System-Testing Cinetheodolites by Stars[*†]

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ABSTRACT: *A new method of system-testing a Mk 5 cinetheodolite was initiated and completed at the Naval Proving Ground, Dahlgren, Virginia, during the period August–October, 1955.*

The method is based on the photogrammetric reduction of data obtained from star exposures made at regular angular intervals, at known chronometer times, and at a known geographic position. Mechanical precision of the rotation axes and the accuracy of direction angles are determined with these data.

The vertical errors are determined from exposures made at regular intervals in a vertical plane having a fixed azimuth, while horizontal errors are determined from exposures made at regular intervals in azimuth having a fixed vertical angle.

The results obtained are given at the end of the report. These data, because of the errors of timing, are considered correct to plus-or-minus 3 seconds of arc. The errors of timing shall be reduced to plus-or-minus 0.2 seconds of arc in subsequent tests.

It is concluded from the results of the test that the Mk 5 is a satisfactory ballistic cinetheodolite, except for an apparent camera-tube-bending effect on elevation readings, and that the method of testing provides a reliable knowledge of the instrument's static errors.

SYNOPSIS

THE cinetheodolite is a particular type of goniometric camera adapted to recording phenomena in motion automatically. A goniometric camera is one that is capable of rotation about two mutually orthogonal axes that are equipped with graduated circles, whereby the direction angles of an exposure may be visually read or photographed.

The space coordinates of an object may be determined with the camera coordinates of the conjugate images, the direction angles of the camera optical axes, and the ground coordinates of the two corresponding camera stations from which the object has been simultaneously photographed. The accuracy of the space coordinates so obtained depends on the accuracy of the given data. Of particular interest here is the accuracy of the goniometric and camera data, or specifically, the given mechanical direction angles and the camera coordinates (x, y, f) of an image.

The *accuracy of the direction angles* is dependent on:

(*1*) The mechanical precision of the goniometric axes (horizontal and vertical axes).

* Statements and opinions advanced in this paper are to be understood as individual expressions of the author and not necessarily those of the Navy Department.

† This is one of the papers obtained by the U. S. Reporter for Commission V of the International Society of Photogrammetry.

(*2*) Centering of the circles with respect to these axes (circle eccentricity).

(*3*) Orientation of reference plane and point.

(*4*) Orthogonality of the axes with respect to each other.

(*5*) Vertical deviation of the telescope tube caused by gravity alone.

(*6*) Dynamic deviations caused by constant and accelerated movements in tracking.

(*7*) Deviations caused by differential expansion and contraction of components arising from temperature change.

Errors arising from (*1*), (*2*), (*3*), (*4*), and (*5*) may be determined from star exposure data. The plate bubbles and vertical index constitute the horizontal reference plane and point.

The *accuracy of camera coordinate data* is dependent on:

(*1*) Magnitude and direction of image displacement owing to aberration characteristics of the lens.

(*2*) Focal length accuracy.

(*3*) Coincidence of the fiducial axes intersection with the point where the optical axis pierces the focal plane.

(*4*) Lens system inclination.

(*5*) Lens element decentering.

(*6*) Film flatness.

(*7*) Film distortion.

Because of the long nominal focal length of 600 millimeters and the narrow cone angle of 3° by 2° errors related to (*4*), (*6*), and (*7*) are trivial if they fall within the range of similar errors found in most metrical cameras. Errors arising from (*7*) are independent of the optical mechanical system. Tests of comparable film exhibit maximum image displacement owing to film distortion of 11 microns at the outer edges and 5 microns in the region of most-used imagery.

In fact, most of the camera data errors are trivial compared to the mechanical errors caused by the length-cone-angle geometry of the camera. Perhaps the most significant of the camera errors are those defined by (*2*). The unimportance of camera errors is contrary to conventional photogrammetric applications where cameras having short focal lengths and wide cone angles are the practice. The difference in significance arises primarily from the fact that ballistic personnel are interested in the coordinates of a rapidly moving object or point, while photogrammetrists are concerned generally in the three-dimensional delineation of a space containing an infinite number of static objects.

In any case, errors defined by (*2*) shall be determined by star exposures insofar as (*2*) is significant, and (*1*) and (*3*) may be ignored for small cone angles.

The bulk of photogrammetric effort is directed toward the delineation of an undulating object surface and is accomplished largely with various types of stereoplotting instruments. This practice has led to an opposition to an analytical approach to the problem on the premise that it is time-consuming, complicated, and impractical. The bulk of the ballistic and missile effort is toward the definition of a series of space coordinates describing the path of a rapidly moving object, and is accomplished by analytical means with the aid of automatic computers. The instrumental approach is in general too slow, and inaccurate. Thus, independent of the photogrammetrists, mathematicians and physicists have developed practical procedures and methods of reducing large quantities of data by analytical means largely owing to the photogrammetrist's inherent opposition to the analytical approach.

It is clear, then, that the method of handling photogrammetric data is dependent on the nature of the problem, the background of the investigators, and the available tools for reducing the data.

If the photogrammetrist is to properly serve his purpose, the photographic methods and procedures employed in the analysis of ballistic data should be encompassed by him in order that the methods and procedures of photographic ballistics and photogrammetry may be exchanged, evaluated, and exploited for advancement of the science of photographic measurements.

There are three approaches to the problem of obtaining accurate data with goniometric photographic recorders of static or moving phenomena.

One approach is to build accuracy into all components of the instrument. This results in an extremely costly instrument that is wholly dependent on the manufacturer for adjustment. This is the general practice in Europe.

Another approach is to include provisions for a series of adjustments where, with a suitable test range, the errors may be adjusted out from time to time. This is time-consuming where observations are being made from day to day in large quantities. This is the general practice of surveyors in this country.

A third approach is to build precision into a minimum number of components, such as the horizontal and vertical axes; then to determine the remaining errors by calibration and to enter them as constants in the analytical reduction of the coordinates of a point. This is a logical outgrowth of data reduction by automatic computers; it, therefore, is the approach of the U. S. Naval Proving Ground in system-testing by stars.

System-Testing By Stars

A. INTRODUCTION

Generally, it is highly objectionable to employ one science requiring a special set of experiences, formulae, and nomenclature to evaluate a precise complex instrument relative to another science that also requires a special set of experiences, formulae, and nomenclature. The specific reference here is the testing of a cinetheodolite for static errors by astronomic methods. Three considerations are offered to overcome the basic objections:

(1) Any system developed and component-tested in the laboratory should be system-tested in the field if the grossness of the field errors does not camouflage the real errors of the system;

(2) Observers or operators of the instrument referred to will logically be more familiar with astronomic methods than optical bench methods; and

(3) The operators have an evaluation tool that frees them from a dependence on laboratory procedures and equipment.

The following errors may be determined from a series of exposures made at regular intervals in a vertical plane.

1. Eccentricity of the vertical circle.
2. Inclination of the horizontal axis (I_X).
3. Inclination of the camera optical axis (I_0).
4. Index error of the vertical circle.
5. Precision of the horizontal axis (X axis).

The geometry of the observing procedure is illustrated in Figure 1. Assume that the exposures are made at a fixed azimuth with vertical angles of 30°, 60°, 90°, 120°, and 150° in accordance with Figure 1.

The following errors may be determined from a series of exposures made at regular intervals in azimuth, with the camera clamped at some fixed vertical angle, say 60°.

1. Eccentricity of the horizontal circle.
2. Inclination of the vertical axis (I_Z).
3. Precision of the vertical axis (Z axis).

The geometry of the observing procedure is illustrated in Figure 2. Assume that exposures are made at a fixed zenith angle with azimuth angles of 0°, 60°, 120°, 180°, 240°, and 300°.

In making the exposures in a vertical plane, each of the zenith angles is preset on the zenith circle with the instrument leveled. Each exposure is referenced to radio time signals at opening and closing. The time interval of exposure de-

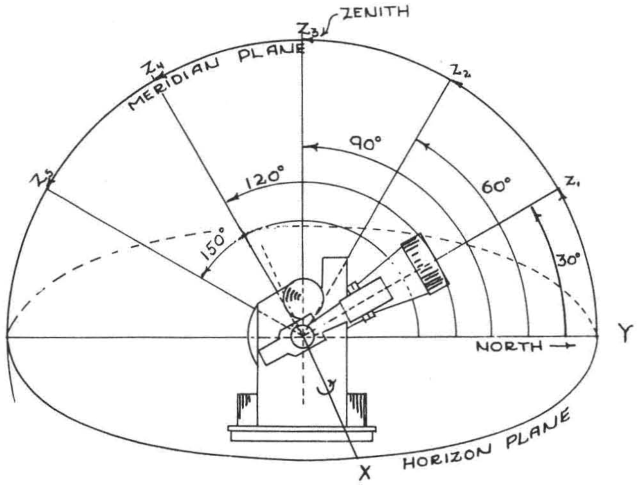

FIG. 1. Geometry of vertical observing procedure.

pends on the relative aperture and focal length. Three stars are identified on each exposure defining a near-equilateral triangle and equal radial distances from the fiducial axes intersection.

The stars are identified by comparison of the negative with a suitable star chart. The approximate right ascension and declination of the principal point are required for each exposure to facilitate positioning the negative on the star

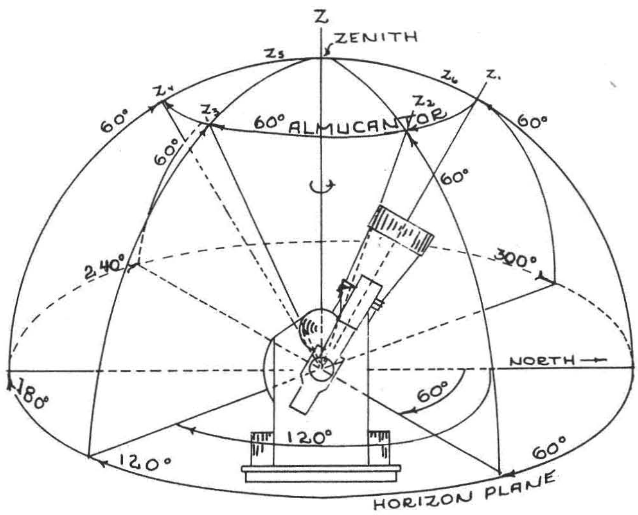

FIG. 2. Geometry of horizontal observing procedure.

chart. The preset azimuth (Az), vertical angle (h), and radio times converted to sidereal time are employed in these preliminary reductions:

1. *Right Ascension* (RA). The right ascension of a star is the angle measured eastward in the plane of the celestial equator from the celestial meridian passing through the vernal equinox to the celestial meridian passing through the star.

$$RA = LST - t \text{ (time)}$$

where

LST is the right ascension of the observer's meridian and is measured in the same manner.

t is the hour angle of the star.

The hour angle of the star is the angle measured westward in the plane of the celestial equator from the observer's meridian to the meridian passing through the star. The hour angle, like the right ascension, may vary from 0° to 360°, or 0^h to 24^h sidereal time. The hour angle is computed with the cosine formula:

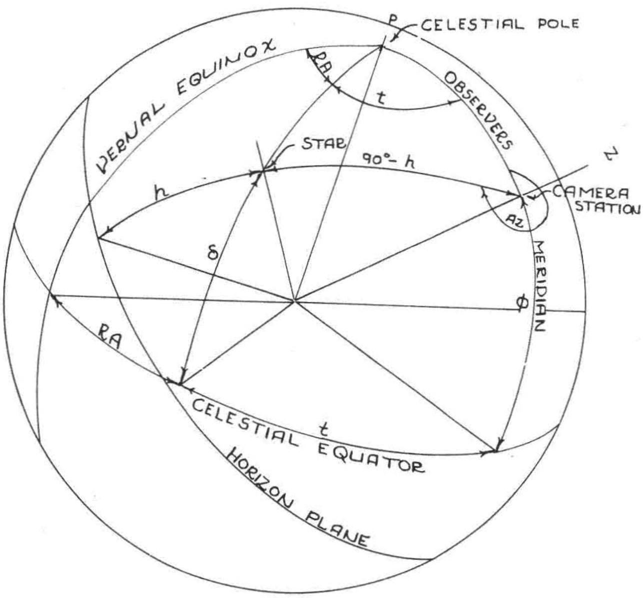

FIG. 3. Astronomic notation.

$$\cos t = \frac{\sin h - \sin \delta \sin \phi}{\cos \delta \cos \phi}$$

where

h = altitude of star,

δ = declination of star, and

ϕ = latitude of observer.

The declination of a star is the angle measured in the plane of the star's meridian above or below the celestial equator. The latitude of the observer is the angle measured in the plane of the observer's meridian above or below the equator. These various notations and the equation for cos t are illustrated in Figure 3.

If the numerator is negative, t is between 90° and 270°. If the observed azimuth is between 0° and 180°, $+\cos t$ is between 270° and 360° and $-\cos t$ is between 180° and 270°. If the observed azimuth is between 180° and 360°, $+\cos t$ is between 0° and 90° and $-\cos t$ is between 90° and 180°. The sign convention is shown in the following table.

SIGN CONVENTION OF t

Az	$+\cos t$	$-\cos t$
0° to 180°	18^h to 24^h	12^h to 18^h
180° to 360°	0^h to 6^h	6^h to 12^h

Then

$$LST = (\text{Radio time} + \text{time zone}) + GST\ 0^h + \text{sol.-sid. corr.} - \lambda\ (\text{time})$$

$$GCT = (\text{Radio time} + \text{time zone})$$

The date of (Radio time+time zone) $= GCT$ is the argument for obtaining $GST\ 0^h$, and is also the argument for obtaining solar to sidereal correction from

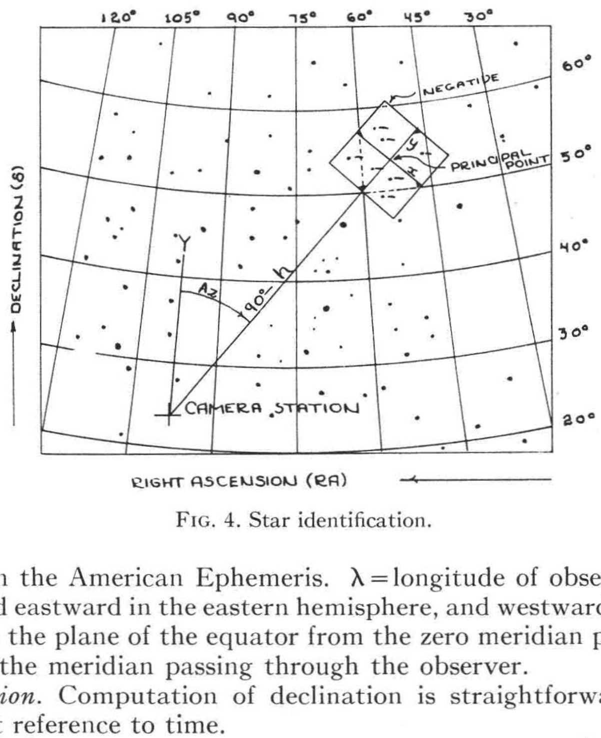

FIG. 4. Star identification.

tables found in the American Ephemeris. λ = longitude of observer and is the angle measured eastward in the eastern hemisphere, and westward in the western hemisphere, in the plane of the equator from the zero meridian passing through Greenwich to the meridian passing through the observer.

2. *Declination.* Computation of declination is straightforward and is obtained without reference to time.

$$\sin \delta = \sin h \sin \phi + \cos h \cos \phi \cos Az$$

Sin δ is negative when the Az is between 90° and 270° and the product $\cos h \cos \phi \cos Az$ is greater than $\sin h \sin \phi$.

Assume, now, that the celestial coordinates of the principal point have been determined. The celestial coordinates of the observer and the principal point are plotted on a star chart. The negative is positioned over the point corresponding to the celestial coordinates of the principal point, and the y fiducial line of the negative oriented on the line drawn from the principal point to the observer's position. This is illustrated in Figure 4.

The coordinates of the observer corresponding to RA and δ are LST and ϕ. The line connecting the plotted position of the principal point and the observer should define an angle with the meridian passing through the observer equal to the observed azimuth, approximately. The star images selected for definition and geometric distribution are easily identified when the position and orientation of the negative on the star chart are established.

After the stars are identified, the camera coordinates are measured referred

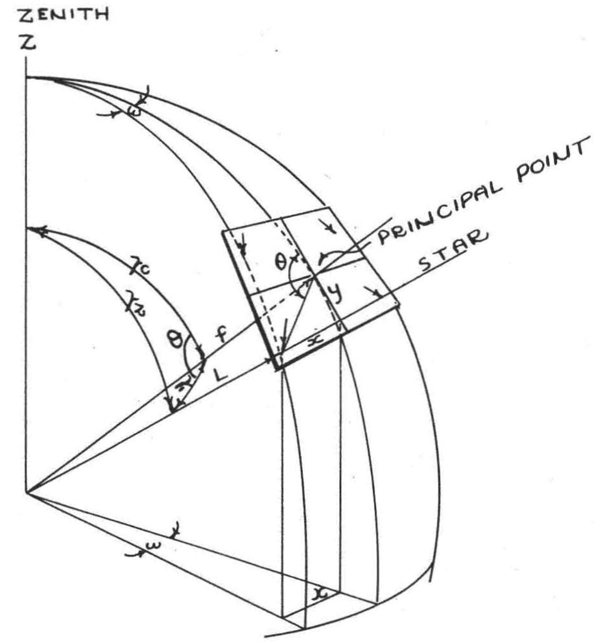

FIG. 5. Geometry of refraction on an oblique exposure.

to the fiducial axes, and the apparent places of the stars for the instant of exposure are computed. The terminals of the star trails are measured in x and y. The mean time of opening and closing is used in the apparent places computation and the mean coordinates of the terminals in photogrammetric computations.

The coordinates of any point are:

$$\frac{xb_1 + xe_1}{2} - xp = x_1$$

$$\frac{yb_1 + ye_1}{2} - yp = y_1$$

where b and e denote beginning and ending, and p, the principal point.

The x and y values are corrected for refraction. The correction is applied so as to increase the absolute values of x and translate all y values away from the zenith.

The equations for the refraction corrections are derived from the cosine formula illustrated in Figure 5.

$$\cos \gamma_z = \cos \gamma_c \cos \gamma_1 - \sin \gamma_c \sin \gamma_1 \cos \theta_1$$

where

γ_z = zenith angle of image,

γ_c = camera zenith angle $(90° - h_0)$,

γ_1 = camera zenith angle of image referred to lens axis, and

θ_1 = the angle in the plane of the film at the principal point between the vertical plane defined by the observer's zenith and the principal point, and the plane defined by the star image and the principal point.

The camera angle γ_1 is obtained with camera data.

$$\cos \gamma_1 = \frac{f}{(x_1^2 + y_1^2 + f^2)^{1/2}}$$

$$\sin \gamma_1 = \frac{(x_1^2 + y_1^2)^{1/2}}{(x_1^2 + y_1^2 + f^2)^{1/2}}$$

$$\cos \theta_1 = \frac{y_1}{(x_1^2 + y_1^2)^{1/2}}$$

where f is the focal length of the camera.

Substituting,

$$\cos \gamma_z (x_1^2 + y_1^2 + f^2)^{1/2} = \cos \gamma_c f - \sin \gamma_c y$$

Equating for y,

$$y = \frac{\cos \gamma_c f - \cos \gamma_z L}{\sin \gamma_c} = \cos \gamma_c f - \frac{\cos \gamma_z L}{\sin \gamma_c}$$

Differentiating y with respect to γ_z,

$$dy = \sin \gamma_z \cdot \frac{L}{\sin \gamma_c} \cdot d\gamma_z$$

Now,

$$d\gamma_c = \tan \gamma_z \left(\frac{983\ B}{460 + T} \right) \tan 1''$$

where

B = barometric pressure in inches, and

T = temperature in Fahrenheit degrees.

Therefore,

$$dy = \frac{\sin^2 \gamma_z}{\cos \gamma_z} \cdot \frac{L}{\sin \gamma_c} \cdot K \qquad (1)$$

where

$$K = \left(\frac{983\ B}{460 + T} \right) \tan 1''$$

The correction in x is obtained from the differentiation of an expression in x:

$$x = L \sin \gamma_z \sin \omega$$

$$dx = L \cos \gamma_z \sin \omega \, d\gamma_z = L \sin \omega \sin \gamma_z K$$

Since

$$x = L \sin \gamma_z \sin \omega,$$

$$dx = x \cdot K \qquad (2)$$

Summarizing,

$$dy = \frac{1 - \cos^2 \gamma_z}{\cos \gamma_z} \cdot \frac{L}{\sin \gamma_z} \cdot K$$

where

$$\cos \gamma_c = \cos \gamma_c \frac{f}{L} - \sin \gamma_c \frac{\nu}{L}$$

$$L = (x^2 + y^2 + f^2)^{1/2}$$

$$K = \left(\frac{983\ B}{460 + T}\right) \tan 1''$$

$$dx = x \cdot K$$

The mean places of the selected star images are computed with the standard equations:

$$\lambda_0 = \lambda_m + (t_0 - 1950)AV_\lambda + \frac{1}{2}(t_0 - 1950)^2\left(\frac{1}{100}\right)SV_\lambda + \left(\frac{t_0 - 1950}{100}\right)^3 3\mathrm{rd}t_\lambda$$

$$\delta_0 = \delta_m + (t_0 - 1950)AV_\delta + \frac{1}{2}(t_0 - 1950)^2\left(\frac{1}{100}\right)SV_\delta + \left(\frac{t_0 - 1950}{100}\right)^3 3\mathrm{rd}t_\delta$$

where

λ_m, δ_m = values of right ascension and declination obtained from the star catalogue.

t_0 = year of the observation.

AV, SV = annual and secular variations in right ascension and declination found in the Boss, Yale, or AGK_2 star catalogues.

3rd$t$ = third order terms in right ascension and declination also found in the star catalogues.

The apparent places for the instant of observation may be computed with Independent star numbers.

$$\lambda = \lambda_0 + f + f' + \tau\mu + \frac{1}{15} g \sin (G + \lambda_0) \tan \delta_0 + \frac{1}{15} h \sin (H + \lambda_0) \sec \delta_0$$

$$\delta = \delta_0 + \tau\mu' + g \cos (G + \lambda_0) + h \cos (H + \lambda_0) \sin \delta_0 + i \cos \delta_0$$

where

$$h = \frac{C}{\sin H}$$

$$g = \frac{B}{\sin G}$$

λ, δ = right ascension and declination for the instant of observation.

τ = fractional part of the year elapsed from the beginning of the Besselian fictitious year.

μ, μ' = proper motions, obtained from the star catalogues.

The values τ, f, f', G, H, i, B, and C are found in the American Ephemeris. All subsequent factors are determined with this fundamental data.

B. CALIBRATION DATA

1. *Procedure for Cones of 10 Degrees or Greater.* The focal length and principal point location are computed with the zenith exposures insofar as they have the least refraction and the most symmetrical data with respect to the principal point. The equations for correcting the images for refraction apply to all but the zenith exposures. The refraction corrections for zenith exposures are derived from a tangent formula:

$$\tan \gamma = \frac{(x^2 + y^2)^{1/2}}{f} = \frac{R}{f}$$

$$\tan \gamma f = R$$

$$f \sec^2 \gamma d\gamma = dR$$

But, since

$$R \sin \theta = x \quad \text{and} \quad R \cos \theta = y,$$

$$dR \sin \theta = dx \quad \text{and} \quad dR \cos \theta = dy$$

or

$$f \sin \theta \sec^2 \gamma d\gamma = dx$$

and

$$f \cos \theta \sec^2 \gamma d\gamma = dy$$

Now

$$d\gamma = \tan \gamma \left(\frac{983B}{460 + T} \right) \tan 1'' = \tan \gamma K$$

Therefore,

$$f \sin \theta \sec^2 \gamma \tan \gamma K = dx$$

$$f \cos \theta \sec^2 \gamma \tan \gamma K = dy$$

or

$$\frac{xK}{\cos^2 \gamma} = dx$$

$$\frac{yK}{\cos^2 \gamma} = dy$$

The reduced coordinates and angles subtended by pairs of images comprise the given data. The angles subtended by three images a, b, and c are computed with trigonometric functions of right ascension and declination.

$$\cos aLb = \sin \delta_a \sin \delta_b + \cos \delta_a \cos \delta_b \cos (\lambda_a - \lambda_b)$$

$$\cos bLc = \sin \delta_b \sin \delta_c + \cos \delta_b \cos \delta_c \cos (\lambda_b - \lambda_c)$$

$$\cos cLa = \sin \delta_c \sin \delta_a + \cos \delta_c \cos \delta_a \cos (\lambda_c - \lambda_a)$$

With these data, the following equations are formed:

$$a_1\Delta x + b_1\Delta y + c_1\Delta z = q_1$$
$$a_2\Delta x + b_2\Delta y + c_2\Delta z = q_2$$
$$a_3\Delta x + b_3\Delta y + c_3\Delta z = q_3$$

where

$$a_1 = xam_1 + xbn_1 \qquad b_1 = yam_1 + ybn_1 \qquad c_1 = f'(m_1 + n_1)$$
$$a_2 = xbm_2 + xcn_2 \qquad b_2 = ybm_2 + ycn_2 \qquad c_2 = f'(m_2 + n_2)$$
$$a_3 = xcm_3 + xan_3 \qquad b_3 = ycm_3 + yan_3 \qquad c_3 = f'(m_3 + n_3)$$

$$La = (xa^2 + ya^2 + f^2)^{1/2} \qquad q_1 = -\ LaLb \sin aLb'\Delta(aLb)$$
$$Lb = (xb^2 + yb^2 + f^2)^{1/2} \qquad q_2 = -\ LbLc \sin bLc'\Delta(bLc)$$
$$Lc = (xc^2 + yc^2 + f^2)^{1/2} \qquad q_3 = -\ LcLa \sin cLa'\Delta(cLa)$$

$$m_1 = 1 - \cos aLb' \frac{Lb}{La} \qquad n_1 = 1 - \cos aLb' \frac{La}{Lb}$$
$$m_2 = 1 - \cos bLc' \frac{Lc}{Lb} \qquad n_2 = 1 - \cos bLc' \frac{Lb}{Lc}$$
$$m_3 = 1 - \cos cLa' \frac{La}{Lc} \qquad n_3 = 1 - \cos cLa' \frac{Lc}{La}$$

$$\cos aLb' = \frac{xaxb + yayb + f^2}{LaLb}$$
$$\cos bLc' = \frac{xbxc + ybyc + f^2}{LbLc}$$
$$\cos cLa' = \frac{xcxa + ycya + f^2}{LcLa}$$

$$\Delta(aLb) = aLb - aLb$$
$$\Delta(bLc) = bLc - bLc'$$
$$\Delta(cLa) = cLa - cLa'$$

and f' is the first approximation of f.

The equations are solved simultaneously for Δx, Δy, and Δz, which are applied as corrections to the x and y coordinates and f'. With the corrected values, new coefficients and constant terms are formed and a second set of simultaneous equations is solved for Δx_2, Δy_2, and Δz_2. The forming of revised coefficients and constant terms coupled with repeated simultaneous solutions is repeated until the differential unknowns and constant terms vanish.

Then, for any image a,

$$xa_c = xa + \sum \Delta x$$
$$ya_c = ya + \sum \Delta y$$
$$f = f' + \sum \Delta z$$

Over three approximations are seldom required. These values of f, $\Sigma\Delta x$ and $\Sigma\Delta y$ are employed in all other exposures. The value of $\Sigma\Delta z$ is a direct measure of the accuracy of the focal plane setting, while $\Sigma\Delta x$ and $\Sigma\Delta y$ are a measure of the accuracy with which the principal point was made to coincide with the fiducial axes intersection.

2. *Procedure for Cones of Less Than 10 Degrees.* The spatial resection method of determining camera calibration data described in the previous section fails when the spatial resection angles are very small. Small cone angles of 5 degrees or less yield extremely small resection angles which in turn produce large computational errors in the coordinates of the principal point. Fortunately, the precise location of the principal point is not significant for small cone angles. For this reason, a method of modified dependencies is introduced for the determination of focal length only. The method of modified dependencies treats the given coordinates of the principal point as correct.

The simultaneous equations are developed from the three general forms:

α form

$$\cos \alpha_1 \cos \alpha_x + \cos \beta_1 \cos \alpha_y + \cos \gamma_1 \cos \alpha_z = \cos \alpha_{s_1}$$
$$\cos \alpha_2 \cos \alpha_x + \cos \beta_2 \cos \alpha_y + \cos \gamma_2 \cos \alpha_z = \cos \alpha_{s_2}$$
$$\cos \alpha_3 \cos \alpha_x + \cos \beta_3 \cos \alpha_y + \cos \gamma_3 \cos \alpha_z = \cos \alpha_{s_3}$$

β form

$$\cos \alpha_1 \cos \beta_x + \cos \beta_1 \cos \beta_y + \cos \gamma_1 \cos \beta_z = \cos \beta_{s_1}$$
$$\cos \alpha_2 \cos \beta_x + \cos \beta_2 \cos \beta_y + \cos \gamma_2 \cos \beta_z = \cos \beta_{s_2}$$
$$\cos \alpha_3 \cos \beta_x + \cos \beta_3 \cos \beta_y + \cos \gamma_3 \cos \beta_z = \cos \beta_{s_3}$$

γ form

$$\cos \alpha_1 \cos \gamma_x + \cos \beta_1 \cos \gamma_y + \cos \gamma_1 \cos \gamma_z = \cos \gamma_{s_1}$$
$$\cos \alpha_2 \cos \gamma_x + \cos \beta_2 \cos \gamma_y + \cos \gamma_2 \cos \gamma_z = \cos \gamma_{s_2}$$
$$\cos \alpha_3 \cos \gamma_x + \cos \beta_3 \cos \gamma_y + \cos \gamma_3 \cos \gamma_z = \cos \gamma_{s_3}$$

where

$$\cos \alpha_n = \frac{x_n}{(x_n^2 + y_n^2 + f^2)^{1/2}}$$

$$\cos \beta_n = \frac{y_n}{(x_n^2 + y_n^2 + f^2)^{1/2}}$$

$$\cos \gamma_n = \frac{f}{(x_n^2 + y_n^2 + f^2)^{1/2}}$$

$$\cos \alpha_{s_n} = \sin GHA_n \cos \delta_n$$
$$\cos \beta_{s_n} = \cos GHA_n \cos \delta_n$$
$$\cos \gamma_{s_n} = \sin \delta_n$$

Division of each equation by $\cos \gamma_n$ gives a modified form:

$$x_1 a_x + y_1 a_y + \cos\alpha_z = \cos\alpha_{s_1} \sec\gamma_1$$
$$x_2 a_x + y_2 a_y + \cos\alpha_z = \cos\alpha_{s_2} \sec\gamma_2$$
$$x_3 a_x + y_3 a_y + \cos\alpha_z = \cos\alpha_{s_3} \sec\gamma_3$$

$$x_1 b_x + y_1 b_y + \cos\beta_z = \cos\beta_{s_1} \sec\gamma_1$$
$$x_2 b_x + y_2 b_y + \cos\beta_z = \cos\beta_{s_2} \sec\gamma_2$$
$$x_3 b_x + y_3 b_y + \cos\beta_z = \cos\beta_{s_3} \sec\gamma_3$$

$$x_1 c_x + y_1 c_y + \cos\gamma_z = \cos\gamma_{s_1} \sec\gamma_1$$
$$x_2 c_x + y_2 c_y + \cos\gamma_z = \cos\gamma_{s_2} \sec\gamma_2$$
$$x_3 c_x + y_3 c_y + \cos\gamma_z = \cos\gamma_{s_3} \sec\gamma_3$$

The unknowns are a_x, a_y, and $\cos\alpha_z$ in the first group; b_x, b_y, and $\cos\beta_z$ in the second group; and c_x, c_y, and $\cos\gamma_z$ in the third group. Only the quantities $\cos\alpha_z$, $\cos\beta_z$, and $\cos\gamma_z$, are needed and these quantities are in error as a function of the error in the assumed f in the formation of the sec γ values.

$$x_1\begin{cases} a_x \\ b_x \\ c_x \end{cases} + y_1\begin{cases} a_y \\ b_y \\ c_y \end{cases} + 1\begin{cases} \cos\alpha_z \\ \cos\beta_z \\ \cos\gamma_z \end{cases} = \sec\gamma_1\begin{cases} \cos\alpha_{s_1} \\ \cos\beta_{s_1} \\ \cos\gamma_{s_1} \end{cases}$$

$$x_2\begin{cases} a_x \\ b_x \\ c_x \end{cases} + y_2\begin{cases} a_y \\ b_y \\ c_y \end{cases} + 1\begin{cases} \cos\alpha_z \\ \cos\beta_z \\ \cos\gamma_z \end{cases} = \sec\gamma_2\begin{cases} \cos\alpha_{s_2} \\ \cos\beta_{s_2} \\ \cos\gamma_{s_2} \end{cases}$$

$$x_3\begin{cases} a_x \\ b_x \\ c_x \end{cases} + y_3\begin{cases} a_y \\ b_y \\ c_y \end{cases} + 1\begin{cases} \cos\alpha_z \\ \cos\beta_z \\ \cos\gamma_z \end{cases} = \sec\gamma_3\begin{cases} \cos\alpha_{s_3} \\ \cos\beta_{s_3} \\ \cos\gamma_{s_3} \end{cases}$$

If the assumed f value were correct, the corresponding values of $\cos\alpha_z$, $\cos\beta_z$, and $\cos\gamma_z$ would be equally correct. In which case, the following equation would be satisfied:

$$\cos^2\alpha_z + \cos^2\beta_z + \cos^2\gamma_z = 1.$$

As a consequence of errors in f,

$$\cos^2\alpha_z + \cos^2\beta_z + \cos^2\gamma_z = \frac{1}{m^2}$$

and

$$m = \frac{1}{(\cos^2\alpha_z + \cos^2\beta_z + \cos^2\gamma_z)^{1/2}}$$

The direction cosines of z may be improved by the factor m.

$$\cos\alpha_z' = m\cos\alpha_z$$
$$\cos\beta_z' = m\cos\beta_z$$
$$\cos\gamma_z' = m\cos\gamma_z$$

whereupon

$$\cos^2\alpha_z' + \cos^2\beta_z' + \cos^2\gamma_z' = 1$$

With adjusted direction cosines, improved sec γ values may be computed:

$$\cos\alpha_{s_1}\cos\alpha_z' + \cos\beta_{s_1}\cos\beta_z' + \cos\gamma_{s_1}\cos\gamma_z' = \cos\gamma_1'$$

$$\cos\alpha_{s_2}\cos\alpha_z' + \cos\beta_{s_2}\cos\beta_z' + \cos\gamma_{s_2}\cos\gamma_z' = \cos\gamma_2'$$

$$\cos\alpha_{s_3}\cos\alpha_z' + \cos\beta_{s_3}\cos\beta_z' + \cos\gamma_{s_3}\cos\gamma_z' = \cos\gamma_3'$$

The entire solution is repeated with revised sec γ values and the "sum-of-the-squares" test repeated until

$$\cos^2\alpha_{z_n} + \cos^2\beta_{z_n} + \cos^2\gamma_{z_n} = 1$$

The sec γ values producing these correct z-direction-cosine values are regarded as having an implicitly correct f component. Therefore,

$$f_1 = \left(\frac{x_1^2 + y_1^2}{\tan^2\gamma_1}\right)^{1/2}$$

$$f_2 = \left(\frac{x_2^2 + y_2^2}{\tan^2\gamma_2}\right)^{1/2}$$

$$f_3 = \left(\frac{x_3^2 + y_3^2}{\tan^2\gamma_3}\right)^{1/2}$$

C. DIRECTION COSINES OF THE CAMERA Z AXIS

The direction cosines of the camera z axis are computed for each exposure.

$$\cos\alpha_{sa}\cos\alpha_z + \cos\beta_{sa}\cos\beta_z + \cos\gamma_{sa}\cos\gamma_z = \cos\gamma_a$$

$$\cos\alpha_{sb}\cos\alpha_z + \cos\beta_{sb}\cos\beta_z + \cos\gamma_{sb}\cos\gamma_z = \cos\gamma_b$$

$$\cos\alpha_{sc}\cos\alpha_z + \cos\beta_{sc}\cos\beta_z + \cos\gamma_{sc}\cos\gamma_z = \cos\gamma_c$$

where

$$\cos\alpha_{sa} = \sin GHA_a\cos\delta_a \qquad \cos\beta_{sa} = \cos GHA_a\cos\delta_a$$

$$\cos\alpha_{sb} = \sin GHA_b\cos\delta_b \qquad \cos\beta_{sb} = \cos GHA_b\cos\delta_b$$

$$\cos\alpha_{sc} = \sin GHA_c\cos\delta_c \qquad \cos\beta_{sc} = \cos GHA_c\cos\delta_c$$

$$GHA = (GCT + GST\ 0^h + GCT\ \text{corr.} - RA)15$$

[RA = right ascension = λ from the apparent place computation (in time).]

$$\cos\gamma_{sa} = \sin\delta_a \qquad \cos\gamma_a = \frac{f}{(xa^2 + ya^2 + f^2)^{1/2}}$$

$$\cos\gamma_{sb} = \sin\delta_b \qquad \cos\gamma_b = \frac{f}{(xb^2 + yb^2 + f^2)^{1/2}}$$

$$\cos\gamma_{sc} = \sin\delta_c \qquad \cos\gamma_c = \frac{f}{(x_c^2 + y_c^2 + f^2)^{1/2}}$$

These equations are solved simultaneously for $\cos\alpha_z$, $\cos\beta_z$, and $\cos\gamma_z$. The geometry of these equations is illustrated in Figure 6.

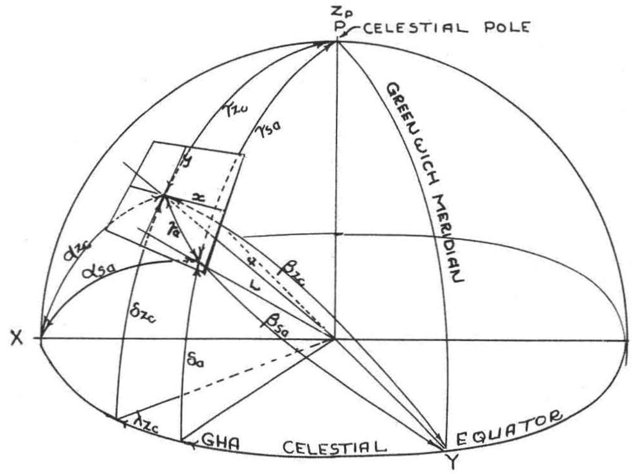

FIG. 6. Astronomic direction angles.

D. INCLINATION OF THE CAMERA OPTICAL AXIS TO THE X AXIS

If the optical axis of the camera is inclined to the X axis, the astronomic coordinates of each exposure define a secondary circle with respect to the X axis. A secondary circle is one which is perpendicular to a pole and parallel to an equator but does not pass through the center of a sphere. Great circles cannot define parallel planes. Secondary circles can. Thus, while the path traced by the exposure is a line with any orientation whatsoever with respect to the celestial coordinate system, it is a secondary circle which defines equal angles to the great circle that is perpendicular to the inclined X axis.

In other words, the arcs defined by the X axis produced to the celestial sphere and each principal point produced to the celestial sphere are equal. With this geometry in mind, we may write an equation for each exposure:

$$\cos\alpha_{z_1}\cos\alpha_X + \cos\beta_{z_1}\cos\beta_X + \cos\gamma_{z_1}\cos\gamma_X = \cos I_1$$

$$\cos\alpha_{z_2}\cos\alpha_X + \cos\beta_{z_2}\cos\beta_X + \cos\gamma_{z_2}\cos\gamma_X = \cos I_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cos\alpha_{z_n}\cos\alpha_X + \cos\beta_{z_n}\cos\beta_X + \cos\gamma_{z_n}\cos\gamma_X = \cos I_n$$

where α_X, β_X, γ_X are the direction angles of the X axis, and I is the angle defined by the camera z axis and the X axis. The following equality would exist if the X axis had no mechanical rotational error:

$$I_0 = \frac{I_1 + I_2 + \cdots + I_n}{n}$$

Let it be assumed that the Least Square value is I_0, which is the value corresponding to a perfectly precise horizontal, or X, axis. Division of the above equations by $\cos I_0$ gives

$$\cos\alpha_{z_1}v + \cos\beta_{z_1}\mu + \cos\gamma_{z_1}\nu = 1$$

$$\cos\alpha_{z2}v + \cos\beta_{z2}\mu + \cos\gamma_{z2}\nu = 1$$

$$\cos\alpha_{z3}v + \cos\beta_{z3}\mu + \cos\gamma_{z3}\nu = 1$$

where

$$v = \frac{\cos \alpha_X}{\cos I_0} \qquad \mu = \frac{\cos \beta_X}{\cos I_0} \qquad \nu = \frac{\cos \gamma_X}{\cos I_0}$$

Normal equations are formed in v, μ, and ν:

$$[\cos \alpha_z \cos \alpha_z]v + [\cos \alpha_z \cos \beta_z]\mu + [\cos \alpha_z \cos \gamma_z]\nu = [\cos \alpha_z]$$
$$[\cos \alpha_z \cos \beta_z]v + [\cos \beta_z \cos \beta_z]\mu + [\cos \beta_z \cos \gamma_z]\nu = [\cos \beta_z]$$
$$[\cos \alpha_z \cos \gamma_z]v + [\cos \gamma_z \cos \beta_z]\mu + [\cos \gamma_z \cos \gamma_z]\nu = [\cos \gamma_z]$$

These equations are solved simultaneously for v, μ, and ν.

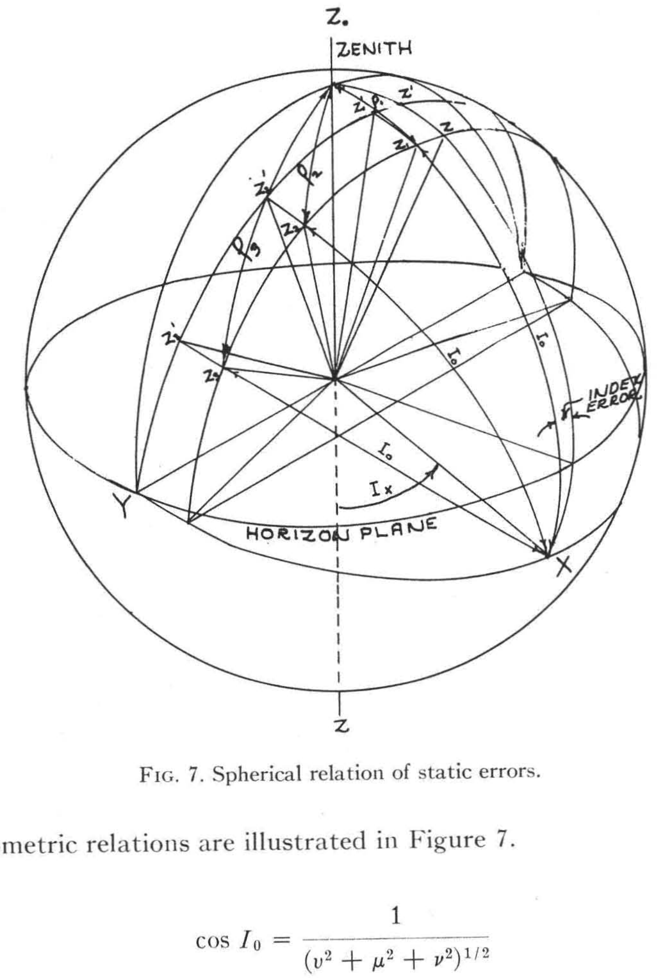

FIG. 7. Spherical relation of static errors.

The geometric relations are illustrated in Figure 7.
Then

$$\cos I_0 = \frac{1}{(v^2 + \mu^2 + \nu^2)^{1/2}}$$
$$\cos \alpha_X = \cos I_0 v$$
$$\cos \beta_X = \cos I_0 \mu$$
$$\cos \gamma_X = \cos I_0 \nu$$

I_0 is the inclination of the camera optical axis to the X axis.

E. ASYMMETRY OF THE X AXIS

Assume there is asymmetry in the X axis. cos. α_X, cos β_X, and cos γ_X are substituted back in the original observation equations to obtain $I_1, I_2, \cdots, I_n$. Then the angular measure of asymmetry in the X axis follows:

$$\Delta\gamma_{X_1} = I_1 - I_0$$

$$\Delta\gamma_{X_2} = I_2 - I_0$$

$$\cdots\cdots\cdots$$

$$\Delta\gamma_{X_n} = I_n = I_0$$

These values are plotted in arc seconds in Y against degrees of zenith angle in X for a graphical representation of the angular asymmetry in the horizontal axis. A plot is shown in Figure 8.

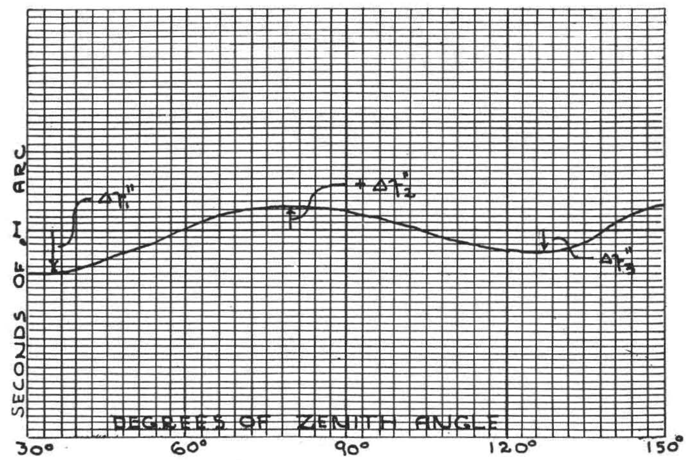

FIG. 8. X axis asymmetry.

F. INCLINATION OF THE X AXIS

1. With Respect to the Vertical. It is assumed that the exposures are made at known geographic coordinates. Let the direction angles of the observer be

$$\alpha_0$$

$$\beta_0$$

$$\gamma_0$$

where

$$\cos \alpha_0 = \sin \lambda \cos \phi$$

$$\cos \beta_0 = \cos \lambda \cos \phi$$

$$\cos \gamma_0 = \sin \phi$$

The inclination of the X axis, I_X, is defined by the vertical and the X axis.

$$\cos \alpha_0 \cos \alpha_X + \cos \beta_0 \cos \beta_X + \cos \gamma_0 \cos \gamma_X = \cos I_X$$

2. With Respect to the Z Axis. Inclination of the X axis to the Z axis is frequently referred to as trunnion tilt, or the lack of orthogonality between the

X and Z axes. The direction cosines obtained by the method of Least Squares for each axis are combined to obtain I_{XZ}:

$$\cos \alpha_X \cos \alpha_Z + \cos \beta_X \cos \beta_Z + \cos \gamma_X \cos \gamma_Z = \cos I_{XZ}$$

G. ECCENTRICITY OF THE VERTICAL CIRCLE, AND INDEX ERROR

The arcs defined by $z_1, z_2, \cdots, z_n$ are computed. Initially,

$$\cos \alpha_0 \cos \alpha_{z_1} + \cos \beta_0 \cos \beta_{z_1} + \cos \gamma_0 \cos \gamma_{z_1} = \cos \rho_1$$

$$\cos \alpha_0 \cos \alpha_{z_2} + \cos \beta_0 \cos \beta_{z_2} + \cos \gamma_0 \cos \gamma_{z_2} = \cos \rho_2$$

$$\cdots\cdots\cdots$$

$$\cos \alpha_0 \cos \alpha_{z_n} + \cos \beta_0 \cos \beta_{z_n} + \cos \gamma_0 \cos \gamma_{z_n} = \cos \rho_n$$

Then the true arcs generated in a plane normal to the X axis may be computed.

$$\cos z'z_1' = \frac{\cos \rho_1 - \cos I_X \cos I_0}{\sin I_X \sin I_0}$$

$$\cos z'z_2' = \frac{\cos \rho_2 - \cos I_X \cos I_0}{\sin I_X \sin I_0}$$

$$\cdots\cdots\cdots$$

$$\cos z'z_n' = \frac{\cos \rho_n - \cos I_X \cos I_0}{\sin I_X \sin I_0}$$

$z'z_1'$ is the index error, and the true arcs generated by the vertical circle are

$$z'z_2' - z'z_1' = \theta_1 + e_1 = \eta_1$$

$$z'z_3' - z'z_1' = \theta_2 + e_2 = \eta_2$$

$$\cdots\cdots\cdots$$

$$z'z_n' - z'z_1' = \theta_{n-1} + e_{n-1} = \eta_{n-1}$$

where the e values are the errors of circle eccentricity.

CIRCLE ECCENTRICITY

Circle eccentricity is defined as the situation where the center of a graduated circle does not coincide with the mechanical axis of rotation that is normal to the plane of the circle. Circle eccentricity causes errors in the observed angle that vary as the sine of the angle referred to the line defined by the circle and mechanical centers. The mean of double-reading circles cancels out the errors caused by circle eccentricity. It may be seen from Figure 9,

$$\text{Direct reading} \quad \theta_D = \eta - e$$

$$\text{Reverse reading} \quad \theta_R - 180^\circ = \eta + e$$

$$\text{Sum} \quad \theta_D + \theta_R - 180^\circ = 2\eta$$

$$\text{Mean} \quad \frac{\theta_D + \theta_R - 180^\circ}{2} = \eta$$

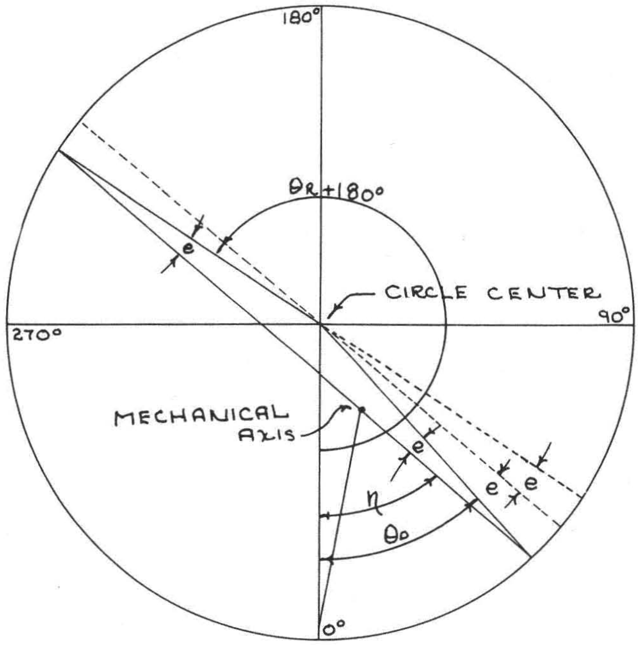

FIG. 9. Double reading circle.

There are several methods of determining the errors, and axis, of circle eccentricity. It is felt that an analytic solution making no assumption regarding sine curve behavior is the more desirable. The analytic solution is illustrated in Figure 10.

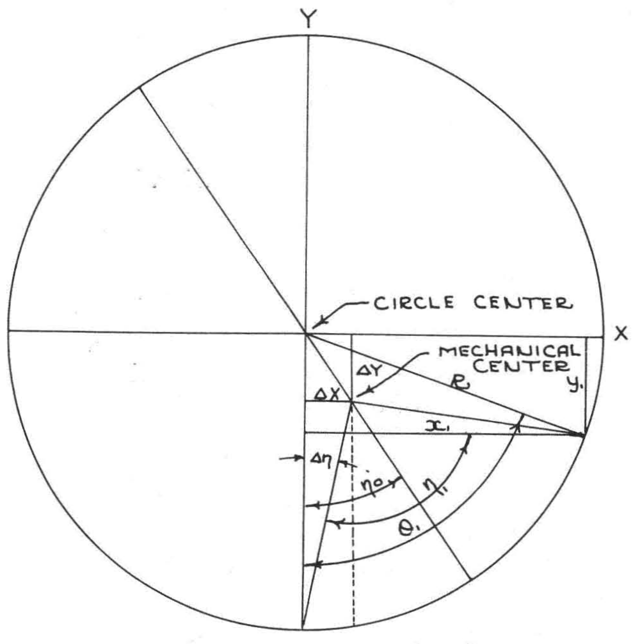

FIG. 10. Circle eccentricity.

It may be seen that

$$x_0 = 0 \qquad y_0 = R$$
$$x_1 = R \sin \theta_1 \qquad y_1 = R \cos \theta_1$$

and

$$\tan (\eta_1 - \Delta\eta) = \frac{x_1 - \Delta x}{y_1 - \Delta y}$$

Since R relates only to scale, we may put $R=1$, in which case,

$$\tan (\eta - \Delta\eta) = \frac{\sin \theta - \Delta x'}{\cos \theta - \Delta y'}$$

where

$$\Delta x' = \frac{\Delta x}{R}$$
$$\Delta y' = \frac{\Delta y}{R}$$

Expanding,

$$(\tan \eta - \tan \Delta\eta)(\cos \theta - \Delta y') = (1 + \tan \eta \tan \Delta\eta)(\sin \theta - \Delta x')$$

Here η=true angle and θ=observed angle.

$$\tan \eta \cos \theta - \tan \eta\Delta y' - \tan \Delta\eta \cos \theta + \tan \Delta\eta\Delta y'$$
$$= \sin \theta - \Delta x' + \tan \eta \tan \Delta\eta \sin \theta - \tan \eta \tan \Delta\eta\Delta x'$$

The unknowns are $\Delta\eta$, $\Delta x'$, and $\Delta y'$. Collecting coefficients of common unknowns,

$$\tan \eta \cos \theta - \sin \theta$$
$$= [\tan \Delta\eta](\tan \eta \sin \theta + \cos \theta) + [\Delta y' - \tan \Delta\eta\Delta x'] \tan \eta - (\Delta x' + \tan \Delta\eta\Delta y')$$

Now

$$\tan \Delta\eta = \frac{\Delta x'}{1 - \Delta y'}$$
$$\tan \Delta\eta - \tan \Delta\eta\Delta y' = \Delta x'$$

Therefore

$$(\Delta x' + \tan \Delta\eta\Delta y') = \tan \Delta\eta$$

Letting

$$\Delta y' - \tan \Delta\eta\Delta x' = v$$
$$\tan \Delta\eta = \nu$$

$$a = \tan \eta$$
$$b = \tan \eta \sin \theta + \cos \theta - 1$$
$$c = \tan \eta \cos \theta - \sin \theta$$

we may write

$$a_1 v + b_1 \nu = c_1$$
$$a_2 v + b_2 \nu = c_2$$
$$\cdots\cdots$$
$$a_n v + b_n \nu = c_n$$

Normal equations are formed in v and ν:

$$[a \cdot a]v + [a \cdot b]\nu = [a \cdot c]$$
$$[a \cdot b]v + [b \cdot b]\nu = [b \cdot c]$$

Solving for v and ν,

$$v = \Delta y' - \tan \Delta\eta \Delta x'$$
$$\nu = \tan \Delta\eta$$

and

$$\tan \Delta\eta(1 - \Delta y') = \Delta x'$$
$$v = \Delta y' - \tan^2 \Delta\eta(1 - \Delta y')$$
$$\Delta y' = \frac{v + \tan \Delta\eta}{1 + \tan^2 \Delta\eta}$$
$$\Delta x' = (1 - \Delta y') \tan \Delta\eta$$

$$\tan \eta_0 = \frac{\Delta x'}{\Delta y'}$$
$$\Delta x = R\Delta x'$$
$$\Delta y = R\Delta y'$$
$$e_1 = (\eta_1 - \Delta\eta) - \theta_1$$
$$e_2 = (\eta_2 - \Delta\eta) - \theta_2$$
$$\cdots\cdots$$
$$e_n = (\eta_n - \Delta\eta) - \theta_n$$

The equations for determining the axis of zero eccentricity by the alternate solution may be derived from Figure 11. In accordance with the sine curve,

$$\sin \eta_0 e = e_0$$

and

$$\sin (\theta_1 - \eta_0)e = e_1$$
$$e_1 = e_1' - e_0$$

Therefore,

$$\sin (\theta_1 - \eta_0)e + e_0 = e_1'$$

Expanding,

$$\sin \theta_1 \cos \eta_0 e - \cos \theta_1 \sin \eta_0 e + e_0 = e_1'$$

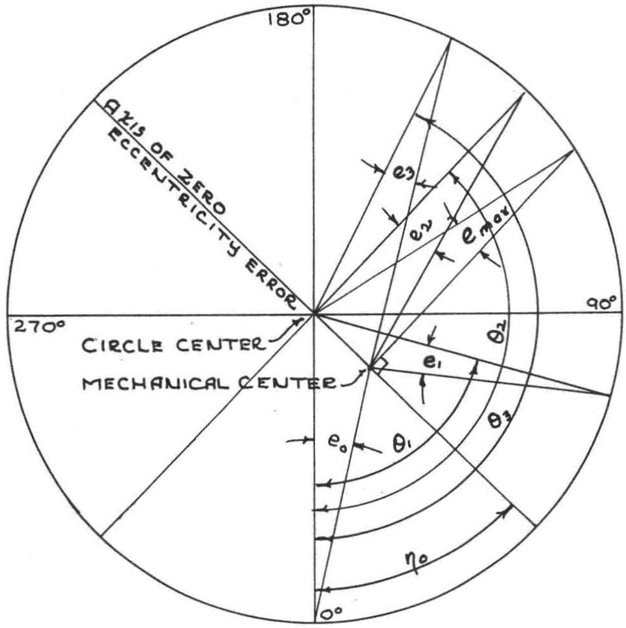

FIG. 11. Circle eccentricity.

or

$$\sin \theta_1(\cos \eta_0 e) + [1 - \cos \theta_1](\sin \eta_0 e) = e_1'$$

A similar observation equation is written for each value of e′:

$$(\sin \theta_1) \cos \eta_0 e + (1 - \cos \theta_1) \sin \eta_0 e = e_1'$$

$$(\sin \theta_2) \cos \eta_0 e + (1 - \cos \theta_2) \sin \eta_0 e = e_2'$$

$$\cdots\cdots\cdots$$

$$(\sin \theta_n) \cos \eta_0 e + (1 - \cos \theta_n) \sin \eta_0 e = e_n'$$

The coefficients in parentheses are the observed vertical angles.
η_0 is the angle of the zero axis.
e_0 is the eccentricity error of the zero θ angle or reference.
e' is the computed error of any observed angle.
e is the maximum eccentricity error referred to the axis of zero eccentricity.
The observation equations are converted to two normal equations: one in ($\cos \eta_0 e$) and one in ($\sin \eta_0 e$).

$$[\sin \theta \cdot \sin \theta] \cos \eta_0 e + [(\sin \theta) \cdot (1 - \cos \theta] \sin \eta_0 e = [\sin \theta \cdot e']$$

$$[\sin \theta \cdot (1 - \cos \theta)] \cos \eta_0 e + [(1 - \cos \theta) \cdot (1 - \cos \theta)] \sin \eta_0 e = [(1 - \cos \theta) \cdot e']$$

These equations are solved simultaneously for $\cos \eta_0 e$ and $\sin \eta_0 e$. Then

$$e = [(\cos \eta_0 e)^2 + (\sin \eta_0 e)^2]^{1/2}$$

$$\tan \eta_0 = \frac{\sin \eta_0 e}{\cos \eta_0 e}$$

and

$$e_0 = \sin \eta_0 e$$

H. ASYMMETRY OF THE VERTICAL AXIS

For the evaluation of the vertical axis asymmetry, azimuth circle eccentricity, and inclination of the Z axis, exposures are assumed to have been made, with the vertical circle clamped, at 60°, or 30°, intervals in azimuth. All details of radio timing, coordinate measurement, refraction correction, star identification, and apparent places are the same.

The 30° almucantor generates a secondary circle that defines equal zenith angles with the mechanical vertical axis of the instrument. The mechanical vertical axis may be neither parallel to the direction of gravity nor perpendicular to the X axis. Any small departure from absolute equality of the zenith angles may be attributed to asymmetry in the vertical axis. The dihedral angles defined by successive altitude circles are true azimuth angles generated by the azimuth rotation of the instrument.

Assume the direction cosines of the camera optical axis have been determined for each of the fixed-zenith-angle exposures.

The direction cosines of the mechanical vertical axis are determined.

$$\cos\alpha_{z_1}\cos\alpha_Z + \cos\beta_{z_1}\cos\beta_Z + \cos\gamma_{z_1}\cos\gamma_Z = \cos\gamma_1$$
$$\cos\alpha_{z_2}\cos\alpha_Z + \cos\beta_{z_2}\cos\beta_Z + \cos\gamma_{z_2}\cos\gamma_Z = \cos\gamma_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cos\alpha_{z_n}\cos\alpha_Z + \cos\beta_{z_n}\cos\beta_Z + \cos\gamma_{z_n}\cos\gamma_z = \cos\gamma_n$$

If there were no asymmetry in the vertical axis,

$$\gamma_1 = \gamma_2 = \cdots = \gamma_n = \gamma_0$$

where

$$\gamma_0 = \frac{\gamma_1 + \gamma_2 + \cdots + \gamma_n}{n}$$

and is the zenith angle for zero asymmetry. The value of γ_0 is obtained by the method of Least Squares. Division by $\cos\gamma_0$ gives

$$\cos\alpha_{z_1}\upsilon + \cos\beta_{z_1}\mu + \cos\gamma_{z_1}\nu = 1$$
$$\cos\alpha_{z_2}\upsilon + \cos\beta_{z_2}\mu + \cos\gamma_{z_2}\nu = 1$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cos\alpha_{z_n}\upsilon + \cos\beta_{z_n}\mu + \cos\gamma_{z_n}\nu = 1$$

Normal equations are formed in υ, μ, and ν.

$$[\cos\alpha_z\cos\alpha_z]\upsilon + [\cos\alpha_z\cos\beta_z]\mu + [\cos\alpha_z\cos\gamma_z]\nu = [\cos\alpha_z]$$
$$[\cos\alpha_z\cos\beta_z]\upsilon + [\cos\beta_z\cos\beta_z]\mu + [\cos\beta_z\cos\gamma_z]\nu = [\cos\beta_z]$$
$$[\cos\alpha_z\cos\gamma_z]\upsilon + [\cos\gamma_z\cos\beta_z]\mu + [\cos\gamma_z\cos\gamma_z]\nu = [\cos\gamma_z]$$

Solving for υ, μ, and ν,

$$\cos\gamma_0 = \frac{1}{(\upsilon^2 + \mu^2 + \nu^2)^{1/2}}$$
$$\cos\alpha_Z = \upsilon\cos\gamma_0$$
$$\cos\beta_Z = \mu\cos\gamma_0$$
$$\cos\gamma_Z = \nu\cos\gamma_0$$

$\cos \alpha_Z$, $\cos \beta_Z$, and $\cos \gamma_Z$ are substituted back in the original observation equations to obtain $\cos \gamma_1$, $\cos \gamma_2$, $\cdots$, $\cos \gamma_n$.

The angular measure of vertical asymmetry is obtained from the differences:

$$\gamma_1 - \gamma_0 = \Delta z_1''$$
$$\gamma_2 - \gamma_0 = \Delta z_2''$$
$$\cdots\cdots$$
$$\gamma_n - \gamma_0 = \Delta z_n''$$

The differences, in arc seconds, are plotted in Y against the azimuth angles, in degrees, in X.

I. INCLINATION OF THE Z AXIS (IZ)

The inclination of the Z axis follows:

$$\cos I_Z = \cos \alpha_0 \cos \alpha_Z + \cos \beta_0 \cos \beta_Z + \cos \gamma_0 \cos \gamma_Z$$

J. HORIZONTAL ECCENTRICITY

Initially, the true horizontal angles generated by the instrument are computed.

$$\cos \eta_{1,2} = \frac{\cos \alpha_{z_1} \cos \alpha_{z_2} + \cos \beta_{z_1} \cos \beta_{z_2}}{\sin \gamma_{z_1} \sin \gamma_{z_2}}$$
$$\cos \eta_{1,3} = \frac{\cos \alpha_{z_1} \cos \alpha_{z_3} + \cos \beta_{z_1} \cos \beta_{z_3}}{\sin \gamma_{z_1} \sin \gamma_{z_3}}$$
$$\cdots\cdots\cdots\cdots$$
$$\cos \eta_{1,n} = \frac{\cos \alpha_{z_1} \cos \alpha_{z_n} + \cos \beta_{z_1} \cos \beta_{z_n}}{\sin \gamma_{z_1} \sin \gamma_{z_n}}$$

and

$$\eta_{1,2} = Az_{1,2} + e_{1,2}'$$
$$\eta_{1,3} = Az_{1,3} + e_{1,3}'$$
$$\cdots\cdots$$
$$\eta_{1,n} = Az_{1,n} + e_{1,n}'$$

where Az is the observed azimuth angle.

The eccentricity of the horizontal circle is determined with values of Az and e' substituted in equations identical to those described under Paragraph G.

K. PLATE BUBBLE ACCURACY

Normally, the value of a division and the reliability of a plate bubble are measured with a level trier. However, it is possible to measure the orientation accuracy of the plate bubbles by making a direct and reverse exposure of the zenith on the same frame. No special attempt is made to adjust the bubbles. The X and Y bubbles are centered and exposure made with the horizontal and vertical circles locked. The horizontal circle is then released, rotated through 180 degrees, and clamped. The X and Y bubbles are recentered and a second exposure made without unclamping the vertical motion or advancing the film.

Thus, two zenithal exposures have been made in which the direction has been established only with the bubbles. If the bubbles are precise aside from (1) inclination of the vertical axis, (2) vertical axis precision, or (3) inclination of the optical axis, small equal zenith angles should be defined. In fact, the small equal zenith angles should correspond to the fixed dihedral angle defined by the tangent plane of the two bubbles and the film plane. Any inequality of the zenith angles may be resolved into the component errors of the X and Y bubbles. The position of the two bubbles is illustrated in Figure 12.

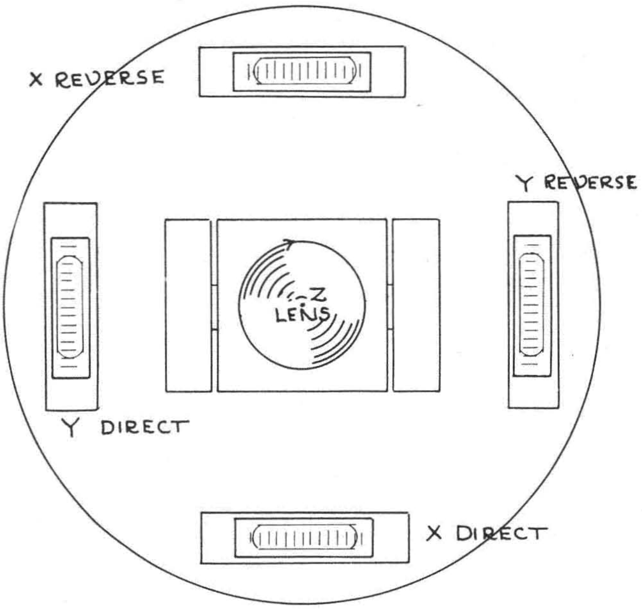

FIG. 12. Repeated centering of the bubbles.

Assume that three direct star images and three reverse star images have been identified, measured on the film, corrected for refraction, and reduced to their apparent places as before. The direction cosines of the direct star images are employed to determine the direction cosines of the principal point in the direct position; the direction cosines of the reverse star images are employed to determine the direction cosines of the principal point in the reverse position. The spherical geometry of the two positions of the principal point is illustrated in Figure 13. With direction cosines of the observer's position and the direction cosines of P_1 and P_2, γ_1 and γ_2 may be computed:

$$\cos\alpha_0 \cos\alpha_{z_1} + \cos\beta_0 \cos\beta_{z_1} + \cos\gamma_0 \cos\gamma_{z_1} = \cos\gamma_1$$

$$\cos\alpha_0 \cos\alpha_{z_2} + \cos\beta_0 \cos\beta_{z_2} + \cos\gamma_0 \cos\gamma_{z_2} = \cos\gamma_2$$

If the bubbles are perfectly precise, $\gamma_1 = \gamma_2$; if not, the measure of unreliability of each bubble is

$$\Delta x_1'' - \Delta x_2'' = e_x''$$

$$\Delta y_1'' - \Delta y_2'' = e_y''$$

where

$$\Delta x_1'' = \sin Az_1 \gamma_1''$$
$$\Delta y_1'' = \cos Az_1 \gamma_1''$$
$$\Delta x_2'' = \sin Az_2 \gamma_2''$$
$$\Delta y_2'' = \cos Az_2 \gamma_2''$$

and

$$\cos Az_1 = \frac{\cos \gamma_{z_1} - \cos \gamma_0 \cos \gamma_1}{\sin \gamma_0 \sin \gamma_1}$$

$$\cos Az_2 = \frac{\cos \gamma_{z_2} - \cos \gamma_0 \cos \gamma_2}{\sin \gamma_0 \sin \gamma_2}$$

If there is no error in the bubbles,

$$Az_1 = 180° + Az_2$$

These equations are illustrated in Figure 14.

It is assumed that the exposures are made with the Y bubble parallel to the observer's meridian and the X bubble normal to the observer's meridian. A more elaborate test would involve repeated exposures with the bubbles shifted successive divisions. From these data, the value of a division and reliability of the bubble may be determined by the method of Least Squares. Consider the X bubble:

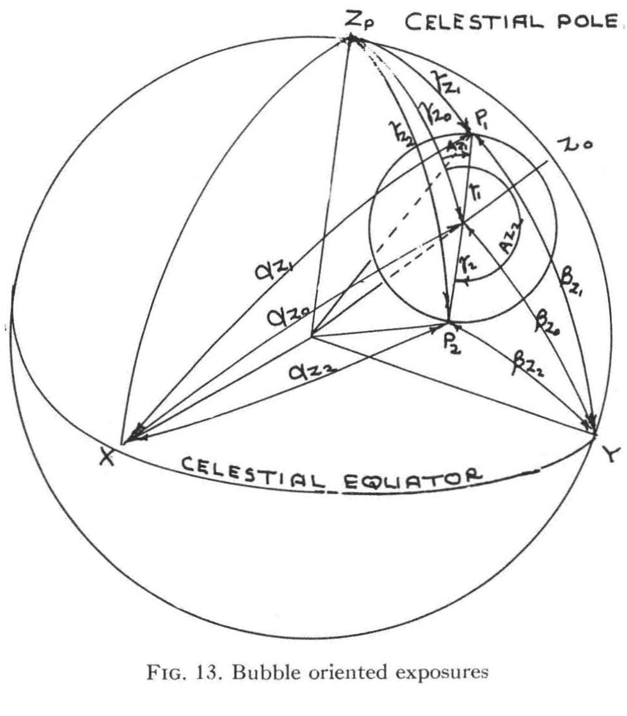

FIG. 13. Bubble oriented exposures

$$\Delta x_1'' = a_x n_1 + b_x$$
$$\Delta x_2'' = a_x n_2 + b_x$$
$$\cdots\cdots\cdots$$
$$\Delta x_n'' = a_x n_n + b_x$$

Normal equations are formed in a and b.

$$[1]b_x + [n]a_x = [\Delta x'']$$
$$[n]b_x + [n^2]a_x = [n\Delta x'']$$
$$a_x'' = \frac{[n\Delta x''][1] - [n][\Delta x'']}{[1][n^2] - [n][n]}$$

which is the most probable value of a division in seconds.

$$a_y'' = \frac{[n\Delta y''][1] - [n][\Delta y'']}{[1][n^2] - [n][n]}$$

A perhaps more expedient method would be to make two exposures on the same frame for two adjacent direct positions of the bubble lying in the meridian and repeating until the bubble has traversed the length of the graduations. Then, repeating the entire operation with the remaining bubble lying in the meridian, the same equations are employed outside the fact that

$$\Delta x'' = \phi - \gamma_{z_1} \text{ for the } X \text{ bubble}$$

and

$$\Delta y'' = \phi - \gamma_{z_2} \text{ for the } Y \text{ bubble.}$$

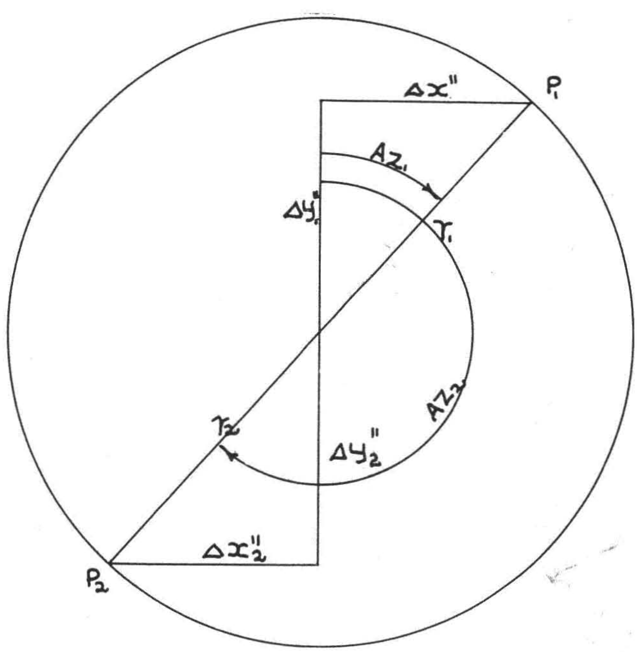

FIG. 14. Bubble error.

Summary of Results from a System-Test of a Cinetheodolite at the U. S. N. Proving Ground

HORIZONTAL SYSTEM

$$I_Z = 11''$$
Maximum Z Axis Asymmetry = 06″
Average Z Axis Asymmetry = 03″

VERTICAL SYSTEM

$$I_X = 00''$$
Maximum X Axis Asymmetry = 13″
Average X Axis Asymmetry = 04″
Index Error = 12″
Telescope Inclination $(90° - I_0) = 33''$
Trunnion Tilt $(I_{XZ}) = 10''$
Focal Length $(f) = 599.87 \pm 0.07$ mm.

CIRCLE ERRORS

(Errors in minutes of arc)

Elevation Circle

Approximate Elevation	*Error*
30°	+1.0*
50°	+0.6
60°	+0.5
70°	+0.3
80°	+0.3
100°	−0.3
110°	−0.6
120°	−0.7
150°	−1.1

Horizontal Circle

Mean Computed Horizontal	*Mean Observed Horizontal*	*Error*‡	*Deviation from Average*
60°04.1	60°04.3	+0.2	0.0
120°08.1	120°08.2	+0.1	−0.1
180°03.5	180°03.7	+0.2	0.0
239°55.4	239°55.7	+0.3	+0.1
299°57.7	299°58.1	+0.4	+0.2
360°00.0	360°00.1	0.0	−0.2
		Average = +0.2	

* Apparent tube bending, because in all cases the camera was sighting a greater distance from the zenith than the elevation circle indicated.

‡ All horizontal pointings were referred to the initial exposure to eliminate unknown errors of an object space reference mark.

Acknowledgment

The author is indebted to Dr. William A. Kemper, Chief, Ballistics Division, U. S. N. Proving Ground, Dahlgren, for his thorough support of the project and repeated cross-examination of the soundness of the methods outlined; to William Milstead, Mechanical Engineer, for his support and ability to create a suitable test situation; and to Doris Rock, Mathematician, for her thorough reduction of the observed data and interpretation of the results in terms of the purpose of the test.