*Determination of Geographic Coordinates, Flight Heights, and True Orientations for Extensions of Strips of Aerial Photographs**

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INTRODUCTION

As THE use of high-speed electronic computers is increased, it is inevitable that a greater number of analytical—as compared to instrumental—computations will be performed in photogrammetry.

One goal of mapping from aerial photographs is to reduce to a minimum the quantity of ground control required. The extension of horizontal and vertical control by analytical photogrammetric methods is constantly becoming more nearly an economic reality. If only two or three photographs in a strip are being considered, the area covered is usually small enough that the effect of earth's curvature and the convergence of meridians is neglected. However, as the extension process is continued, so that long strips of aerial photographs are considered, factors such as earth curvature and convergence of meridians are no longer negligible. The situation is aggravated with high-altitude photography since this commonly means smaller-scale photography and a greater distance covered by a strip of photographs.

A method is presented herein for determining the geographic coordinates of both exposure stations and ground control pass points. These coordinates are the latitude and longitude as well as the true heights above sea level for the computed points as based on any adopted spheroid. At the same time the orientation elements of the photographs at each exposure station are determined. This orientation is most commonly expressed as tilt, swing, and azimuth of the photograph.

It is convenient, especially on long strips of photographs, to have the true azimuth of the photograph rather than some grid azimuth determined. On extensive areas it is common practice to have the positions of widely separated control points specified in terms of geographic coordinates, that is, in terms of latitude ϕ, longitude λ, and elevation above mean sea level h. Hence it is desirable that the positions of points sought be expressed in the same terms of geographic coordinates. Herein, mean sea level is indicated by the surface of the reference spheroid. If the surface of the geoid be used as the surface of mean sea level, a correction should be added. Of course, once the geographic coordinates of the exposure stations and of the ground control have been computed, any conventional geodetic process may be followed in converting these to a particular system of coordinates for a desired type of map projection.

GEOCENTRIC COORDINATES

The geographic coordinates and the elevation above sea level of the ground control points are first transferred to a rectangular coordinate system. This

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rectangular coordinate system is established with its origin at the center of the reference ellipsoid of the earth. The $+Z$ axis is toward the north pole, the $+X$ and $+Y$ axes lie in the plane of the equator with the $+X$ axis in the meridian plane through Greenwich. The relation between geocentric coordinates and geographic coordinates was developed by Professor Earl Church.[1] The formulas to convert the geographic coordinates and elevation of a point to its corresponding rectangular coordinates are given as:

$$X = \frac{a \cos \phi \cos \lambda}{\sqrt{1 - e^2 \sin^2 \phi}} + h \cos \phi \cos \lambda \qquad (1)$$

$$Y = -\frac{a \cos \phi \sin \lambda}{\sqrt{1 - e^2 \sin^2 \phi}} - h \cos \phi \sin \lambda \qquad (2)$$

$$Z = \frac{a(1 - e^2) \sin \phi}{\sqrt{1 - e^2 \sin^2 \phi}} + h \sin \phi \qquad (3)$$

where,

a = semi-major axis of the earth
b = semi-minor axis of the earth

$e^2 = \frac{a^2 - b^2}{a^2}$ = eccentricity of the reference spheroid of the earth.

ϕ, λ, and h = geographic coordinates and elevation above sea level of the ground control point.

For the Clark spheroid of 1866,

a = 20, 925, 832 ft.
b = 20, 854, 892 ft.

This X, Y, and Z coordinate system is called the geocentric coordinate system to distinguish it from any local space survey coordinate systems on the ground which will be introduced later.

Figure 1 shows the relationships existing between the X, Y, and Z geocentric coordinates of a point and the corresponding ϕ, λ, and h geographic coordinates and elevation above sea level for the point.
In which

X, Y, Z = geocentric coordinate system
M, N, K = local survey coordinate system
x, y, z = photographic coordinate system
a, b = semi-major and semi-minor earth axes
L = exposure station
f = focal length of lens
n and N' = nadir point on the photograph and on the ground respectively
p = principal point of the photograph
ϕ, λ = latitude and longitude of the nadir point N'
h = flight height
t, s, α = tilt, swing, and azimuth of the photograph

If three or more ground control points with their geographic coordinates and elevations above sea level are given at the beginning of strip of photographs, it

[1] Professor Earl Church, "Coordinate Determination of Ground Points on Aerial Photographs." (Air Force Problem 48), Problem Report No. 25, June 15, 1951.

is possible to determine the coordinates of the exposure stations and the orientation elements of successive photographs in the strip. This is performed by operating geometrically with the intersection of the bundles of rays from the point of the exposure station, through the photo image points, to the corresponding ground points. Use of electronic computers enables these computations to be

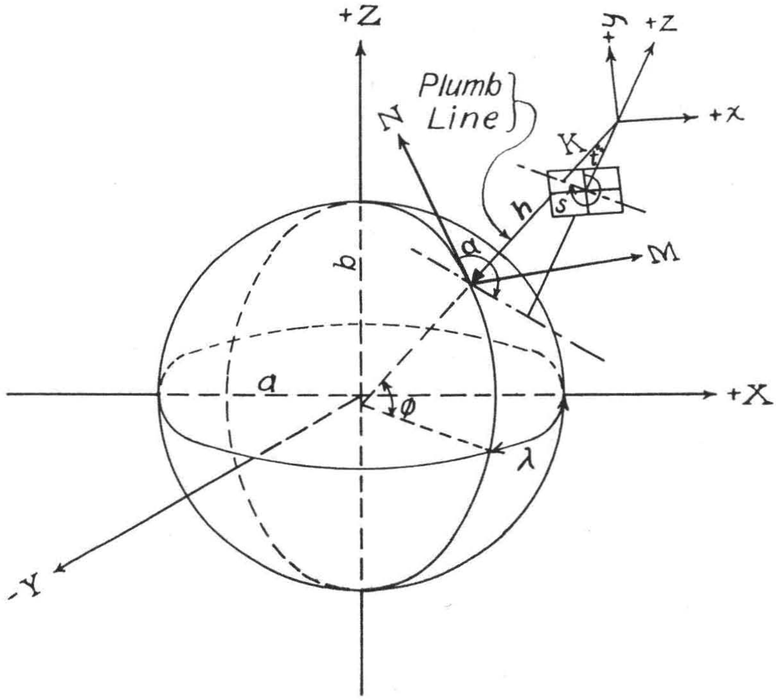

Fig. 1. Relation between geocentric, local survey and photographic coordinate systems.

performed in a matter of minutes or even of seconds. Coordinates of pass points for successive photographs are, of course, also computed.

Geographic Coordinates

If the exposure stations of the photographs in the strip are solved using the geocentric coordinates of the ground control points at the beginning of the strip, then the newly determined coordinates of the exposure stations of the photographs in the strip will also refer to this geocentric coordinate system. By reversion of equations (1), (2), and (3), the following equations are obtained.[2]

$$\tan \lambda = -\frac{Y}{X} \tag{4}$$

$$\tan \phi = \frac{Z}{\sqrt{X^2 + Y^2}} \frac{(a^2 + ah)}{(b^2 + ah)} \tag{5}$$

[2] Phase I—Interim Technical Report, "Development of Computational Procedure Suitable for Use with Electronic Computing Equipment for Bridging Horizontal and Vertical Control in Military Mapping," by Cornell University, Engineer Research & Development Laboratories, July 1, 1955.

and

$$h = \frac{Z}{\sin \phi} - \frac{a(1 - e^2)}{\sqrt{1 - e^2 \sin^2 \phi}} \tag{6}$$

in which again

$$e^2 = \frac{a^2 - b^2}{a^2}$$

The geographic coordinates and the elevation above sea level for all the exposure stations can be solved by the method of successive approximation from these equations. Equation (4) can be solved directly. The solution of (5) and (6) follows a process of successive approximation by initially assigning an approximate quantity for h in (5). After an approximate ϕ is obtained from equation (5), then since e^2 is a known constant for any given spheroid, the values are substituted in equation (6) to compute a new approximate value for h. By alternately using equations (5) and (6), the latitude, ϕ, and elevation of the exposure station above sea level, h, will converge to their correct values. Usually no more than two successive approximations are necessary.

By the above method, the geographic coordinates of all the exposure stations as well as their elevations above sea level can be obtained. Therefore, the curvature of the earth has been take care of automatically.

Some variations of equations (1) through (6) should be noted.

$$Z = \left[\frac{a(1 - e^2)}{\sqrt{1 - e^2 \sin^2 \phi}} + h\right] \sin \phi \tag{3a}$$

and

$$X^2 + Y^2 = \frac{a^2 \cos^2 \phi}{1 - e^2 \sin^2 \phi} + \frac{2ha \cos^2 \phi}{\sqrt{1 - e^2 \sin^2 \phi}} + h^2 \cos^2 \phi \tag{3b}$$

hence

$$\sqrt{X^2 + Y^2} = \left[\frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} + h\right] \cos \phi \tag{3c}$$

Dividing (3a) by (3c) gives exactly

$$\tan \phi = \left[\frac{Z}{\sqrt{X^2 + Y^2}}\right]\left[\frac{a^2 + ah\sqrt{1 - e^2 \sin^2 \phi}}{b^2 + ah\sqrt{1 - e^2 \sin^2 \phi}}\right] \tag{5a}$$

Thus it is noted that equation (5) as used is an approximate form. However, the approximation is small and equation (5) is satisfactory for most precisions. When photography is encountered such that ϕ is much greater than 45° then the relationship $1/\tan \phi = \cot \phi$ should be used.

Equation (6) for flight altitude is seen from Figure 1 to be derived as a difference between the length of two normals terminating in the major axis.[3] Inspection shows that as the photography approaches the equator, both Z and $\sin \phi$ approach zero. Equation (6) is not sufficiently accurate for the range

[3] "Geodesy," by George L. Hosmer, John Wiley & Sons, Inc., New York, 1930, page 172.

$2° > \phi > -2°$. A general, though more elaborate, formula is obtained as the difference between the lengths from the origin of the geocentric coordinates to the given point and to the surface of the spheroid where h equals zero.

$$h = \sqrt{X^2 + Y^2 + Z^2} - a\sqrt{1 - \frac{e^2 \sin^2 \phi(1 - e^2)}{1 - e^2 \sin^2 \phi}} \tag{6a}$$

Space Orientation

The next step is to determine the nadir points of the photographs and the principal planes of the photographs with respect to true north. That is to say, the true azimuth of the principal plane of each photograph in the strip will be sought.

The orientation elements between each photographic coordinate system and the geocentric coordinate system can be obtained by any of the known methods of analytical aerial triangulation. The more common of these solutions include:

(1) a combined solution to solve for the geocentric coordinates of the exposure station and the three independent orientation elements simultaneously (the Herget solution);

(2) a bifurcated solution to solve for the geocentric coordinates of the exposure station, first, and then the orientation elements of the photograph (the Church solution);

(3) a solution, first, for the relative orientation between each two adjacent photographs in the strip, second, for the absolute orientation elements which are the orientations between the photographic coordinate system and the geocentric coordinate system, and finally, third, for the geocentric coordinates of the exposure stations in the strip (the British Ordnance Survey Method).

The computed direction cosines between the geocentric coordinate axes and the photographic coordinate axes can be shown in Table 1.

Table 1. Symbolized Table of Direction Cosines

	x	y	z
X	u_1	v_1	w_1
Y	u_2	v_2	w_2
Z	u_3	v_3	w_3

In this table, u^2 for instance, symbolizes the known direction cosine between the geocentric Y axis and the photographic x axis.

For any given photograph, tilt, swing, and azimuth relate the nadir point, the principal plane of the photograph, and the true azimuth or angle between the principal plane and true north. To obtain tilt, swing, and azimuth, a local space coordinate system is introduced at each exposure station. This local space coordinate system is assumed:

(1) to have its $+Y$ axis in the meridian plane of the exposure station of the photograph in the strip with its positive direction toward the north;

(2) to have its $+Z$ axis in coincidence with the normal line drawn from the surface of the ellipsoid through the exposure station upward; and

(3) to have its $+X$ axis perpendicular to the $Z-Y$ plane with its positive direction toward the east.

The direction cosines between the geocentric and local space coordinate systems will be:

$$\left.\begin{array}{lll}\cos MX = +\sin\lambda & \cos NX = -\sin\phi\cos\lambda & \cos KX = +\cos\phi\cos\lambda \\ \cos MY = +\cos\lambda & \cos NY = +\sin\phi\sin\lambda & \cos KY = -\cos\phi\sin\lambda \\ \cos MZ = 0 & \cos NZ = +\cos\phi & \cos KZ = +\sin\phi\end{array}\right\} \quad (7)$$

In which ϕ and λ = geographic coordinates of the exposure station of any one photograph in the strip, whose tilt, swing, and azimuth are sought:

M, N, and K indicate the local X, Y, and Z axes respectively.

MX, MY, and MZ = the angles made by the space coordinate X axis with the geocentric coordinate X, Y, and Z axes respectively.

NX, NY, and NZ = the angles made by the space coordinate Y axis with the geocentric coordinate X, Y, and Z axes respectively.

KX, KY, and KZ = the angles made by the space coordinate Z axis with the geocentric coordinate X, Y, and Z axes respectively.

From the known orientation elements between the photographic coordinate axes and the geocentric coordinate axes as shown in table (1) and from the known orientation elements between the geocentric coordinate axes and the local space coordinate axes as given in Equation (7), the orientation elements between the photographic coordinate axes and the corresponding local space coordinate axes can be computed. Only five of the nine elements of orientation have a direct relation to the tilt, swing, and azimuth of the photograph. These five equations are:

$$\left.\begin{array}{l}\cos Kx = \cos KX \cdot u_1 + \cos KY \cdot u_2 + \cos KZ \cdot u_3 \\ \cos Ky = \cos KX \cdot v_1 + \cos KY \cdot v_2 + \cos KZ \cdot v_3 \\ \cos Mz = \cos MX \cdot w_1 + \cos MY \cdot w_2 + \cos MZ \cdot w_3 \\ \cos Nz = \cos NX \cdot w_1 + \cos NY \cdot w_2 + \cos NZ \cdot w_3 \\ \cos Kz = \cos KX \cdot w_1 + \cos KY \cdot w_2 + \cos NZ \cdot w_3\end{array}\right\} \quad (8)$$

The quantities on the right side of this equation are known from the relations of Table 1 and equation (7). In the equation (8), cos Kx and cos Ky denote the direction cosine elements of the local space coordinate Z axis with reference to the photographic x and y axes respectively. Cos Mz, cos Nz, and cos Kz denote the photographic coordinate z axis with reference to the local space coordinate X, Y, and Z axes respectively.

The introduction of the above local space coordinate system for each photograph in the strip is only for the purpose of solving the five quantities on the left side of equation (8) in order to obtain the tilt, swing, and true azimuth of each of the photographs in the strip. The following equations give tilt, t, swing, s, and azimuth, α:

$$\left.\begin{array}{l}\cos t = \cos Kz \\ \tan s = \cos Kx/\cos Ky \\ \tan \alpha = \cos Mz/\cos Nz\end{array}\right\} \quad (9)$$

Since cos Kz is always positive, therefore, cos t is always positive. The proper quadrant of s and α can be determined by the criterion:

$$\left.\begin{array}{l}\text{Sin } s = -\cos Kx/\sin t \text{ and } \cos s = -\cos Ky/\sin t \\ \text{Sin } \alpha = -\cos Mz/\sin t \text{ and } \cos \alpha = -\cos Nz/\sin t\end{array}\right\} \quad (10)$$

This method of solving tilt, swing, true azimuth, and flight height of each photograph in a strip can be extended theoretically to an indefinite length of strip. The only prerequisite is that the extension of control by any method has previously been completed for each photograph. It should be noticed that the corrections for the curvature of the earth and for the convergence of meridians will automatically be taken care of by equations (6) and the third equation of (9).

In the course of extension, the new pass points, whose locations on the ground are given in geocentric coordinates, can be transferred to the geographic coordinates ϕ and λ and the ground elevation above sea level, h. These quantities can further be transferred to any map projection or can be applied to the mapping by means of photogrammetric stereo-plotters.

If ten significant figures are used in the computations of ϕ, λ, and h, then ϕ and λ will be determined to the nearest 100th of one second of arc, and h will be determined to the nearest foot. Only eight significant figures are required to provide computations of tilt, swing, and azimuth to comparable accuracy.

The high speed electronic computer gives promise that the photogrammetric extension and control problem may be solved more rapidly, accurately, and economically than by the mechanical-optical method.