1 is the third picture in the strip since the first model $(-1, 0)$ is assumed to have been rigorously adjusted and absolutely oriented. Of course, also the first model $(-1, 0)$ can be numerically adjusted.

The last term of (20) refers to the measurements of the elevations in the scale transfer points only, and has no direct connection with the corrections from the relative orientation. In order to determine the standard error of the correction ΔX_n, the last term is of particular interest.

Substituting the expressions (1)–(5) into (15)–(20) we obviously can express the corrections of the elements of orientation along the strip as direct functions of the measured y-parallaxes.

We obtain:

$$\Delta\kappa_n = \frac{1}{3b}\sum_{i=1}^{i=n}(p_{31}-p_{33}+p_{11}-p_{13}+p_{51}-p_{53})_{i-1,i} \tag{21}$$

$$\Delta\phi_n = \frac{h}{2db}\sum_{i=1}^{i=n}(p_{51}-p_{53}-p_{11}+p_{13})_{i-1,i} \tag{22}$$

$$\Delta\omega_n = \frac{h}{4d^2}\sum_{i=1}^{i=n}(-2p_{31} - 2p_{33} + p_{11} + p_{13} + p_{51} + p_{53})_{i-1,i} \tag{23}$$

$$\begin{aligned}\Delta by_n = &\frac{1}{3}\sum_{i=1}^{i=n}(n-i)(p_{31} - p_{33} + p_{11} - p_{13} + p_{51} - p_{53})_{i-1,i}\\ &+ \frac{1}{12d^2}\sum_{i=1}^{i=n}\{(3h^2 + 2d^2)(p_{11} + p_{51} - 2p_{31}) - 2(3h^2 + 4d^2)p_{33}\\ &+ (3h^2 - 2d^2)(p_{13} + p_{53})\}_{i-1,i}\end{aligned} \tag{24}$$

$$\begin{aligned}\Delta bz_n = &\frac{h}{2d}\sum_{i=1}^{i=n}(n-i)(p_{11} - p_{13} - p_{51} + p_{53})_{i-1,i}\\ &+ \frac{h}{2d}\sum_{i=1}^{i=n}(p_{53} - p_{13})_{i-1,i}\end{aligned} \tag{25}$$

$$\begin{aligned}\Delta X_n = &\frac{h}{2d}\sum_{i=1}^{i=n}\left\{\frac{h}{b}\left(1 + \frac{b^2}{h^2}\right) + \frac{b}{h}(n-i)\right\}(p_{11} - p_{13} - p_{51} + p_{53})_{i-1,i}\\ &+ \frac{b}{2d}\sum_{i=1}^{i=n}(2n - 2i + 1)(p_{13} - p_{53})_{i-1,i}\end{aligned} \tag{26}$$

The subscript $_{i-1,\,i}$ denotes that the y-parallaxes are measured in that model where i is the right picture and $_{i-1}$ the left one, for the case of base i n. If there are elevation differences left between the scale transfer points from model to model, the last term of (20) should also be included in the expression (26).

Since all corrections (21)–(26) are expressed in terms of directly measured quantities, the standard error of the corrections can easily be expressed in terms of the standard error of the measurements with the aid of the special law of error propagation.* The same result will be obtained if the general law of error propagation* is applied to the expressions (15)–(20). For this purpose we need the weight-correlation-numbers corresponding to the expressions (1)–(5). These data have in [2] been obtained directly from the normal equations. We shall here briefly demonstrate the determination of the accuracy of the correction.

According to the definition in [1] of dH in expression (20), this term is the difference between two elevation measurements. Consequently, the standard error m_H can be expressed as $m_h\sqrt{2}$ where m_h is the standard error of one of the elevation measurements. But the standard error of the elevation measurements depends upon the base-height ratio and the standard error of the correction of the x-parallaxes. If the x-parallaxes are corrected stereoscopically and the y-parallaxes monocularly, the ratio between the standard errors of these operations is normally assumed to be 1:2. If the standard error of the y-parallax measurements is μ, a simple computation shows the relation

$$m_H = \frac{h\mu}{b\sqrt{2}} \tag{27}$$

Since in this way we have determined the relation between the differentials of expression (20), we can determine the weight-number of (20) under the assumption that the measurements of the x- and y-parallaxes are free from correlation.

* These expressions seem to have been introduced by *Tienstra* and are frequently used by *Roelofs*. They mean essentially the laws for error propagation in functions of uncorrelated quantities and in functions of correlated quantities, respectively. See [14].

Using the same technique as demonstrated in [1], by application of the special law of error propagation to the expressions (21)–(26), including the last term of (20), we obtain:

$$Q_{\kappa_n\kappa_n} = \frac{2n}{3b^2} \tag{28}$$

$$Q_{\phi_n\phi_n} = \frac{h^2 n}{b^2 d^2} \tag{29}$$

$$Q_{\omega_n\omega_n} = \frac{3h^2 n}{4d^4} \tag{30}$$

$$Q_{by_n by_n} = \frac{(2n-1)(n-1)n}{9} + \frac{n(9h^4 + 8d^4 + 12h^2d^2)}{12d^4} + \frac{(n-1)n}{3} \tag{31}$$

$$Q_{bz_n bz_n} = \frac{h^2}{d^2}\left\{\frac{(2n-1)(n-1)n}{6} + \frac{n}{2} + \frac{(n-1)n}{2}\right\} \tag{32}$$

$$Q_{X_nX_n} = \frac{h^4}{b^2d^2}\left(1 + \frac{b^2}{h^2}\right)^2 n - \frac{h^2}{d^2}\left(1 + \frac{b^2}{h^2}\right) n + \frac{b^2}{6d^2}(2n^2+1)n + \frac{(2n+1)(n+1)n}{12}. \tag{33}$$

The standard errors of the corrections of the elements of arbitrary pictures along the strip can now be found as the product between the standard error of the y-parallax measurements and the square roots of the respective weight numbers. Of course, we have assumed the coefficients b, d and h to be constant for all models, and the same standard error of the y-parallax measurements in all models. No particular difficulties are introduced concerning the principles if the individual models are different, but the formulas get more complicated.

Similar derivations of correction formulas and the accuracy of the corrections can be performed in the same way for the case that 9 or 15 orientation points have been used or if convergent photography is used. See Diagrams 1–6.* The fundamental expressions have been derived in [3], [4] and [5]. The convergent photography is assumed in accordance with the well known American method. The errors of the interior orientation or of the transformation of the data of the exterior orientation (particularly the angle between the cameras in the same exposure station) are *not* included in the formulas.

The corrections of the model coordinates and the accuracy of the corrections can in a similar way be obtained from the following formulas (see [8] for instance):

$$\begin{aligned}\Delta x_{i-1,i} = &-y\Delta\kappa_{i-1} + h\left(1 + \frac{x^2}{h^2}\right)\Delta\phi_{i-1} - \frac{xy}{h}\Delta\omega_{i-1} - \frac{x}{h}\Delta bz_{i-1} + \Delta X_{i-1} \\ &+ \frac{x}{b}\left\{y(\Delta\kappa_{i-1} - \Delta\kappa_i) + \frac{h^2 + (x-b)^2}{h}\Delta\phi_i - \frac{h^2 + x^2}{h}\Delta\phi_{i-1}\right. \\ &\left.- \frac{(x-b)y}{h}\Delta\omega_i + \frac{xy}{h}\Delta\omega_{i-1} - \frac{x-b}{h}\Delta bz_i + \frac{x}{h}\Delta bz_{i-1} + \Delta X_i - \Delta X_{i-1}\right\}\end{aligned} \tag{34}$$

(*continued on page 232*)

* The standard error μ of the y-parallax measurements has in the diagrams been chosen 0.02 mm. *Special determination of μ should be performed in each model.* See [4], [5], [13] and the appendix. Modern cameras and high precision platters have proved to give the value 0.006 mm. as an average. The influence of the inner orientation in connection with aerial triangulation is not yet sufficiently investigated.

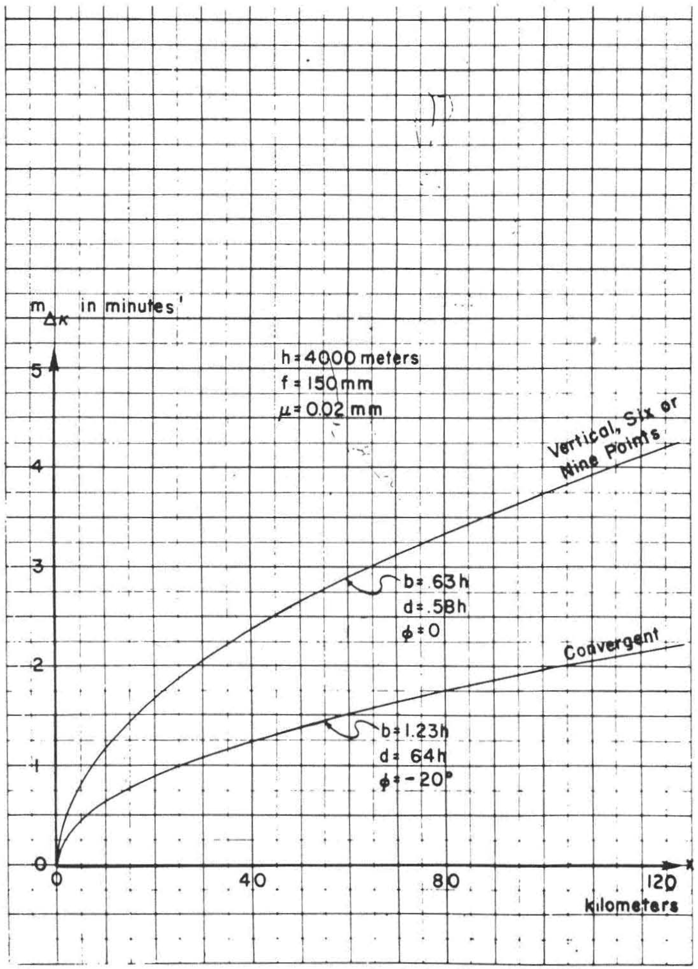

DIAGRAM 1. Standard error m_{Δ_κ} as a function of the length of the strip.

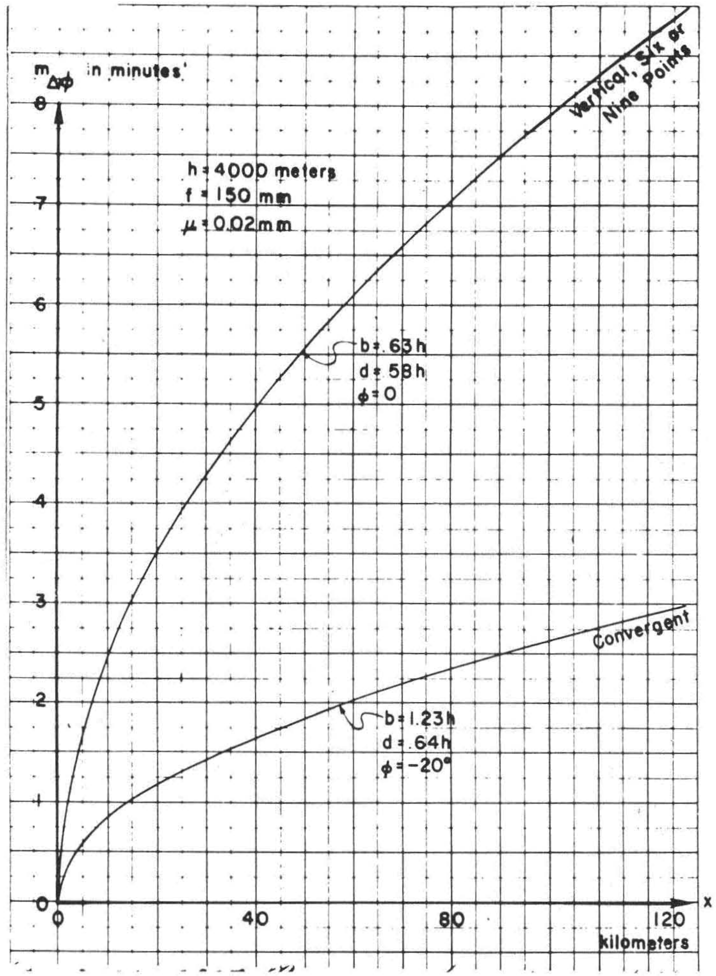

DIAGRAM 2. Standard error m_{Δ_ϕ} as a function of the length of the strip.

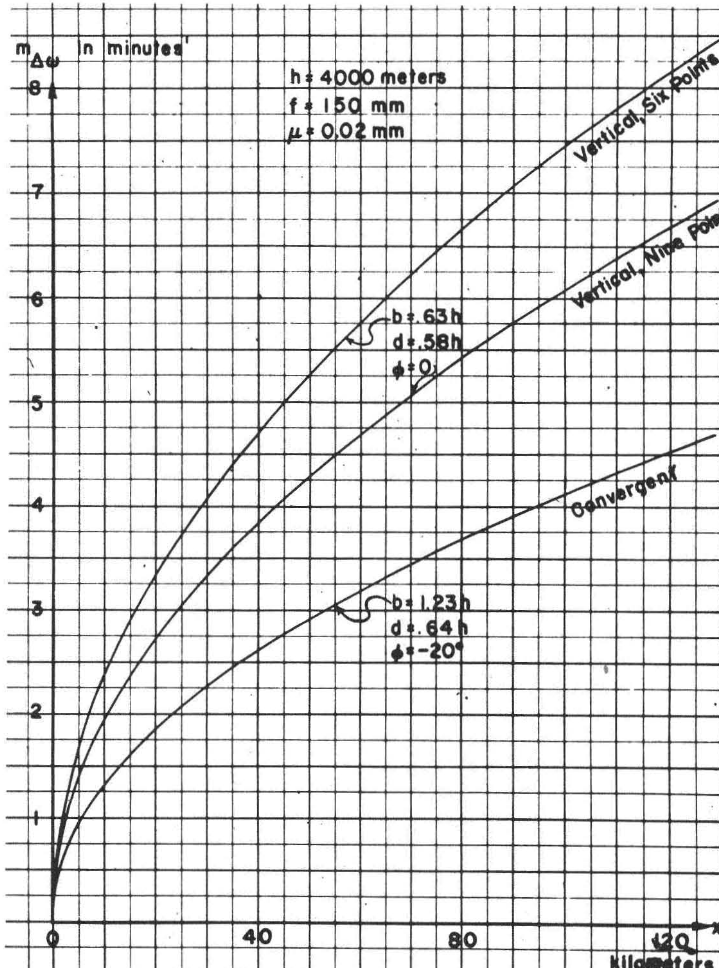

DIAGRAM 3. Standard error m_{Δ_ω} as a function of the length of the strip.

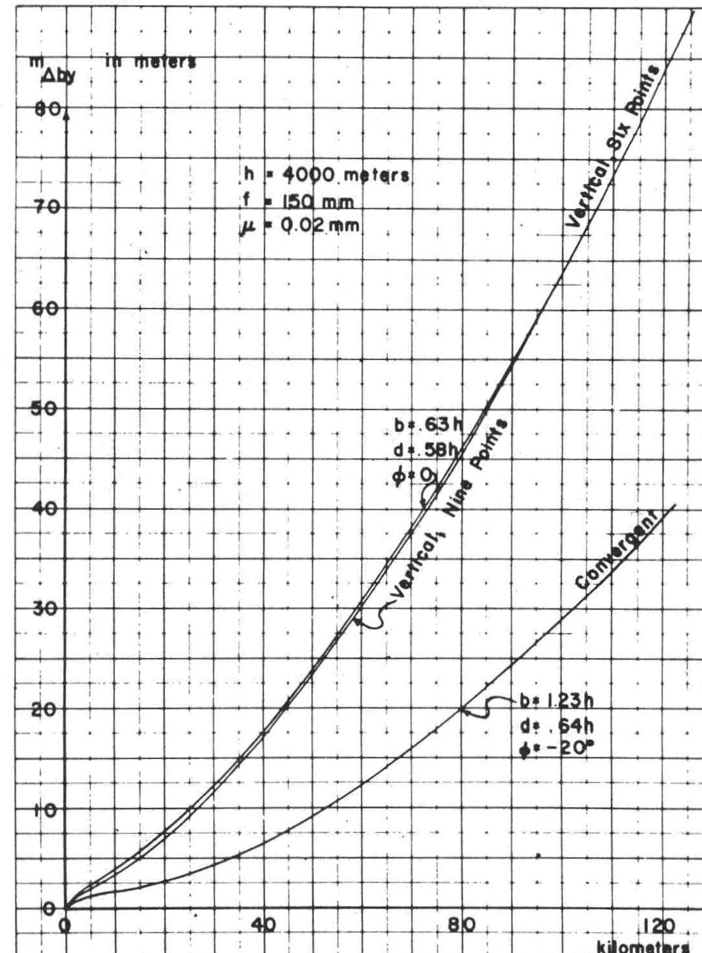

DIAGRAM 4. Standard error $m_{\Delta_{by}}$ as a function of the length of the strip.

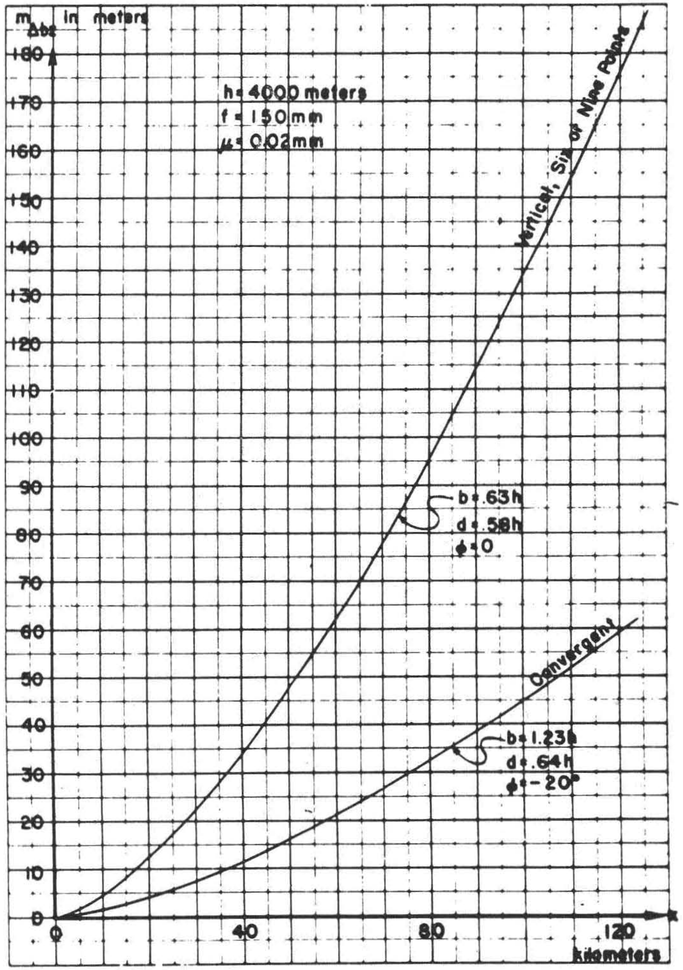

DIAGRAM 5. Standard error $m_{\Delta b3}$ as a function of the length of the strip.

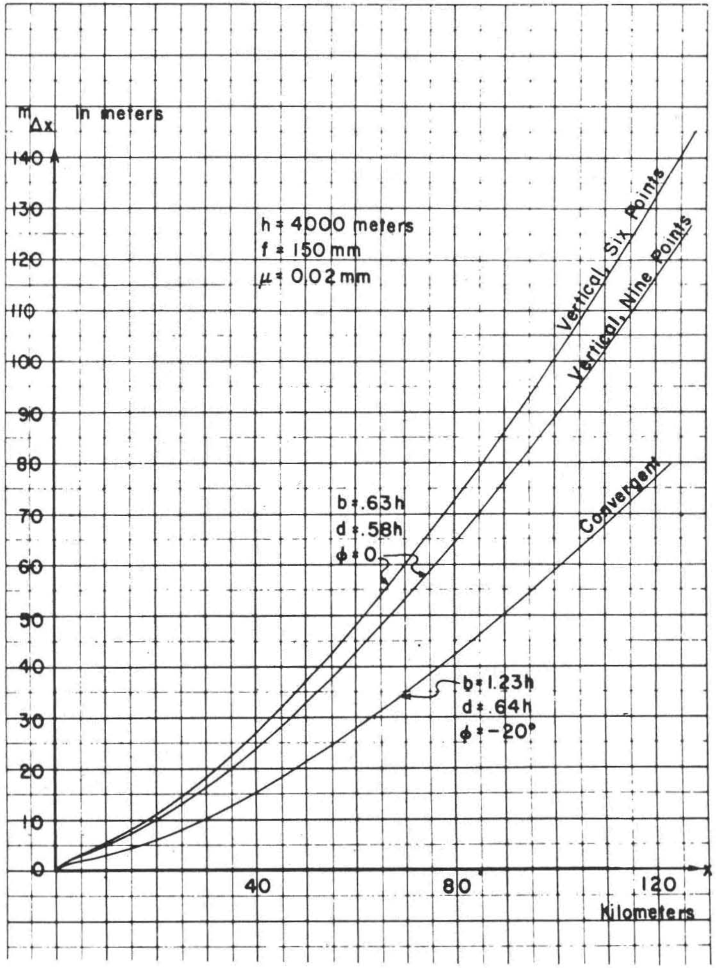

DIAGRAM 6. Standard error $m_{\Delta X}$ as a function of the length of the strip.

$$\Delta y_{i-1,i} = \frac{1}{2}\left\{x\Delta\kappa_{i-1} + (x-b)\Delta\kappa_i + \frac{xy}{h}\Delta\phi_{i-1} + \frac{(x-b)y}{h}\Delta\phi_i - \left(1+\frac{y^2}{h^2}\right)h\Delta\omega_{i-1} - \left(1+\frac{y^2}{h^2}\right)h\Delta\omega_i - \frac{y}{h}(\Delta bz_{i-1} + \Delta bz_i) + \Delta by_{i-1} + \Delta by_i\right\} - \frac{y}{h}\Delta h_{i-1,i} \qquad (35)$$

$$\Delta h_{i-1,i} = \frac{hy}{b}(\Delta\kappa_i - \Delta\kappa_{i-1}) + \frac{h^2 + x^2}{b}\Delta\phi_{i-1} - \frac{h^2 + (x-b)^2}{b}\Delta\phi_i - \frac{xy}{b}\Delta\omega_{i-1} + \frac{(x-b)y}{b}\Delta\omega_i - \frac{x}{b}\Delta bz_{i-1} - \left(1 - \frac{x}{b}\right)\Delta bz_i + \frac{h}{b}(\Delta X_{i-1} - \Delta X_i). \qquad (36)$$

With the aid of (1)–(5) the corrections can be expressed as direct functions of the measured y-parallaxes. The weight numbers of the corrections can be determined as indicated above.

Rearrangements and simplifications of the formulas (34)–(36) can, of course, be performed.

4. CORRECTION OF THE SYSTEMATIC ERRORS

Systematic errors can affect the y- and x-parallaxes. Consequently the functions of the parallaxes, as for instance, the elements of orientation and the coordinates of the models will be systematically influenced. If systematic sources of errors are known and also the functions with which the errors propagate, corrections can then be computed in a similar way as has been demonstrated above. For instance, if there is radial distortion present, the ϕ and bz will primarily be affected, because also those y-parallaxes in the orientation points 11, 51, 13 and 53, that are caused by the distortion, will be corrected with these elements. The systematic errors of ϕ and bz will cause model deformations in addition to those which are directly caused by the distortion. The determination of the latter deformations is an elementary problem that frequently has been treated in the photogrammetric literature.

Obviously it is of the greatest importance to determine the sources of the systematic errors as accurately as possible. The most important systematic error in photogrammetry is probably the distortion of the pencils of rays in connection with the photography. Great attention has been paid to this disturbance and several methods for the determination of the distortion in the aerial camera and for the correction of this distortion in the plotting instrument have been developed. But it still seems very doubtful if this source of error is correctly treated up to now. The distortion of the camera is, for instance, normally determined under quite different circumstances than are present at the time of photography in the air. The influence of the temperature, vibrations, the atmospheric refraction and the earth's curvature can not normally be taken into account when the camera is investigated on the ground. Furthermore, the correction of the distortion in the instruments does not normally seem to be investigated in the instrument, itself; at least, only a very few reports on this subject have been published. It is therefore probable that systematic errors of this kind play a much more important role in the photogrammetric procedure than in normally assumed. In [13] a method is described that seems to be of value for the treatment of some systematic errors in photogrammetry. The method of least squares is here used as a powerful tool for the determination of systematic errors, and also gives important information concerning the accuracy of the determination. See also [11].

As soon as the systematic errors and their influence upon the parallaxes are determined, the errors of the elements of orientation can be found from (1)–(5), (15)–(20), and corresponding formulas for other orientation procedures. The influences upon the model and strip coordinates can be found from (12)–(14) and (34)–(36) for the case of the normal six orientation points. Of course, the measured y-parallaxes can immediately be corrected before further computations.

Also the accuracy of the corrections can be found if the accuracy of the determination of the systematic errors is known.

After the corrections, with respect to the residual y-parallaxes from the preliminary relative orientation and known systematic errors, the remaining discrepancies can be regarded as caused mainly by accidental errors. Correct adjustment procedures under such circumstances are well known from the literature. See for instance [1], [9] or [10].

From (34)–(36) the error propagation in uncontrolled aerial triangulation strips ("open" aerial triangulation) after numerical corrections can be investigated with well-known methods. A considerably simplified method for the adjustment of discrepancies in control points, in arbitrary positions within the triangulation strip and for the determination of the accuracy after the adjustment, has been worked out by the author and will be published later on.

The fundamental operation of the principles of the numerical corrections in aerial photogrammetry, the measurements of y-parallaxes, and some results of the measurements are demonstrated in the Appendix. The measurements in this case were performed with a mirror stereoscope and parallax bar under very simple circumstances. Exactly the same procedure can be applied to all kinds of stereoscopic plotters.

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Appendix

Summary of Check Measurements of Pictures from Three Cameras

Photographed by U.S. Coast and Geodetic Survey

1. General Procedure

The y-parallaxes were measured in the contact diapositives with mirror stereoscope and parallax bar. Fifteen points were normally used; these were located in accordance with Figure 3. In some cases additional symmetric points were used.

Thirty-one and thirty-three are the principal points. The diapositives were rotated 90 degrees in order to make the y-parallaxes appear as x-parallaxes. From the y-parallaxes of 11, 13, 31, 33, 51, and 53, the relative orientation was computed with adjustment formulas (1)–(5). The square sum $[vv]$ was also computed (6). From the determined corrections to the elements of the relative orientation the corresponding corrections to the measured y-parallaxes in all points were computed and added to the results of the measurements. The residuals were thus obtained. These residuals are of great interest, since they indicate how well the relative orientation could have been performed in a stereoscopic plotter.

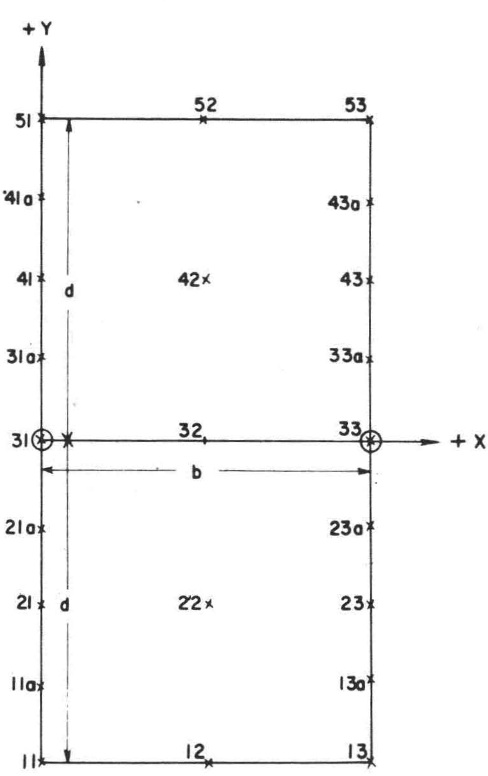

FIG. 3. Locations and notations of model points for y-parallax measurements.

Furthermore, the square sum of the residuals and the standard errors have been computed as if the relative orientation had been performed by the parallaxes in 9 or 15 points. Particularly the square sum of the residuals in 9 points (8) is of interest since the standard error of the observations can be determined by a better accuracy than from 6 points only.

2. The Results of the Measurements

The measurements have been performed by Mr. J. Dowdy and Mr. W. Mahoney, both students of the Ohio State University. The instrument used was a Wild mirror stereoscope and parallax bar. Since two sets of the contact diapositives were available, the measurements by the operators can be regarded as completely independent. The measurements in each point were repeated five times and the average was used for the computations.

The results of the computations are shown under 2.1, 2.2 and 2.3.

The distortion and asymmetry effect can easily be found from the systematism of the residual y-parallaxes in points which are symmetrical around $x=b/2$ and $y=0$. There are traces of distortion in the cameras A and C, of which the camera A is the best one. In camera B there is a very pronounced distortion effect. The distortion can also be expressed as a "ϕ-curve."

The photographed area is comparatively flat.

3. SUMMARY

In summary, the results are rather interesting. The agreement between $d\phi_2$, $d\omega_2$ and dbz_2 is as good as can be expected from the determined standard errors.

A series of measurements of x-parallaxes in a number of elevation control points was also performed. After correction of the model deformations due to $d\phi_2$ and $d\omega_2$ and a numerical absolute orientation, the elevation accuracy proved to be fully comparable to what normally is expected in high precision plotters, and in some cases is better. Known radial distortion must of course be corrected for separately.

The absolute values of ϕ and ω were also determined in this connection and agreed very well for the two operators. The nadir points and isocenters can of course easily be determined in the pictures for radial triangulation.

Under the condition that the pictures are taken with small deviations from the vertical (automatic devices will probably keep the deviations within 10′–15′) and comparatively flat ground with good detail, this very simple procedure can become of great value for among other purposes:

1. Checking of aerial photographs immediately after the photography.
2. Determination of the actual distortion influence.
3. Determination of the absolute values of ϕ and ω and also bz.
4. Elevation measurements of high precision.
5. Radial triangulation (The most important systematic source of error can be practically eliminated.)

2.1 DIAPOSITIVES NO. 1689–1690. CAMERA A

$b=100$ mm. $d=100$ mm. $h=150$ mm.

Point	Dowdy			Mahoney		
	Micrometer Readings, Average, mm.	y-Parallax, mm.	Residual Value, mm.	Micrometer Readings, Average, mm.	y-Parallax, mm.	Residual Value, mm.
11	10.778	*+0.340*	−0.002	5.202	*+0.322*	−0.002
12	11.028	+0.590	−0.012	4.894	+0.014	−0.005
13	11.254	*+0.816*	+0.002	4.576	*−0.304*	+0.002
21	10.498	+0.060	+0.022	4.928	+0.048	+0.022
22	10.838	+0.400	0.000	4.712	−0.168	−0.004
23	11.172	+0.734	−0.015	4.478	−0.402	−0.010
31	10.438	*0.000*	+0.004	4.880	*0.000*	+0.004
32	10.820	+0.382	+0.019	4.728	−0.152	−0.014
33	11.240	*+0.802*	−0.004	4.548	*−0.332*	−0.004
41	10.544	+0.106	−0.002	4.998	+0.118	+0.008
42	11.010	+0.572	+0.008	4.910	+0.030	−0.003
43	11.486	+1.048	+0.008	4.816	−0.064	−0.008
51	10.822	*+0.384*	−0.002	5.316	*+0.436*	−0.002
52	11.374	+0.936	+0.001	5.274	+0.394	+0.012
53	11.928	*+1.490*	+0.002	5.256	*+0.376*	−0.002

Dowdy:

$d\kappa_2 = -0.007947$ rad.
$d\phi_2 = -0.004725$ rad. $= -16'$
$d\omega_2 = 0.005348$ rad. $= 18'$
$dby_2 = 0.004$ mm.
$dbz_2 = 0.506$ mm.

$\mu_\epsilon = 0.006$ mm.
$\mu_9 = 0.010$ mm.
$\mu_{15} = 0.010$ mm.

Mahoney:

$d\kappa_2 = +0.003393$ rad.
$d\phi_2 = -0.004245$ rad. $= -14{,}5'$
$d\omega_2 = 0.005602$ rad. $= 19'$
$dby_2 = 1.176$ mm.
$bdz_2 = 0.510$ mm.

$\mu_\epsilon = 0.006$ mm.
$\mu_9 = 0.008$ mm.
$\mu_{15} = 0.009$ mm.

2.2 DIAPOSITIVES NO. 23–24. CAMERA B
$b=90$ mm. $d=90$ mm. $h=153$ mm.

Point	Dowdy Parallax Average, mm.	Dowdy Reduced Parallax, mm.	Dowdy Residual Value, mm.	Mahoney Parallax Average, mm.	Mahoney Reduced Parallax, mm.	Mahoney Residual Value, mm.
11	7.334	+*.644*	−.002	10.136	+*.645*	−.002
12	7.184	+.494	−.017	10.270	+.779	−.012
13	7.000	+*.310*	−.002	10.381	+*.890*	+.002
11a	7.248	+.558	−.036	—	—	—
13a	6.856	+.166	+.030	—	—	—
21	7.118	+.428	−.056	9.908	+.417	−.044
22	6.918	+.228	−.013	9.986	+.495	+.006
23	6.716	+.026	+.032	10.037	+.546	+.084
21a	6.914	+.224	−.027	—	—	—
23a	6.560	−.130	+.024	—	—	—
31	6.690	*.000*	−.004	9.491	*.000*	+.003
32	6.516	−.174	+.024	9.652	+.161	−.025
33	6.390	−*.300*	+.004	9.764	+*.273*	−.003
41	6.156	−.534	+.049	8.983	−.508	+.044
42	6.078	−.612	−.006	9.188	−.303	−.024
43	5.992	−.698	−.054	9.380	−.111	−.079
41a	6.406	−.284	+.052	—	—	—
43a	6.214	−.476	−.035	—	—	—
51	5.620	−*1.070*	+.002	8.462	−*1.029*	−.002
52	5.512	−1.178	−.012	8.582	−.909	+.020
53	5.380	−*1.310*	−.002	8.742	−*.749*	+.002
51a	5.882	−.808	+.045	—		
53a	5.746	−.994	−.025	—		

The distortion amounts, determined from symmetrical points, agree well with corresponding quantities, determined from the distortion curve for a Metrogon lens.

Dowdy	Mahoney
$d\kappa_2 = -\ .003237$	$d\kappa_2 = -\ .002953$
$d\phi_2 = -\ .000888 = -3'$	$d\phi_2 = -\ .000325 = -1'$
$d\omega_2 = -\ .003901 = -13'$	$d\omega_2 = -\ .003727 = -13'$
$dby_2 = -\ .301$ mm.	$dby_2 = -\ .839$ mm.
$dbz_2 = -1.377$ mm.	$dbz_2 = -1.393$ mm.
$\mu_6 = .008$ mm.	$\mu_6 = .006$ mm.
$\mu_9 = .013$ mm.	$\mu_9 = .014$ mm.
$\mu_{15} = .029$ mm.	$\mu_{15} = .039$ mm.

The standard errors for 9 and 15 points probably include in this case comparatively large amounts of distortion, which is a systematic error. This agrees very well with the results from the residual y-parallaxes. Of course, the expression "standard error" should not be used in this connection. "Mean square value of the residuals" might be better.

6. Preparation of the relative orientation in stereoscopic plotters.
7. Approximate aerial triangulations.

Terrestrial control data are not necessary for the points 1, 2 and 6.

If the flying altitude is relatively well known, only elevation control data are necessary for the points 3, 4 and 5 (water surfaces may sometimes be sufficient).

The accuracy of the determined data can in all cases be computed from formulae that are so simple that an ordinary operator can perform the computations without knowing the theory involved.

If there are large deviations from the vertical, large elevation differences,

2.3 DIAPOSITIVES NO. 74–75. CAMERA C

$b=90$ mm. $d=90$ mm. $h=153$ mm.

Point	Dowdy			Mahoney		
	Parallax Average, mm.	Reduced Parallax, mm.	Residual Value, mm.	Parallax Average, mm.	Reduced Parallax, mm.	Residual Value, mm.
11	7.894	+*1.136*	−.002	11.364	+*1.157*	−.000
12	7.478	+ .720	−.004	11.209	+1.002	+.001
13	7.054	+ *.296*	+.002	11.057	+ *.850*	+.000
11a	7.526	+ .768	+.000	10.975	+ .768	+.016
13a	6.824	+ .066	−.006	10.825	+ .618	−.008
21	7.208	+ .450	+.008	10.666	+ .460	+.008
22	6.922	+ .164	+.004	10.644	+ .438	+.008
23	6.628	− .130	+.008	10.628	+ .421	+.002
21a	6.954	+ .196	+.007	10.417	+ .210	−.004
23a	6.498	− .260	+.012	10.513	+ .307	−.013
31	6.758	*.000*	+.003	10.207	*.000*	+.001
32	6.574	− .184	+.026	10.299	+ .093	+.017
33	6.442	− *.316*	−.003	10.426	+ *.220*	−.001
41	6.532	− .226	−.002	9.960	− .247	+.002
42	6.500	− .258	−.003	10.199	− .008	+.003
43	6.464	− .294	−.000	10.478	+ .271	−.036
41a	6.606	− .152	+.012	10.057	− .150	+.000
43a	6.422	− .336	+.002	10.426	+ .219	−.020
51	6.522	− *.236*	−.002	9.939	− *.268*	−.000
52	6.628	− .130	−.012	10.318	+ .111	−.008
53	6.710	− *.048*	+.002	10.680	+ *.473*	+.000
51a	6.490	− .268	+.007	9.915	− .292	+.008
53a	6.556	− .202	+.004	10.563	+ .356	+.012

Dowdy

$d\kappa_2 = + .003585$ rad.
$d\phi_2 = - .009709 = -35{,}5'$
$d\omega_2 = + .008406 = 29'$
$dby_2 = +1.605$ mm.
$dbz_2 = - .292$ mm.

$\mu_6 = .006$ mm.
$\mu_9 = .012$ mm.
$\mu_{15} = .008$ mm.

Mahoney

$d\kappa_2 = - .002424$ rad.
$d\phi_2 = - .009900 = -34'$
$d\omega_2 = .008374 = 28{,}5'$
$dby_2 = +1.062$ mm.
$dbz_2 = - .320$ mm.

$\mu_6 = .001$ mm.
$\mu_9 = .008$ mm.
$\mu_{15} = .012$ mm.

and the orientation points cannot be chosen in the ideal positions, the procedure will of course get more complicated. However, a system of repeated procedures can be used in many cases; that gives good results. Large elevation differences and irregular location of the orientation points require normally a numerical solution of the normal equations, which of course can be performed.

Finally some examples of the obtainable accuracy of some important data will be given.

We assume the procedure as described above to be used for the determination of elevations on the ground and the final tilts ϕ and ω. We assume only three elevation control points located in the points 11, 51 and 33 of the model (Figure 1).

The standard error of the final elevations on the ground is then in general (see [12]):

$$M_h = \frac{\mu h}{b}\sqrt{\frac{x^2(x-b)^2}{b^2d^2} + \frac{3x^2y^2}{4d^4} + \frac{3x^2}{2b^2} + \frac{y^2}{2d^2} - \frac{x}{b} + \frac{3}{2}}$$

μ is the standard error of the y-parallax and the x-parallax measurements.

The mean square value of the elevation errors for the neat model area is obtained as

$$M_h = \mu\sqrt{\frac{7h^2}{60d^2} + \frac{5h^2}{3b^2}}.$$

For wide-angle cameras and 60% overlap we obtain

$$M_h = 2.1\mu.$$

For $\mu = 0.01$ mm. (camera A above) and the scale factor h/f we find

$$M_h = \frac{h}{f}\,0.021 \text{ mm.}$$

For $f = 150$ mm. and $h = 3{,}000$ m. we find

$$M_h = 0.42 \text{ m. (around 1.5 feet).}$$

This means

$$M_h = \frac{h}{7{,}000}.$$

The standard errors of the final ϕ and ω can be expressed as follows (see [12]):

$$M_{\phi_1} = M_{\phi_2} = \frac{\mu h}{b}\sqrt{\frac{3}{2b^2} + \frac{1}{2d^2}}$$

$$M_{\omega_1} = \mu h\sqrt{\frac{2}{3b^4} + \frac{3}{4d^4} + \frac{1}{2b^2d^2}}$$

$$M_{\omega_2} = \mu\,\frac{h}{b}\sqrt{\frac{2}{3b^2} + \frac{1}{2d^2}}.$$

For wide-angle cameras and 60% overlap, $\mu = 0.01$ mm. and $f = 150$ mm., we find that the standard errors of the final angles ϕ and ω are around 1′. It has to be noted, however, that this figure refers to the accuracy of the relative and absolute orientation only, and that the errors of the inner orientation may play a certain part. Particularly systematic asymmetries of the pencils of rays in the camera are dangerous. Also the rotation of the pictures under the mirror stereoscope has to be performed with high accuracy.

Anyhow, it is obvious that a very good accuracy can be obtained in the determination of certain data in aerial photogrammetry with the aid of the principles of numerical corrections, even if very simple instruments and methods are used. Also in the photogrammetric education and training, these principles may have considerable interest.