Relative Orientation of Photographs Taken from the Same Station

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ABSTRACT: *A proposed direct solution of the relative orientation of two photographs from the same station is critically examined. An improved solution is developed but both solutions are noted to be defective in relation to real data.*

THE relative orientation of two overlapping photographs exposed at the same station is treated by von Gruber in Chapter II of his essays (1). It does not seem that the German photogrammetrist envisaged a direct solution to the problem such as that given by Prof. A. H. Faulds in his recent paper on the subject (2).

If we have two rectangular systems of reference S and S', the transformation of direction-cosines is given by

$$l' = a_{11}l + a_{12}m + a_{13}n,$$
$$m' = a_{21}l + a_{22}m + a_{23}n,$$
$$n' = a_{31}l + a_{32}m + a_{33}n,$$

where the co-efficients $a_{11}, a_{12} \cdots a_{33}$ are governed by six independent rigorous conditions which are well known. Hence the orthogonal matrix

$$\begin{pmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{pmatrix}$$

depends on three independent quantities only and it follows that two rays, with direction-cosines in both systems, provide more than enough data for the determination of this matrix. Faulds' method (which depends on three points observed on both photographs) utilizes redundant data and fails to make use of the rigorous conditions between the elements of the unknown matrix.

If the directions 1 and 2 correspond to two points coordinated on both photographs we have

$$l_1' = a_{11}l_1 + a_{12}m_1 + a_{13}n_1 \qquad 1.1$$
$$l_2' = a_{11}l_2 + a_{12}m_2 + a_{13}n_2 \qquad 1.2$$
$$m_1' = a_{21}l_1 + a_{22}m_1 + a_{23}n_1 \qquad 2.1$$
$$m_2' = a_{21}l_2 + a_{22}m_2 + a_{23}n_2 \qquad 2.2$$
$$n_1' = a_{31}l_1 + a_{32}m_1 + a_{33}n_1 \qquad 3.1$$
$$n_2' = a_{31}l_2 + a_{32}m_2 + a_{33}n_2 \qquad 3.2$$

in which 1.1, 2.1, 3.1 refer to the first ray and 1.2, 2.2, 3.2 to the second. These six relations are not independent because of the conditions between the direction cosines, and in fact 3.1 and 3.2 contribute no additional information. Hence we have four equations 1.1, 1.2, 2.1, 2.2 to determine the three independents of the matrix, and it follows that there is some redundancy even with two points. In fact it can be seen that the direction-cosines must satisfy the condition

$$l_1'l_2' + m_1'm_2' + n_1'n_2' = l_1l_2 + m_1m_2 + n_1n_2,$$

in order that measurements in S' be consistent with those in S. In Faulds' method it would be necessary to impose two further conditions before it could be assumed that the method of solution would yield an orthogonal matrix. Thus for a real data a rigorous one-shot solution is not quite the simple matter that it appears to be.

The author's own solution described below is still open to the objections outlined above, but since it is based on two common points instead of three, it is less vulnerable than Faulds' to the errors of observation. Assuming for the moment that the observations on the two points are consistent, so that the last condition is satisfied, the problem now is to find a third ray whose direction-cosines are perfectly consistent with those of rays 1 and 2. (We require this ray in order to maintain the linearity of the equations.) The necessary ray is given by the vector product of the unit vectors defined by the rays 1 and 2, and we bring it into play by means of the following theorem. "The orthogonal transform of the vector product of two vectors is equal to the vector product of their transforms." This follows at once from the purely rotational character of the orthogonal matrix.

Hence, if we write

$$l_3 = m_1n_2 - m_2n_1, \qquad l_3' = m_1'n_2' - m_2'n_1',$$

$$m_3 = n_1l_2 - n_2l_1, \qquad m_3' = n_1'l_2' - n_2'l_1',$$

$$n_3 = l_1m_2 - l_2m_1, \qquad n_3' = l_1'm_2' - l_2'm_1',$$

Then from the theorem we get

$$l_3' = a_{11}l_3 + a_{12}m_3 + a_{13}n_3 \qquad 1.3$$

$$m_3' = a_{21}l_3 + a_{22}m_3 + a_{23}n_3 \qquad 2.3$$

$$n_3' = a_{31}l_3 + a_{32}m_3 + a_{33}n_3 \qquad 3.3$$

and combining 1.3 with 1.1 and 1.2, we get a set which may be solved for a_{11}, a_{12}, a_{13}. Similarly a_{21}, a_{22}, a_{23} are determined from the set 2.1, 2.2, 2.3 while a_{31}, a_{32}, a_{33} are found from 3.1, 3.2, 3.3.

In view of the possible inconsistency mentioned above, it would be best to determine a_{12} and a_{13} only from the first set while a_{23} alone is determined from the second. The remaining elements are then found from the conditions.

$$\text{i.e. } a_{11} = \pm\sqrt{(1 - a_{12}^2 - a_{13}^2)},$$

$$a_{33} = \pm\sqrt{(1 - a_{13}^2 - a_{23}^2)},$$

$$a_{21} = -(a_{33}a_{12} + a_{11}a_{13}a_{23})/(a_{11}^2 + a_{12}^2),$$

$$a_{22} = \pm\sqrt{(1 - a_{21}^2 - a_{23}^2)},$$

$$a_{31} = a_{12}a_{23} - a_{22}a_{13},$$

$$a_{32} = a_{13}a_{21} - a_{11}a_{23},$$

The signs of the radices are usually determinate, being fixed by the relationships between the co-ordinate axes of the two photographs.

When the matrix has been determined, the co-ordinates on one photograph may be converted into their values on the other by

$$x' = f' \cdot (a_{11}x + a_{12}y + a_{13}f)/(a_{31}x + a_{32}y + a_{33}f),$$

$$y' = f' \cdot (a_{21}x + a_{22}y + a_{23}f)/(a_{31}x + a_{32}y + a_{33}f).$$

Although the direct solution appears to be of academic interest only, in the case of twin-camera photography, it may be useful at the start of a flight to provide a good first approximation to the relative orientation for each simultaneous pair. Thereafter it would seem preferable to use differential methods to obtain improved values for each pair, if necessary using redundant data and least square techniques.

WORKED EXAMPLE

The following example (which uses fictitious data) was computed by a colleague Mr. S. Hull of the Ordnance Survey Photogrammetric Group. It will be noticed that the table of direction-cosines and vector components leads directly into the simultaneous solutions of the sets (1.1, 1.2, 1.3) and (2.1, 2.2, 2.3). The solution scheme is a simple one based on alternate division and subtraction which leads directly to the values of a_{13} and a_{23}. The first stage of the back-solution gives a_{12}.

DETERMINATION OF MATRIX FROM PHOTO 23 TO PHOTO 23′

Photographic Co-ordinates

Photo	Point 1		Point 2		
23	x_1 +50.16	y_1 +47.83	x_2 −52.73	y_2 +41.87	$f=150.64$
23′	x_1' +64.91	y_1' +170.68	x_2' −80.73	y_2' +156.95	$f'=151.13$

DIRECTION-COSINES AND COMPONENTS OF VECTOR PRODUCT

Pt.	l	m	n	l'	m'	n'
1.	+.30250	+.28845	+.90846	+.27384	+.72006	+.63759
2.	−.31957	+.25375	+.91295	−.34744	+.67546	+.65042
3.	+.03282	−.56648	+.16894	+.03767	−.39963	+.43515

+1.00000	+ .953556	+3.003178	+ .905258	+ 2.380364	*Solution*
+1.00000	− .794036	−2.856808	+1.087211	− 2.113653	
+1.00000	−17.260207	+5.147471	+1.147776	−12.176417	
	− 1.747592	−5.859985	+0.181953	− 4.494017	
	−18.213763	+2.144294	+0.242518	−14.556781	
	+ 1.000000	+3.353177	−0.104116	+ 2.571548	
	+ 1.000000	+0.117729	−0.013315	+ 0.799226	
		+3.470906	−0.090801	+ 1.772322	
		+1.000000	−0.02616	+ 0.51062	
			a_{13}	a_{23}	

$$a_{12} = -0.104116 - 3.353177a_{13} = -0.013315 + 0.117729a_{13} = -0.01640$$

The Completed Matrix

$$\begin{pmatrix} +.99952, & -.01640, & -.02616 \\ +.02746, & +.85936, & +.51062 \\ +.01411, & -.51109, & +.85941 \end{pmatrix}$$

REFERENCES

1. Von Gruber, O., "Photogrammetry—Collected Essays and Lectures." Chapter II. English translation—London 1932.
2. Fauld, A. H., "Determination of Relative Orientation for Two Overlapping Photographs Taken at a Common Exposure Station." PHOTOGRAMMETRIC ENGINEERING, Vol. XXII, No. 2. April 1956.

An Analysis of Errors Using the Graduation Process

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ABSTRACT: *The principles of graduation can be applied to the analysis of errors in certain photogrammetric operations. In case of aerial triangulation this procedure makes possible separating the combined effect of the errors of systematic character from the errors of accidental nature. For this purpose the relatively inaccurate data given by some of the methods for direct determination of the elements of outer orientation of an aerial camera may be used provided that these data are affected by accidental errors only. The graduation procedure was found to be successful when applied using data obtained with the Airborne Profile Recorder. Successful applications using horizon camera, statoscope, solar periscope, Shoran, etc. data seem feasible.*

INTRODUCTION

THE graduation or smoothing process is a well-known procedure in statistics and experimental physics. Its main purpose is to find a smooth graphical curve representing a set of observations which due to various errors are dispersed. However, in the analysis of errors in photogrammetry and in connection with other surveying problems, the possibilities of graduation have seldom if ever been utilized. In the field of photogrammetry there are many problems which offer interesting possibilities for applications of this technique. In the following pages a brief outline of the principles of graduation and its application to the analysis of errors and to some photogrammetric problems are described.

THE PROBLEM OF GRADUATION

The problem of graduation is easiest to explain in terms of graphical representation. Suppose that as a result of observations or experiments, a set of values of a variable y, $(y_1, y_2, \cdots y_n)$ is obtained corresponding to equidistant values of its argument x, $(x_1, x_2, \cdots x_n)$. Suppose further that it is known, *a priori*, that y is a continuous function of x over the whole range of observations. Then

$$y = F(x)$$