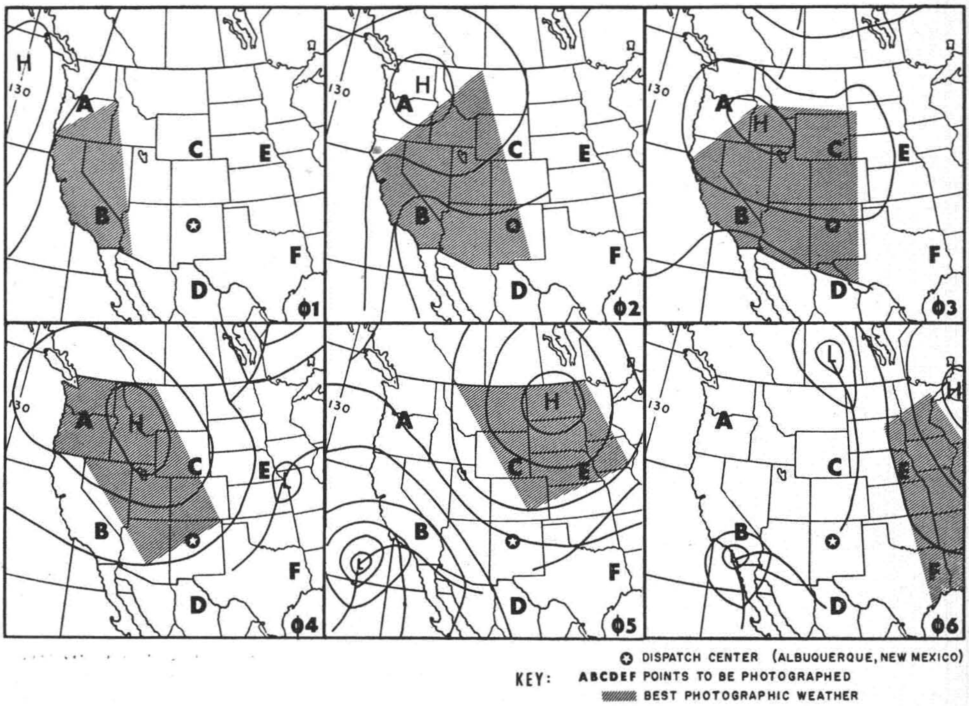

FIG. 7.—Scheduling in Phase with the Weather. Type *F* (Fall and Winter, Zones 3 & 4).

Whatever the problems of any particular operating group, whatever the criteria imiting a particular operation, a form of service can be prepared which will help to minimize difficulties if it cannot entirely eliminate them.

A Contribution to the Problem of Analytical Aerial Triangulation

H. G. JERIE, *Wild Heerbrugg, Ltd., Heerbrugg, Switzerland*

INTRODUCTION

A NUMBER of papers have been published on the problem raised by the analytical determination of the outer orientation elements of aerial photographs and the spatial coordinates of certain points of interest from measurements made with precision stereocomparators. Although these methods up until now were limited almost exclusively to experimental work, the ever growing applications of automatic computers open up wide new prospects for them.

This paper will describe a development of the method presented by Schröder [1] and will present a method designed directly for computing the elements of absolute orientation by using "dependent pairs," instead of "independent stereopairs" (as earlier was the case) which require subsequent work for their mutual correlation. The data made available by H. A. L. Shewell [2] show that the method used by the British Ordnance Survey is all but identical to that of Schröder.

Another approach to the direct computation of the elements of absolute orientation is provided by a comparison of distances on the photographs, as developed by Earl Church [3], and offered in a paper recently published by A. M. Wassef and in which the orientation elements are expressed as Eulerian Angles [4].

In this paper, the elements usually needed to define camera position will be used. In addition, it will deal with the problem posed by aerial photographs taken in mountainous terrain.

The methods offered to date—all of these are based on step-by-step approximations—would then give series which converge very poorly, and in some instances might even prove quite useless.

1. The Influence of Changes in the Outer Orientation Elements of a Camera on the Picture Coordinates of a Point

Starting from the standard position of the camera, with the origin of the coordinate system at the center of projection, and assuming the usually chosen

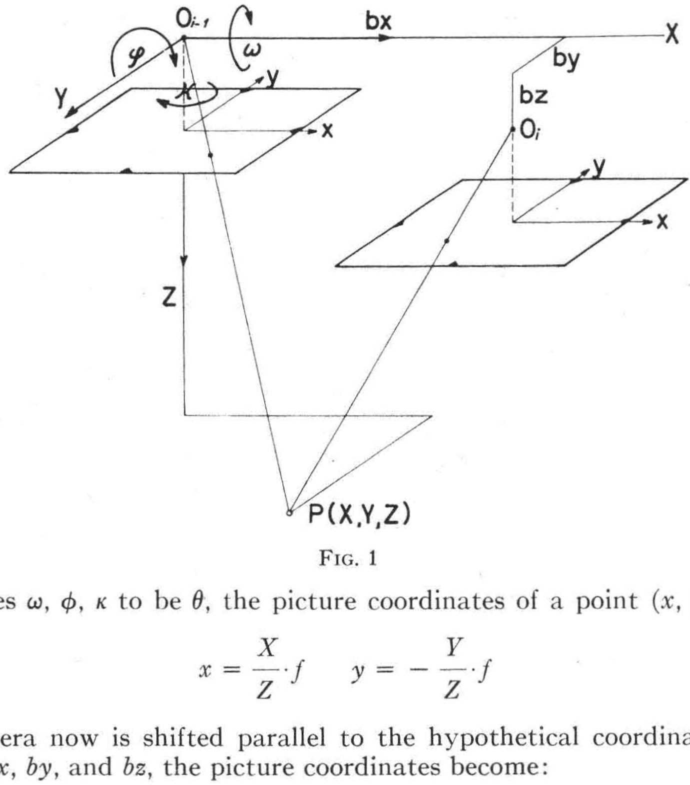

FIG. 1

angle values ω, ϕ, κ to be θ, the picture coordinates of a point (x, y), will be:

$$x = \frac{X}{Z}\cdot f \qquad y = -\frac{Y}{Z}\cdot f \tag{1}$$

If the camera now is shifted parallel to the hypothetical coordinate axes by distances bx, by, and bz, the picture coordinates become:

$$x^{(1)} = \frac{X - bx}{Z}\cdot f \qquad y^{(1)} = -\frac{Y}{Z}\cdot f \tag{2}$$

$$x^{(2)} = \frac{X - bx}{Z}\cdot f = x^{(1)} \qquad y^{(2)} = -\frac{Y - by}{Z}\cdot f = y^{(1)} + \frac{by}{Z}\cdot f \tag{3}$$

$$x^{(3)} = \frac{X - bx}{Z - bz} \cdot f = x^{(2)} \left(1 - \frac{bz}{Z}\right)^{-1} \qquad y^{(3)} = -\frac{Y - bz}{Z - bz} \cdot f = y^{(2)} \left(1 + \frac{bz}{Z}\right)^{-1} \tag{4}$$

If the camera now be turned, in the order, by angles, ω, ϕ, κ about the relevant axes, the picture coordinates are changed, step-by-step, as follows:

$$x^{(4)} = \frac{x^{(3)} \sec \omega}{1 - \frac{y^{(3)}}{f} \cdot \tan \omega} \qquad y^{(4)} = \frac{y^{(3)} + f \cdot \tan \omega}{1 - \frac{y^{(3)}}{f} \cdot \tan \omega} \tag{5}$$

$$x^{(5)} = \frac{x^{(4)} + f \cdot \tan \phi}{1 - \frac{x^{(4)}}{f} \cdot \tan \phi} \qquad y^{(5)} = \frac{y^{(4)} \cdot \sec \phi}{1 - \frac{x^{(4)}}{f} \cdot \tan \phi} \tag{6}$$

$$x^* = x^{(5)} \cos \kappa - y^{(5)} \sin \kappa \qquad y^* = y^{(5)} \cos \kappa + x^{(5)} \sin \kappa \tag{7}$$

The values of x^* and y^* are thus equal to the picture coordinates actually measured in the stereocomparator.

2. Correction of the Measured Picture Coordinates

Conversely, and if the orientation elements are known, the measured picture coordinates x^* and y^* can be transformed into corrected picture coordinates $x^{(1)}$, $y^{(1)}$ by changing about formulae (3) to (7).

$$x^{(5)} = x^* \cos \kappa + y^* \sin \kappa \qquad y^{(5)} = \lambda^* \cos \kappa - x^* \sin \kappa \tag{8}$$

$$x^{(4)} = \frac{x^{(5)} - f \cdot \tan \phi}{1 + \frac{x^{(5)}}{f} \cdot \tan \phi} \qquad y^{(4)} = \frac{y^{(5)} \cdot \sec \phi}{1 + \frac{x^{(5)}}{f} \cdot \tan \phi} \tag{9}$$

$$x^{(3)} = \frac{x^{(4)} \cdot \sec \omega}{1 + \frac{y^{(4)}}{f} \cdot \tan \omega} \qquad y^{(3)} = \frac{y^{(4)} - f \cdot \tan \omega}{1 + \frac{y^{(4)}}{f} \cdot \tan \omega} \tag{10}$$

Some transformation must be carried out in order to reverse formulae (3) and (4) for the purpose of substituting the new unknowns by/bx and bz/bx for by and bz, in order to eliminate the influence of the scale on the computation of the orientation elements. (For derivation see Appendix 1).

$$x^{(1)} = \frac{x^{(3)} \left(1 - \frac{bz}{bx} \cdot \frac{x}{f}\right)}{\left(1 - \frac{bz}{bx} \cdot \frac{x^{(3)}}{f}\right)} \tag{12}$$

$$y^{(1)} = y^{(3)} \left(1 - \frac{bz}{bx} \cdot \frac{x - x^{(1)}}{f}\right) - \frac{by}{bx} \cdot (x - x^{(1)}) \tag{13}$$

Thus x^*, y^* mean the measured picture coordinates on the photograph for which a determination is to be made, while $x^{(1)}$ and $y^{(1)}$ represent the coordinates of the same point but corrected for ω, ϕ, κ, by, bz. x, y are the corrected picture coordinates of the same points in the picture, for which a determination already had been made.

3. Parallax Equation

The requirement for the proper relative orientation of a picture with respect to the preceding one is

$$y^{(1)} = y$$

This process is identical with those customarily used in all analytical methods; it is impossible to express the unknowns directly as functions of the measured picture coordinates x^*, y^*, and of x, y, corrected coordinates of the earlier picture. Therefore, approximate formulae are set up for the picture ordinate differences as a function of the unknown sought.

Approximate values of the orientation elements can be calculated by means of these parallax equations.

These, in turn, are used to correct the measured picture coordinates according to precise formulae (8, 9, 10, 12, 13).

From the corrected picture coordinates, new refinements of the orientation elements can now be computed. This process is continued until no more improvements can be achieved by computation.

In practice, what is done by computation is that taken place automatically in spatial plotting instruments when carrying out the relative orientation.

Following development in series and neglecting terms of higher order, formulae (8), (9), (10), and (13) give the "parallax equation":

$$p = y^* - y = x^* \cdot \kappa + \frac{x^* y^*}{f} \cdot \phi + f \cdot \left(1 + \frac{y^{*2}}{f^2}\right) \omega + y^* \cdot \frac{x - x^*}{f} \cdot \frac{bz}{bx} + (x - x^*) \cdot \frac{by}{bx} \tag{14}$$

4. Control Points

The principal point and two other points on the main vertical, having identical picture ordinates if possible, and the corresponding points on the adjacent photographs are taken as control points.

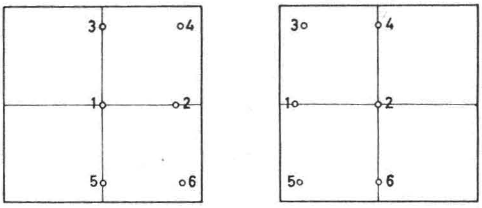

Fig. 2

5. Stereocomparator

The most suitable instrument for these measurements seems to be a stereocomparator having an x-parallax measuring range of the size of the photographs, Dove prisms and possibly a device for the optical interchange of pictures, allowing use of the "base inside"—"base outside" method. Thus, the influences of eccentrically placed photographs and parallax errors at the zero point with respect to scale transfer will disappear.

The y-parallaxes can be measured more exactly by means of the Dove prisms.

6. Computation of Approximate Values of Orientation Elements

If the control points are chosen in the above manner, six equations are available for computing the approximate values of the five unknowns. This system of equations becomes simplified due to symmetry in the arrangement of the control points, thereby making possible computing the unknowns by the method of the least squares—the manner is similar to that used with the formulae for relative orientation given by Hallert [5].

A single additional observation makes this method symmetrical, the accuracy is increased, especially the determination of ω, and some means of verification are provided.

Where the terrain is practically flat, one can use:

$$x - x^* = b/y/ \cdots \text{(for point 3–6)} = d$$

If $y^* - y = p$ is taken, one obtains the following formulae, the well known derivation of which will not be repeated here:

$$\begin{aligned}
\kappa &= -\frac{1}{3b}(p_1 + p_3 + p_5 - p_2 - p_4 - p_6) \\
\phi &= -\frac{f}{2bd}(p_3 - p_5 - p_4 + p_6) \\
\omega &= -\frac{f}{4d^2}(2p_1 + 2p_2 - p_3 - p_4 - p_5 - p_6) \qquad (15) \\
\frac{bz}{bx} &= \frac{f}{2bd}(p_4 - p_6) \\
\frac{by}{bx} &= \frac{1}{3b}(p_2 + p_4 + p_6) - \frac{1}{3b}\left(3f + \frac{2d^2}{f}\right)\omega
\end{aligned}$$

7. Computation of Approximate Values for the Orientation Elements in the Case of Mountainous Terrain

The assumption $(x - x^*) = b = \text{constant}$ no longer applies. Even with comparatively small height differences ($\Delta h \sim 5 - 10\%$ of h for normal angle photographs), the method of successive approximations does not afford satisfactory convergence; in other words a great many approximations are required. In certain cases, in the presence of large height differences, the series used may fail entirely to converge.

This can be avoided, in the computation of approximate values, if formulae are used which allow for such height differences. This action is similar to that in generalizing Hallert's method of orientation to take in any terrain [6].

This procedure is all the more justified by the computation of the approximate values, even in such a case, being only a fraction of the total computation work; the computation of the corrected picture coordinates by means of accurate formulae takes much more time. Consequently the number of repetitions of the computations is the determining factor.

If the control points are chosen as described above and $x - x^* = b$ is taken as a variable, the parallax formula (14) for the six points is modified to:

$$p_1 = -b_1 \cdot \kappa \qquad + \qquad f \cdot \omega \qquad + b_1 \frac{by}{bx}$$

$$p_3 = -b_3 \cdot \kappa - b_3 \cdot \frac{d}{f} \cdot \phi + f\left(1 + \frac{d^2}{f^2}\right)\omega + b_3 \frac{d}{f} \cdot \frac{bz}{bx} + b_3 \frac{by}{bx}$$

$$\begin{aligned}
p_5 &= -b_5\cdot\kappa + b_5\frac{d}{f}\phi + f\left(1+\frac{d^2}{f^2}\right)\omega - b_5\frac{d}{f}\cdot\frac{bz}{bx} + b_5\frac{by}{bx} \\
p_2 &= \qquad\qquad + \qquad f\cdot\omega \qquad\qquad + b_2\frac{by}{bx} \\
p_4 &= \qquad + f\left(1+\frac{d^2}{f^2}\right)\omega + b_4\frac{d}{f}\cdot\frac{bz}{bx} + b_4\frac{by}{bx} \\
p_6 &= \qquad + f\left(1+\frac{d^2}{f^2}\right)\omega - b_6\frac{d}{f}\cdot\frac{bz}{bx} + b_6\frac{by}{bx}
\end{aligned} \tag{16}$$

For the subsequent approximations

$$b = x - x^{(1)} \text{ applies instead of } b = x - x^*$$

To obtain reasonably simple normal equations, a little trick will help. Each correction equation is weighted by factor $1/b_i$. To be sure, there is no theoretical need for this action, but it makes solving the system of standard equations easier, without injecting errors of any measurable influence, because the values of b_i change only slightly. On the other hand, from the standpoint of the theory of errors, the manner in which the closing error, which apparently is attributable to one single supplemental observation, is distributed over the five indispensable ones, is of no consequence. However, it is the only way in which the weighting of the findings exerts an influence.

By introducing additional designations:

$$\frac{d}{f} = \kappa \quad\text{and}\quad 1 + \frac{d^2}{f^2} = 1 + \kappa^2 = \kappa$$

the correction equations take the form:

$$\begin{aligned}
v_1 &= -\kappa \qquad\qquad\qquad + \frac{by}{bx} + \frac{f}{b_1}\cdot\omega - \frac{P_4}{b_1} \\
v_3 &= -\kappa - k\cdot\phi + k\cdot\frac{bz}{bx} + \frac{by}{bx} + \frac{f}{b_3}\kappa\omega - \frac{P_3}{b_3} \\
v_5 &= -\kappa + k\cdot\phi - k\cdot\frac{bz}{bx} + \frac{by}{bx} + \frac{f}{b_5}\kappa\omega - \frac{P_5}{b_5} \\
v_2 &= \qquad\qquad\qquad\qquad + \frac{by}{bx} + \frac{f}{b_2}\cdot\omega - \frac{P_2}{b_2} \\
v_4 &= \qquad\qquad + k\cdot\frac{by}{bx} + \frac{by}{bx} + \frac{f}{b_4}\kappa\omega - \frac{P_4}{b_4} \\
v_6 &= \qquad\qquad - k\cdot\frac{bz}{bx} + \frac{by}{bx} + \frac{f}{b_6}\kappa\omega - \frac{P_6}{b_6}
\end{aligned} \tag{17}$$

After setting up and solving the standard equations one obtains, similar to what was achieved by the derivation shown as [6]:

$$\omega = \omega_1\cdot\frac{u^2}{u^2+v^2} + \omega_2\cdot\frac{v^2}{u^2+v^2} \tag{18a}$$

whereby

$$\omega_1 = \frac{1}{u}\left(2\frac{P_1}{b_1} - \frac{P_3}{g_3} - \frac{P_5}{b_5}\right), \qquad \omega_2 = \frac{1}{v}\left(2\frac{P_2}{b_2} - \frac{P_4}{b_4} - \frac{P_6}{b_6}\right) \tag{18b}$$

and

$$u = \left(2\frac{f}{b_1} - \frac{f}{b_3}\kappa - \frac{f}{b_6}\kappa\right), \qquad v = \left(2\frac{f}{b_2} - \frac{f}{b_4}\kappa - \frac{f}{b_6}\kappa\right) \tag{18c}$$

thereafter the values Pi/bi are corrected for ω-influence

$$\begin{aligned} \frac{\overline{P}i}{bi} &= \frac{Pi}{bi} - \frac{f}{bi}\cdot\omega & i &= 1, 2 \\ \frac{\overline{P}i}{bi} &= \frac{Pi}{bi} - \frac{f}{bi}\cdot\kappa\omega & i &= 3, 4, 5, 6. \end{aligned} \tag{18d}$$

and there is obtained the other elements

$$\begin{aligned} \phi &= -\frac{1}{2k}\left(\frac{\overline{P}_3}{b_3} - \frac{\overline{P}_5}{b_5} - \frac{\overline{P}_4}{b_4} + \frac{\overline{P}_6}{b_6}\right) \\ \kappa &= -\frac{1}{3}\left(\frac{\overline{P}_1}{b_1} + \frac{\overline{P}_3}{b_3} + \frac{\overline{P}_5}{b_5} - \frac{\overline{P}_2}{b_2} - \frac{\overline{P}_4}{b_4} - \frac{\overline{P}_6}{b_6}\right) \\ \frac{by}{bx} &= +\frac{1}{3}\left(\frac{\overline{P}_2}{b_2} + \frac{\overline{P}_4}{b_4} + \frac{\overline{P}_6}{b_6}\right) \\ \frac{bz}{bx} &= +\frac{1}{2k}\left(\frac{\overline{P}_4}{b_4} - \frac{\overline{P}_6}{b_6}\right). \end{aligned} \tag{18e}$$

8. Computation of bx (Scale Transfer)

$$bx = Z\cdot\frac{x - x^{(1)}}{f}\cdot \tag{19}$$

Calculation, from the corrected picture coordinates, of one or several points, of which the height Z is known from the preceding stereopair, for instance the principal point of the common photograph.

9. Computation of by and bz

from calculated by/bx and bz/bx ratios

10. Computation of the Model Coordinates of All Control Points

$$Z = bx\cdot\frac{f}{x - x^{(1)}}, \qquad X = \frac{x}{f}Z, \qquad Y = -\frac{y}{f}Z \tag{20}$$

The model coordinates are referred to the projection center of the first picture of each model.

11. Strip Coordinates

The model coordinates in a uniform strip-system, referred, for instance, to the projection center of the first picture of a strip, are obtained from the model coordinates, as mentioned under 10, by the addition of constants which correspond to the sum of base components bx, by and bz, of all the preceding models.

12. Practical Procedure

For the preparation of the triangulation strip, the control points are pricked only once on the picture where they are on the main vertical. Marking the same points in the adjacent pictures is unnecessary because measuring is done stereoscopically.

The picture coordinates are read in a stereocomparator. When doing this, it is desirable, as already mentioned, to leave one picture in a plate carrier for the purpose of measuring the two following models to which it is common.

Calculation: For the first model, the elements of relative orientation are computed, either in the form of independent stereopairs, by the Schröeder method, or by "dependent pairs" as shown above, whereby the elements of the first picture are provisionally assumed to be zero.

After introducing an approximate value for bx, the spatial coordinates of the given control points are computed (20). Accurate scale value is determined by comparison of spatial distances.

This gives the final values of the base components.

After renewed calculation of the spatial coordinates, it is possible to compute the required model rotation Φ and Ω, either by means of the well-known graphical construction for absolute orientation in spatial plotting instruments, or, if more than three control points are given, by an adjustment in which each point is corrected as follows:

$$v_i = h_0 + X_i\Phi + Y_i\Omega - \Delta Z$$

and by finding the most probable values for Φ, Ω and h_0.

In general Φ and Ω will be sufficiently small so that the influence of the higher order terms which were omitted may be neglected. The same applies to errors which arise through not making allowance for the requirement that one of the two rotations must be carried out about a secondary axis.

By addition of Φ and Ω, new values for ϕ and ω are derived from the elements of relative orientation. The base must also be subject to this rotation.

$$\begin{aligned} \phi_1 &= \phi_1' + \Phi \qquad \phi_2 = \phi_2' + \Phi \\ \omega_1 &= \omega_1' + \Omega \qquad \omega_2 = \omega_2' + \Omega \\ bx &= bx' \cdot \cos\Phi - bz' \cdot \sin\Phi \\ by &= by' \cdot \cos\Omega + bz' \cdot \sin\Omega \\ bz &= bz' \cdot \cos\Phi - bx' \cdot \sin\Phi \end{aligned} \tag{21}$$

Thus the elements of outer orientation of the first and second photograph are found. These, in turn, serve to correct the measuring points of the second picture according to formulae (8, 9, 10, 12, 13).

The differences between the measured picture ordinates of the 3rd picture and the corrected ordinates of the 2nd are set up.

$$P = Y^*[\text{3rd picture}] - Y[\text{2nd picture}].$$

From these parallaxes, the approximate values of the required orientation elements can be found by means of formulae (15) and (18).

These approximate values are needed for the first correction of the picture coordinates measured on the 3rd picture (formulae 8, 9, 10, 12, 13).

By means of the differences between these corrected coordinates of the 3rd picture and of those of the 2nd one, refinements can be applied to the approximate values of the orientation elements.

The originally measured picture coordinates are reduced once more to allow for the corrected values of the orientation elements, and so on.

This procedure is to be repeated until the remaining ordinate differences (y-parallaxes) are within the required tolerances.

The coordinates of all other points measured on the 3rd picture are reduced simultaneously with the determination of the final orientation elements.

bx then is calculated from one or more points known from the preceding model with respect to its coordinates scale transfer (19).

Thus, by and bz, as well as the model coordinates of all measured points, can now be found (formulae 20).

Finally, these model coordinates must be based on a uniform strip system e.g. on the projection center of the first photograph. This simply requires the addition of the sum of the base components of all previous models to the computed model coordinates.

These strip coordinates can now be made subject to one of the usual adjustment methods, in the same manner as if they had been obtained by aerial triangulation in a spatial plotting instrument.

Appendix 1

The well known equation for the normal, or standard case, can be derived from (1) and (2):

$$Z = bx \cdot \frac{f}{x - x^{(1)}} \tag{11}$$

if this is introduced into (4) following transformation:

$$x^{(3)} = x^{(1)} \cdot \frac{1}{1 - \dfrac{bz}{Z}}$$

$$x^{(1)} = x^{(3)} \cdot \left(1 - \frac{bz}{Z}\right)$$

$$x^{(1)} = x^{(3)} \cdot \left(1 - \frac{bz}{bx} \cdot \frac{x - x^{(1)}}{f}\right)$$

$$x^{(1)} = x^{(3)} \cdot \left(1 - \frac{bz}{bx} \cdot \frac{x}{f}\right) + x^{(1)} \cdot \frac{bz}{bx} \cdot \frac{x^{(3)}}{f}$$

one obtains, following the grouping of all $x^{(1)}$:

$$x^{(1)} = x^{(3)} \cdot \frac{1 - \dfrac{bz}{bx} \cdot \dfrac{x}{f}}{1 - \dfrac{bz}{bx} \cdot \dfrac{x^{(3)}}{f}} \tag{12}$$

Similarly one obtains from (4)

$$y^{(3)} = -\frac{Y - by}{Z - bz} \cdot f$$

$$y^{(3)}\left(1 - \frac{bz}{Z}\right) = -\left(\frac{Y}{Z} \cdot f - \frac{by}{Z} \cdot f\right) = y^{(1)} + \frac{by}{Z} \cdot f$$

and, from the above:

$$y^{(1)} = y^{(3)}\left(1 - \frac{bz}{Z}\right) - \frac{by}{Z} \cdot f$$

if (11) now is brought back into the equation, then:

$$y^{(1)} = y^{(3)}\left(1 - \frac{bz}{bx} \cdot \frac{x - x^{(1)}}{f}\right) - \frac{by}{bx}(x - x^{(1)}). \qquad (13)$$

Literature Cited

1. Schröder, F.: "Die Rechnerische Orientierung von Luftbildaufnahmen auf Grund von Messungen am Stereokomparator und ihr Verhaltnis zu den Optisch-Mechanischen Verfahren," Diss. T. H., Hannover, 1949.
2. Shewell, H. A. L.: *Photogrammetric Record*, 1952/2.
3. Earl Church: "Theory of Photogrammetry," Bull. Nr. 19.
4. Wassef, A. M.: "Some Recent Development in Analytical Photogrammetry: The Use of Eulerian Angles and Computational Procedure," *Photogrammetria*, 1953–54/2.
5. Hallert B.: "Ueber die Herstellung Photogrammetrischer Pläne," Diss. T. H., Stockholm, 1944.
6. Jerie, H. G.: "Beitrag zu Numerischen Orientierungsverfahren für Gebirgiges Gelande," *Photogrammetria*, 1953–54/1.