The Completed Matrix

$$\begin{pmatrix} +.99952, & -.01640, & -.02616 \\ +.02746, & +.85936, & +.51062 \\ +.01411, & -.51109, & +.85941 \end{pmatrix}$$

REFERENCES

1. Von Gruber, O., "Photogrammetry—Collected Essays and Lectures." Chapter II. English translation—London 1932.
2. Fauld, A. H., "Determination of Relative Orientation for Two Overlapping Photographs Taken at a Common Exposure Station." PHOTOGRAMMETRIC ENGINEERING, Vol. XXII, No. 2. April 1956.

An Analysis of Errors Using the Graduation Process

U. V. HELAVA,
Division of Applied Physics,
National Research Council of Canada, Ottawa

ABSTRACT: *The principles of graduation can be applied to the analysis of errors in certain photogrammetric operations. In case of aerial triangulation this procedure makes possible separating the combined effect of the errors of systematic character from the errors of accidental nature. For this purpose the relatively inaccurate data given by some of the methods for direct determination of the elements of outer orientation of an aerial camera may be used provided that these data are affected by accidental errors only. The graduation procedure was found to be successful when applied using data obtained with the Airborne Profile Recorder. Successful applications using horizon camera, statoscope, solar periscope, Shoran, etc. data seem feasible.*

INTRODUCTION

THE graduation or smoothing process is a well-known procedure in statistics and experimental physics. Its main purpose is to find a smooth graphical curve representing a set of observations which due to various errors are dispersed. However, in the analysis of errors in photogrammetry and in connection with other surveying problems, the possibilities of graduation have seldom if ever been utilized. In the field of photogrammetry there are many problems which offer interesting possibilities for applications of this technique. In the following pages a brief outline of the principles of graduation and its application to the analysis of errors and to some photogrammetric problems are described.

THE PROBLEM OF GRADUATION

The problem of graduation is easiest to explain in terms of graphical representation. Suppose that as a result of observations or experiments, a set of values of a variable y, $(y_1, y_2, \cdots y_n)$ is obtained corresponding to equidistant values of its argument x, $(x_1, x_2, \cdots x_n)$. Suppose further that it is known, *a priori*, that y is a continuous function of x over the whole range of observations. Then

$$y = F(x)$$

The y values can be plotted against x values. If the observations are free from errors, all the plotted points lie on a smooth curve representing the equation $y=F(x)$. This is not the case in practice. The plotted points are displaced from their proper positions due to the errors of observation. The curve $y=F(x)$ is therefore not precisely indicated and only an approximation of its shape and position is possible. To eliminate personal judgment inherent in such an approximation the graduation procedure can be used. Then the graduated values of y will fulfill an equation $y'=F'(x)$ which is an approximation of $y=F(x)$.

The problem of graduation also arises in connection with interpolation. If the original set of observations is free from errors, their values can be used for interpolation of y for any intermediate x. This means that if a table of differences if formed,

$\Delta^1 y_1 = y_2 - y_1$	$\Delta^2 y_1 = \Delta^1 y_2 - \Delta^1 y_1$	$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	etc
$\Delta^1 y_2 = y_3 - y_2$	$\Delta^2 y_2 = \Delta^1 y_3 - \Delta^1 y_2$	$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$	.
.	.	.	.
.	.	.	.
etc	etc	etc	

the differences are found to be regular. However, if the observations are erroneous, the differences are irregular and cannot be used for interpolation. The problem of graduation is to find another set of y values $(y_1', y_2', \cdots, y_n')$ which gives regular differences. The terms of the new set should not of course differ too much from the original values. How much they may, is an important detail which will be dealt with later on in this paper.

Some Methods of Graduation

Many different methods of graduation are developed for practical applications; some of them are mentioned below. The derivations of the formulas will not be given here but instead a brief explanation of the main ideas. Most of these methods are designed for equidistant argument intervals, but can be extended to the case of unequal x-intervals by applying "divided differences." (For derivation of formulas and for "divided differences" see 1, 2, 3, 4).

The simplest of the smoothing methods is based on assumption that $F(x)$ can be approximated between three consecutive points by a linear function, or by a straight line in the graphical representation. A linear function is fitted to the three values using the method of least squares in order to find the graduated value of the middle ordinate. This gives a formula:

$$y_n' = 1/3(y_{n-1} + y_n + y_{n+1}) \tag{1}$$

This is actually the average value of three consecutive ordinates.

The same consideration can be extended to the case of unequal x intervals, assuming of course that the total x range is still so small that a linear approximation is justified. We obtain

$$y_{gr}' = 1/3(y_{n-1} + y_n + y_{n+1}) \tag{2}$$

which refers to

$$x_{gr}' = 1/3(x_{n-1} + x_n + x_{n+1})$$

The procedure of graduation is easy to perform using this method of linear approximation either numerically or graphically. The graphical procedure consists of constructing the center of gravity of the triangle formed by the three original points.

If the smoothness of the curve is not sufficient, the procedure may be repeated using the data obtained from the single application. The result of the second application may be processed a third time and so on, until the required smoothness is obtained.

The method explained above is simple and fast to apply and it is usually well justified. However, if the curve representing $F(x)$ is not flat, but possesses considerable curvatures, the linear approximation is not acceptable. Then a parabolic function can be used for approximation of $F(x)$ for a number of consecutive observations. For adjustment of a parabola at least four points are needed. It is however more convenient to take an odd number of points, say, five, seven or nine, because in this case one of the points is in the middle. The value of the ordinate of the middle point is established by fitting a parabola to the observations using the method of least squares. For five points, for instance, a formula is obtained:

$$y_n' = 1/35[-3(y_{n-2} + y_{n+2}) + 12(y_{n-1} + y_{n+1}) + 17y_n] \tag{3}$$

For seven and nine points

$$y_n' = 1/21[-2(y_{n-3} + y_{n+3}) + 3(y_{n-2} + y_{n+2}) + 6(y_{n-1} + y_{n+1}) + 7y_n] \tag{4}$$

$$\begin{aligned} y_n' = 1/231[&-21(y_{n-4} + y_{n+4}) + 14(y_{n-3} + y_{n+3}) + 39(y_{n-2} + y_{n+2}) \\ &+ 54(y_{n-1} + y_{n+1}) + 59y_n] \end{aligned} \tag{5}$$

Similar formulas can be computed for a greater number of points and for higher order parabolas.

An increased number of points or a second application of the graduation formulas will produce a smoother curve. However, the possible advantages of these procedures for practical application are diminished by the fact that the computations are quite laborious, and very much the same result can often be obtained by using the simpler method previously described.

A very elegant method of graduation is based on the work of Professor Whittaker. In his method no assumption is made as to the equation of the curve representing $F(x)$ except that the curve should be a smooth one. The sum of squares of the third differences of graduated values $\sum(\Delta^3 y')^2$ is used as a measure of smoothness and the sum of the squares of the residuals $\sum(y-y')^2$ is regarded as a measure of the "closeness of fit." Graduation is performed in such a way that expression $\sum(\Delta^3 y')^2+\epsilon\sum(y-y')^2$ is a minimum. The fundamental equations were derived from the theory of probability, and the resulting curve is a compromise between "smoothness" and "closeness of fit." The parameter ϵ is a measure of the relative weights assigned to the "smoothness" $\sum(\Delta^3 y')^2$ and "closeness of fit" $\sum(y-y')^2$. A small value of ϵ produces a very smooth curve which does not follow closely the observations, whereas a large value of ϵ produces graduated terms which are close to the observed ones but with more irregular third differences.

The graduation formula for this method is:

$$y_n' = k_0 y_n + k_1(y_{n-1} + y_{n+1}) + k_2(y_{n-2} + y_{n+2}) + \cdots \tag{6}$$

The values of coefficients k and the number of terms to be included are dependent upon the value of ϵ. A table of coefficients is published in ref. p. 314.

Application of the Graduation Process to the Analysis of Errors

Let us suppose that two different sets of observations (y_1', y_2', $\cdots$, and y_1'', y_2'', $\cdots$) are made of the same unknowns Y. Suppose further that the two sets possess different kinds of observational errors: for instance, the first group, is affected mainly by accidental errors, while the second is dominated by a systematic error which is a continuous function of the same argument, $E_y''=F(x)$.

For each value of the variable we have two observations:

$$\begin{aligned} &\text{a)} \quad y' = Y + E_{y'} \\ &\text{b)} \quad y'' = Y + E_{y''} \quad \text{or} \quad y'' = Y + F(x) \end{aligned} \tag{7}$$

Therefore

$$y'' - y' = F(x) - E_{y'}$$

or

$$(y'' - y') + E_{y'} = F(x) \tag{8}$$

The equation (8) indicates that the differences $(y''-y')$ can be regarded as observations of $F(x)$, each of them being affected by an accidental error E_y'. From these observations an approximation of $F(x)$ can be determined using the graduation technique. It is then possible to establish the values of the original unknown using equation (7)b.

A Criterion for the Degree of Smoothing

It was stated previously that the graduated values should not differ too much from the corresponding original observations. This statement is rather loose and does not define precisely how far to proceed with the smoothing. A more exact criterion can be found by studying the residuals $y-y'$.

Let Y denote the "true" value of which y is an observation. The "true error" of each observation is then $y-Y$. If a series of observations including n terms, is available, the mean square error of the series can be computed.

$$\sqrt{\frac{\sum_1^n (y - Y)^2}{n}} \tag{9}$$

If the graduated values y' were a perfect representation of the values Y, then each one of the residuals $y-y'$ would be precisely equal to the corresponding error $y-Y$. Therefore the mean square value of residuals:

$$m' = \sqrt{\frac{\sum_1^n (y - y')^2}{n}} \tag{10}$$

should be equal to m. Thus $m'^2/m^2=1$ and

$$\frac{\sum_1^n (y - y')^2}{m^2} = n \tag{11}$$

or, if we employ the conception of weight, $W=1/m^2$:

$$\sum_{1}^{n} (y - y')^2 W = n \tag{12}$$

This formula together with the fact that the final result must be as smooth a curve as possible gives the criterion for graduation. In other words, the final result is the smoothest curve which satisfies the equation (12)*. For the application of this criterion however the accuracy of the original observations must be known since it is needed for determination of the weight or weights. Sometimes this accuracy is independently available. Often however, this is not the case and a proper smoothing would not be possible. Fortunately the series of observations itself provides in many cases a means for determining the weights as will be shown later.

Direct Determination of the Mean Square Error of Observations

The determination of the mean square error of the original observations is based on an assumption that the correct curve can be very accurately approximated over a certain number of the argument values, by an analytical curve of a certain order. In this case the differences which are of a higher order should vanish. For most smoothing problems, and especially for those of photogrammetry, it is sufficient to assume that the third differences are equal to zero, or very small.

Let us assume for the moment that the third differences are equal to zero. This means, that any consecutive four points should lie on a second-order parabola. In case of e.g. aerial triangulation this is theoretically the case. In the third difference four points are involved as can be seen from the general equation for direct computation of the third differences:

$$\Delta^3 y_n = y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3} \tag{13}$$

All observations, however, are affected by accidental errors. Suppose that their mean square error is $\pm m_y$. The mean square error of the third difference is therefore

$$M_{\Delta^3} = \sqrt{m_y^2 + 9m_y^2 + 9m_y^2 + m^2} = \sqrt{m_y^2 \cdot 20} = m_y\sqrt{20} \tag{14}$$

Since the true value of the third difference is zero, the third differences computed from the observations provide a simple means to estimate the mean square error of the original data. It should be noted that a single determination of M_{Δ^3} is greatly affected by the local errors of the original data. It is obvious that an average of many determinations is more reliable.

Let us now assume that the third differences are not zero, but "very small." We assume further that we have a fairly large number (N) of observations of the third differences. In practice this is often the case. The true error E of the third difference is

$$E_{\Delta^3} = \Delta^3 y - \Delta^3 F(x)$$

$$\sum_{1}^{N} (E_{\Delta^3})^2 = \sum_{1}^{N} (\Delta^3 y - \Delta^3 F(x))^2$$

* In photogrammetric bridging $\sum \Delta^3 y'$ should be theoretically equal to zero.

$$\sum_{1}^{N} (E_{\Delta^3})^2 = \sum_{1}^{N} (\Delta^3 y)^2 - \sum_{1}^{N} 2\Delta^3 y_{\Delta^3} F(x) + \sum_{1}^{N} (\Delta^3 F(x))^2 \tag{15}$$

Since the errors are supposedly accidental the term in the middle will be very small; the larger the number of observations, the smaller the term. In addition, the third differences of $F(x)$ were supposed to be very small when compared with the corresponding values derived from the observations.

The ratio of the square sums is still smaller and we can write:

$$\sum_{1}^{N} (E_{\Delta^3})^2 \cong \sum_{1}^{N} (\Delta^3 y)^2 \tag{16}$$

$$M_{\Delta^3}^2 = \frac{\sum_{1}^{N} (E_{\Delta^3})^2}{N}$$

therefore

$$m_y \cong \sqrt{\frac{\sum_{1}^{N} (\Delta^3 y)^2}{N \cdot 20}} \tag{17}$$

Equation (17) allows us to estimate the mean square error of the original observations. If necessary, the accuracy of this estimation can be improved by applying formula (15) after once performing a graduation in which $F'(x)$ can be used instead of $F(x)$, for estimation of the terms ignored in the equation (16).

Practical Use of the Criterion

The criterion for graduation given above can be used in connection with different methods of smoothing. A certain graduation process is applied, and then the sum of the weighted squares of residuals is computed on the assumption that the smoothed curve is free from errors. This result may not be close to the number of observations, but gives in any case an idea about the degree of smoothing obtained. Following this, the smoothing procedure is modified while bearing in mind the requirement of the criterion. In practice, the modification can be made on the basis of the principle of iteration. For instance, in the case of linear approximation, the procedure is repeated using the values obtained from the previous application of the method. In the case of parabolic approximation, more points can be included while for the probability method a different set of coefficients is used in order to get the required degree of graduation.

Finally a curve which satisfies the criterion may be obtained and is accepted as the final approximation of $F(x)$. In general, no individual one of the curves so produced will be immediately acceptable, but nevertheless the criterion indicates that the proper curve lies between two of them. The final curve based on the residuals involved can then be derived by interpolation.

According to the present experience of the author, the final result is practically the same regardless of which one of the graduation methods is used, provided the final curve is fairly flat. The linear method is the easiest to apply. On the other hand, the probability method is the most nearly correct and the safest, especially if sharp changes of curvature are to be expected.

Applications to Photogrammetry

Photogrammetry lends itself very well to the application of the graduation process. In many photogrammetric operations the same quantity is determined by two series of observations—one being affected by systematic errors or accumulative accidental errors, and the other by non-accumulative accidental errors only. Application of the graduation process leads to a proper separation of errors and permits determination of the unknown quantity. It is interesting to note that this simple and very effective way of establishing the final values has, as far as we know, been overlooked in photogrammetric procedures, despite the fact that some of these offer an ideal application field.

Radar Profile Data

The ideas explained above were applied to an analysis of errors in long distance bridging. In this bridging Radar Profile Data were used for the control of elevations and the scale of bridging (5). The smoothing process was applied for the analysis of errors in elevation. The results are shown in Figure No. 1a. Repeated linear smoothing led to the required result. The same data were later processed by using parabolic formulae and the probability method. The results are represented in Figures No. 1b and 1c.

As can be seen from the figures, the procedure leads to very promising results. The remaining errors may be explained by small systematical errors in the Radar Profile Data and accidental errors in the triangulation. It should be noted that the final result of graduation is substantially the same in all three figures.

Adjustment of elevations is not the only place where the graduation process can be applied in bridging, utilizing Radar Profile Data. The clearance, or distance from airplane to terrain is also given by the Radar Profile equipment,and can be compared with the clearances or variations of clearances obtained from aerial triangulation. This possibility will be tested in the near future.

Horizon Camera Data

In the horizon camera method the horizon is photographed in two perpendicular directions simultaneously with the survey photograph. The variations of the attitude of the survey camera can be derived by measuring the horizon photographs. The results of such measurements are fairly accurate as such, and especially free from systematic errors. Therefore, these data are very suitable for combining with aerial triangulation in order to separate errors by the graduation process.

Statoscope Data

A number of instruments exist which use air pressure to indicate variations of flight altitude. All of them are included here under the name statoscope. The statoscopes refer their indications to an isobaric surface. Once the shape and inclination of this surface have been established, the residual errors of the statoscope data are mainly accidental. The systematic errors of the b_z values of an aerial triangulation can therefore be analyzed by applying the smoothing process outlined above.

Radio Positioning Methods

It seems to the author that the combination of a radio positioning method such as Shoran, Hiran, Decca, etc., with aerial triangulation would lead to a considerable improvement in the accuracy attainable by any of the methods

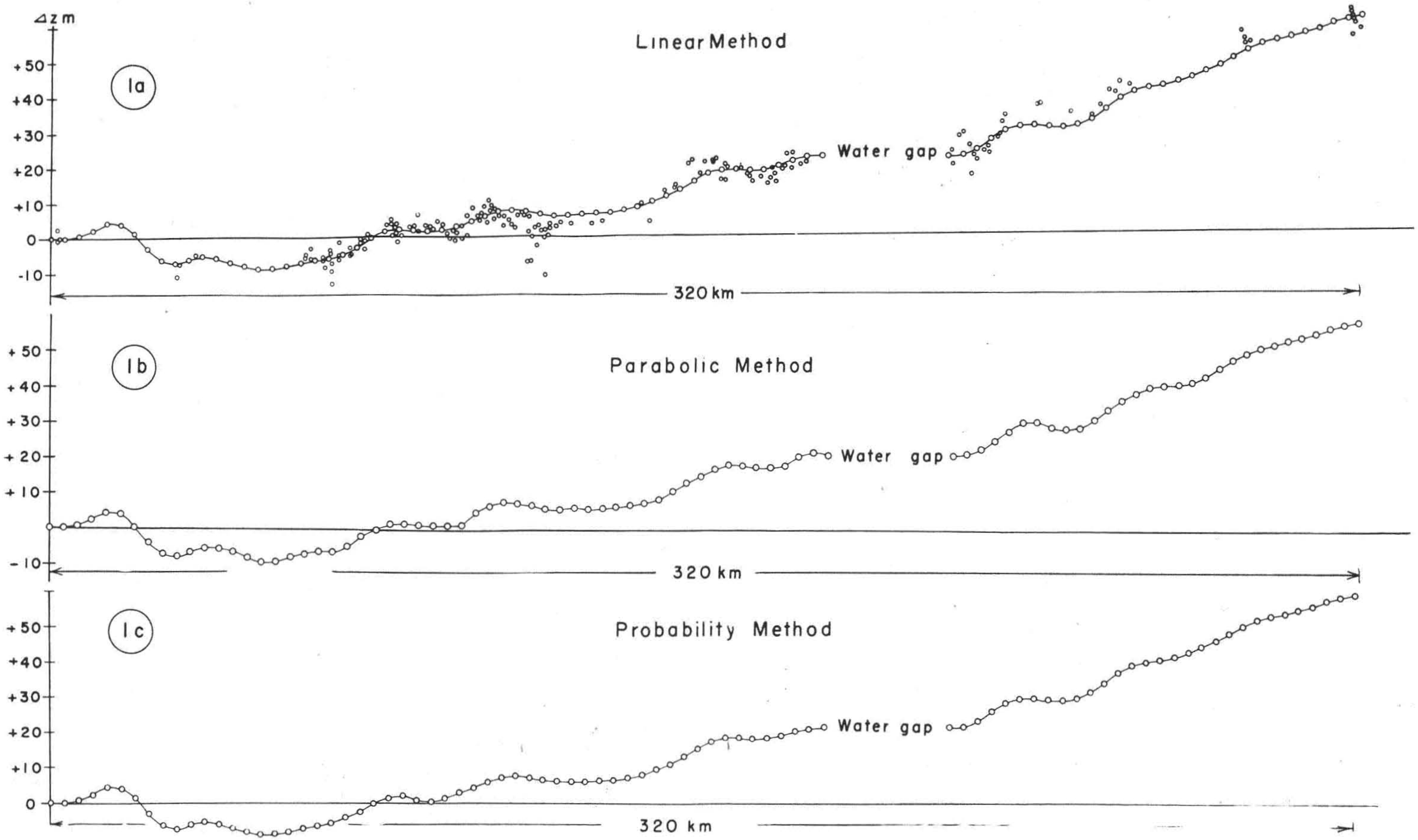

FIG. 1. The linear method (1a), the parabolic method (1b), and the probability method (1c). o—o = Final curve. ∘ = Ground control point.

alone. The errors of Shoran information after processing for photogrammetric use for instance, may be assumed mainly accidental over a considerable area. Starting from this assumption, several possibilities of applying graduation can be developed by combining the Shoran co-ordinates with the photogrammetric ones.

For example, let us consider an aerial triangulation between two geodetic points. The Shoran equipment will provide the relative coordinates of all the air stations. These values can be compared with the corresponding photogrammetric values, i.e. with the "traversing" coordinates obtained by adding up the b_x and b_y components of the bridging. The whole system can be tied into the geodetic coordinates by using the points on first and last model of the bridging. If, in addition, the Radar Profile data are available, the procedure can be extended to a three dimensional "traverse." Some other methods, including a block adjustment, are also conceivable.

Conclusions

The graduation process enables a separation of accidental and systematic errors if two sets of observations possessing different types of errors are available. The effect of systematically accumulating accidental errors is also greatly reduced in certain cases. This offers the possibility of increasing the accuracy of many photogrammetric and other surveying operations. The application is still new in this field but the promising results obtained so far allow the expectation that the theory described above may be a subject for further development and many practical usages.

Bibliography

1. Whittaker & Robinson: "The Calculus of Observations," Blackie & Son Ltd., London and Glasgow, Fourth edition, 1946.
2. Willers, F. A., "Methoden der Praktischen Analysis," Walter de Gruyter & Co., Berlin und Leipzig, 1928.
3. Henderson, Robert, "Mathematical Theory of Graduation," Actuarial Studies No. 4, The Actuarial Society of America, 1938.
4. Wolfenden, Hugh H., "The Fundamental Principles of Mathematical Statistics," The Actuarial Society of America, 1942.
5. Blachut, T. J., "Airborne Controlled Method of Aerial Triangulation," *Photogrammetria*, Congress Issue, 1956.

OMI Corporation of America

Ottico Meccanica Italiana, Rome, has announced the formation of OMI Corporation of America to handle the merchandising in the U. S. and Canada of the complete instruments and products manufactured by "Nistri," O.M.I., Rome. The new corporation will maintain its executive offices at 286 Fifth Avenue, New York. It is establishing the show rooms and map compilation plant at 512 North Pitt Street, Alexandria, Va. In charge of the Alexandria Office will be S. Jack Friedman formerly Chief of the Photogrammetric Section, Ft. Belvoir, Va. The O.M.I. Corporation of America intends to actively engage in giving detailed information on the Nistri instruments throughout North America.