Determination of Relative Orientation for Two Overlapping Photographs Taken at a Common Exposure Station

ARTHUR H. FAULDS, *Assistant Professor of Civil Engineering, Syracuse University*

ABSTRACT: *A simple computational scheme is developed for the precise determination of the relative orientation of two overlapping photographs exposed from what may be considered a common exposure station. The solution of nine simultaneous equations is required, but because of the repetitive nature of the constant coefficients the problem becomes quite easy. A sample computation is included to show the necessary steps.*

IN SYRACUSE UNIVERSITY Bulletin Number 15, *Revised Geometry of the Aerial Photograph*, Professor Earl Church developed geometric expressions for the relative orientation between the vertical exposure and each oblique wing exposure of trimetrogon photography. These expressions are based on the assumptions that the exposure stations of the three photographs are common, the interlocking angles are known, and that the fiducial y-axes of the photographs are exactly coincident. In practice, of course, these conditions are rarely, if ever, realized and so the expressions indicate the simple geometric relations which prevail only under ideal conditions.

It would seem desirable at times to evaluate precisely the orientation of one photograph with respect to another in the trimetrogon process, or in any other photographic scheme where two or more overlapping photographs are exposed from what may be considered a common exposure station. Analysis of relative orientation in dual or multiple camera arrangements for convergent photography or in multiple lens cameras would fall into this category. Furthermore, once relative orientation had been established for two or more such photographs it would be possible to treat the photography as a single exposure for such purposes as analytical tilt determination, space resection and intersection, and determination of rectifier settings. For example, in connection with the tilt determination problem where the positions and elevations of three ground points are required for control, it would be possible to calculate the scale data and exterior orientation data for the vertical photograph of a trimetrogon triplet, even though one, two, or all three ground control points appeared in the wing exposures. It is for these reasons that this paper is concerned with the development of general expressions for determining the relative orientation of any two overlapping photographs taken at a common exposure station regardless of what that relative orientation may be.

Figure 1 represents the vertical photograph and one high oblique wing photograph of trimetrogon photography having a common exposure station designated by L. The assumption regarding a common exposure station is not strictly true, of course, but in the many cases where the individual cameras are placed in a multiple camera mount, or are situated close together in the aircraft

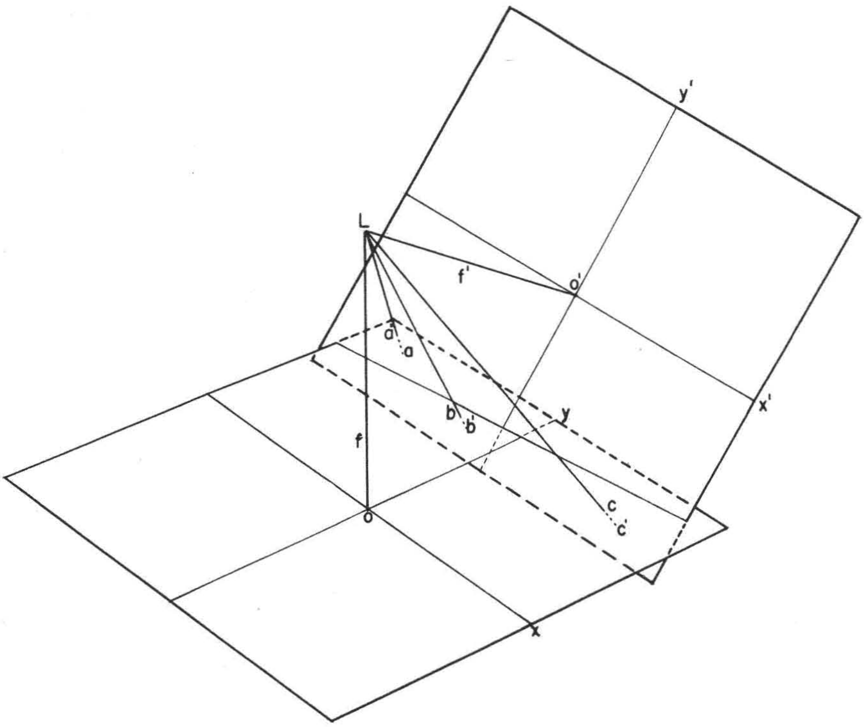

FIG. 1

in separate mounts, this assumption is valid from a practical viewpoint. In the figure, points a, b, and c and a', b', and c' represent identifiable images of ground points A, B, and C which appear in the area of overlap of the vertical and oblique exposures, respectively. The ground survey coordinates of these points are not needed. It is only necessary that they provide sharp images on the two photographs to permit accurate measurements of their positions, and that they be spaced approximately as shown in the figure. The geometric or fiducial axes of the vertical photograph are designated by x, y, and z and those of the oblique photograph by x', y', and z'. For the purpose of this discussion, we will assume that the problem consists of finding the relative orientation of the oblique photograph with respect to the vertical.

The direction cosines of the lines aL, bL, and cL in space with respect to the geometric axes of the vertical photograph may be found from the relations

$$\cos maL = \frac{-x_a}{La} \qquad \cos mbL = \frac{-x_b}{Lb} \qquad \cos mcL = \frac{-x_c}{Lc}$$

$$\cos naL = \frac{-y_a}{La} \qquad \cos nbL = \frac{-y_b}{Lb} \qquad \cos ncL = \frac{-y_c}{Lc}$$

$$\cos kaL = \frac{f}{La} \qquad \cos kbL = \frac{f}{Lb} \qquad \cos kcL = \frac{f}{Lc}$$

$$La = \sqrt{x_a^2 + y_a^2 + f^2} \qquad Lb = \sqrt{x_b^2 + y_b^2 + f^2} \qquad Lc = \sqrt{x_c^2 + y_c^2 + f_c^2}$$

where x_a, y_a, x_b, etc. represent the measured coordinates, corrected for lens distortion and film shrinkage, of the photo images of ground points A, B, and C, and f is the focal length of the aerial camera.

In a similar manner we may obtain the direction cosines of the lines La', Lb', and Lc' in space with respect to the geometric axes of the oblique photograph. Thus:

$$\cos ma'L = \frac{-x_a'}{La'} \qquad \cos mb'L = \frac{-x_b'}{L_b'} \qquad \cos mc'L = \frac{-x_c'}{Lc'}$$

$$\cos na'L = \frac{-y_a'}{La'} \qquad \cos nb'L = \frac{-y_b'}{Lb'} \qquad \cos nc'L = \frac{-y_c'}{Lc'}$$

$$\cos ka'L = \frac{f'}{La'} \qquad \cos kb'L = \frac{f'}{Lb'} \qquad \cos kc'L = \frac{f'}{Lc'}$$

$$La' \quad \sqrt{x_a' + y_a' + f'} \qquad Lb' = \sqrt{y_b' + y_b' + f'} \qquad Lc' = \sqrt{x_c' + y_c' + f'}$$

where x_a', y_a', etc. represent the measured coordinates, corrected for distortions, of the images of ground points A, B, and C on the oblique photograph, and f' is the focal length of the corresponding camera.

Since we have assumed a common exposure station for the two photographs in question, lines La', Lb', and Lc' are exactly coincident with lines La, Lb, and Lc, respectively, the only difference being that each group of lines is referred to a different set of geometric axes. Consequently, if we let the matrix

$$\begin{matrix} \cos mx' & \cos nx' & \cos kx' \\ \cos my' & \cos ny' & \cos ky' \\ \cos mz' & \cos nz' & \cos kz' \end{matrix}$$

represent the unknown orientation of the fiducial axes of the oblique photograph with respect to the fiducial axes of the vertical photograph, then, according to the basic principles of solid analytic geometry, we may write the following nine equations:

$$\begin{aligned}
\cos maL &= \cos ma'L \cos mx' + \cos na'L \cos nx' + \cos ka'L \cos kx' \\
\cos naL &= \cos ma'L \cos my' + \cos na'L \cos ny' + \cos ka'L \cos ky' \\
\cos kaL &= \cos ma'L \cos mz' + \cos na'L \cos nz' + \cos ka'L \cos kz' \\
\cos mbL &= \cos mb'L \cos mx' + \cos nb'L \cos nx' + \cos kb'L \cos kx' \\
\cos nbL &= \cos mb'L \cos my' + \cos nb'L \cos ny' + \cos kb'L \cos ky' \\
\cos kbL &= \cos mb'L \cos mz' + \cos nb'L \cos nz' + \cos kb'L \cos kz' \\
\cos mcL &= \cos mc'L \cos mx' + \cos nc'L \cos nx' + \cos kc'L \cos kx' \\
\cos ncL &= \cos mc'L \cos my' + \cos nc'L \cos ny' + \cos kc'L \cos ky' \\
\cos kcL &= \cos mc'L \cos mz' + \cos nc'L \cos nz' + \cos kc'L \cos kz'
\end{aligned}$$

The solution of these equations for the nine unknown elements of the orientation matrix is quite simple because the repetitive nature of the constant coefficients permits determination of the unknowns in groups of three.

Once numerical values have been established for the nine elements of the orientation table, the relative tilt may be found from the relations,[1]

$$\cos t = \cos kz'$$

or

$$\sin t = \sqrt{\cos^2 mz' + \cos^2 nz'}$$

or

$$\sin t = \sqrt{\cos^2 kx' + \cos^2 ky'}$$

The direction of the relative tilt on the oblique photograph measured clockwise from the $+y'$ axis (comparable to the swing angle) is determined from

$$\tan s = \frac{\cos mz'}{\cos nz'}$$

with s assigned to the proper quadrant in accordance with the algebraic signs of

$$\sin s = \frac{-\cos mz'}{\sin t} \qquad \cos s = \frac{-\cos nz'}{\sin t}$$

Lastly, the direction of relative tilt on the oblique photograph with respect to the $+y$ fiducial axis of the vertical picture (comparable to azimuth of the principal plane) may be calculated from

$$\tan \alpha = \frac{\cos kx'}{\cos ky'}$$

with α assigned to the proper quadrant in accordance with the algebraic signs of

$$\sin \alpha = \frac{-\cos kx'}{\sin t} \qquad \cos \alpha = \frac{-\cos ky'}{\sin t}$$

These three quantities, t, s, and α are sufficient to define uniquely the relative orientation of the two photographs.

In the solution of this problem, if some measure of validity of the results is desired, any number of points greater than three may be selected and the calculation performed in accordance with the principles of the Method of Least Squares.

Sample Computation

The following sample computation for determination of the relative orientation of a trimetrogon high oblique wing photograph with respect to the corresponding vertical exposure will serve to illustrate the simplicity of computation as well as the method for calculation.

VERTICAL PHOTOGRAPH:

Point	x	y	f	Length	$\cos m$	$\cos n$	$\cos k$
a	-91.440	$+91.440$	152.40	aL 199.87073	$+.45749571$	$-.45749571$	$+.76249285$
b	$+93.785$	$+93.785$	152.40	bL 202.03221	$-.46420817$	$-.46420817$	$+.75433517$
c	0	$+96.253$	152.40	cL 180.25094	0	$-.53399445$	$+.84548798$

[1] The reader may verify these identities by referring to the aforementioned Syracuse University Bulletin Number 15 or to Bulletin Number 19, *Theory of Photogrammetry*, by Professor Earl Church.

OBLIQUE PHOTOGRAPH:

Point	z'	y'	f'	Length	cos m	cos n	cos k
a'	−94.790	−96.686	152.40	a'L 203.86046	+.46497492	+.47427539	+.74757018
b'	+89.189	−84.764	152.40	b'L 195.87081	−.45534606	+.43275464	+.77806387
c'	− 1.752	−87.113	152.40	c'L 175.54915	+.00998011	+.49623140	+.86813294

For both photographs a check of the calculated values for each point is immediately available because of the Pythagorean condition which requires that the sum of the squares of the direction cosines for any line be exactly equal to one. In other words, for each point

$$\cos^2 m + \cos^2 n + \cos^2 k = 1$$

SIMULTANEOUS EQUATIONS

			x'	y'	z'
+46497492 cos m	+47427539 cos n	+74757018 cos k	= +45749571	− 45749571	+ 76249285
−45534606 cos m	+43275464 cos n	+77806387 cos k	= −46420817	− 46420817	+ 75433517
+ 998011 cos m	+49623140 cos n	+86813294 cos k	= 0	− 53399445	+ 84548798
+ 998011	+49623140	+86813294	= 0	− 53399445	+ 84548795
+ 998011	+ 1017973	+ 1604567	= + 981958	− 981958	+ 1636596
− 998011	+ 948496	+ 1705332	= − 1017435	− 1017435	+ 1653325
	+48605167	+85208727	= − 981958	− 52417487	+ 82912202
	+50571636	+88518626	= − 1017435	− 54416880	+ 86202123
	+48605167	+85208727	= − 981958	− 52417487	+ 82912202
	+48605167	+85076595	= − 977872	− 52300889	+ 82850169
		+ 132132 cos k	= − 4086	− 116598	+ 62033
		cos k	= −.03092362	−.88243575	+.46947749

cos kx' −.03092362 cos ky' = −.88243575 cos kz' = +.46947749

With these values, which also must satisfy the Pythagorean condition, we may at once calculate relative tilt and the angle (α) between the y-axes of the two photographs. Thus,

$$\cos t = \cos kz' = + \overset{.46}{\cancel{64}}.947749$$

$$t = 61° \, 59' \, 59''$$

$$\tan \alpha = \frac{\cos kx'}{\cos ky'} = \frac{-.03092362}{-.88243575} = + .03504352$$

$$\alpha = 2° \, 00' \, 25''$$

Numerical values for cos nx', cos ny', and cos nz' are obtained by substituting appropriate values in the formulas

		x'	y'	z'
+48605167	cos n +85208727	cos k = − 981958	−52417487	+82912202
+50571636	+88518626	= −1017435	−54416880	+86202123

Thus,

cos $\overset{n}{\cancel{m}}x'$		cos ny'		cos nz'	
− 981958	− 1017435	− 52417487	− 54416880	+ 82912202	+ 86202123
+ 2634962	+ 2737316	+ 75191227	+ 78112000	− 40003579	− 41557502
+ 1653004	+ 1719881	+ 22773740	+ 23695120	+ 42908623	+ 44644621
+.03400881	+.03400881	+.46854566	+.46854565	+.88279962	+.88279962

Similarly, numerical values for cos mx', cos my', and cos mz' may be calculated by substituting known values in the original equations. In this case

$$\cos mx' = +\ .99894390$$

$$\cos my' = -\ .04308369$$

$$\cos mz' = -\ .01540865$$

and with these values, the direction of the relative tilt on the oblique photograph may be determined from the relation

$$\tan s = \frac{\cos mz'}{\cos nz'} = -\ .01745430 = 179°\ 00'\ 01''$$

The above calculation was based on data from a pair of fictitious photographs prepared for this purpose. It is interesting to note that the calculated values agree with the known values to within less than thirty seconds of arc in all cases.

	Correct Value	Calculated Value
t	62° 00′ 00″	61° 59′ 59″
s	179° 00′ 00″	179° 00′ 01″
α	2° 00′ 00″	2° 00′ 25″

*The Galileo Santoni Stereosimplex Model III**

GEORGE D. HARDY, *Photogrammetric Engineer, Abrams Aerial Survey Corporation, Lansing, Michigan*

BEFORE starting my remarks on the Stereosimplex, I will describe the background and reasons for the design of this instrument.

Like all other instruments produced by the Galileo Corporation in Florence, Italy, the Model III was designed by Professor E. Santoni. He for a number of years has been one of the foremost advocates of the production and use of first-order plotting instruments.

Professor Santoni made the following comments on the general criteria which inspired the design of this new plotting instrument: "With the remarkable advance in lens design and mechanical arrangement, the aerophotogrammetric cameras have considerably improved in recent years. The taking of pictures has nowadays become an easy and reasonably cheap task. A few hours flight is today sufficient to get photographic record of very wide areas."

"Also, in the production of first-order plotting instruments a considerable progress has been made, so that the use of aerial views has become much more rigorous than it was some years ago; technical and high precision maps are produced out of pictures taken from higher altitudes than those used about ten years ago; while, when using first-order instruments for aerial bridging, which is the foundation of every aerial triangulation, a degree of precision is achieved, which was not possible a few years ago.

"Notwithstanding this, the application of photogrammetry for systematic works of a precise nature is not so wide as was to be expected. On the contrary, the remarkable bulk of work done using methods of lower order accuracy leads many unquali-

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