the use of the UNIVAC. The project consisted of five strips of photography, at a scale of 1:10,000, with each strip twenty-two models in length. There were four bands of control points, spaced approximately seven models apart. The bands are considered normal to the line of flight, with the control points located in the center of the side-lap between flights. Thus there were eight control points for each flight. The basis for comparing the accuracy of results was the degree of fit to the vertical control and the difference between independently adjusted elevations for passpoints common to adjacent strips. In the graphical adjustment of the project the correction curves were so constructed as to reduce the errors on the majority of the control points to zero. A mean of the resulting differences on passpoints common to adjacent strips was 0.82 meter. The computed adjustment resulted in residual errors on the control of less than one meter for four of the five strips. On the fifth strip the maximum residual error on a vertical control point was −1.8 meters; however, no vertical control on this strip could be proven to be in error. A mean of the resulting differences on passpoints common to adjacent strips was 0.97 meter. It was apparent from examining the pattern of residual errors that the greater deviation in fitting the assumed error surface to the actual error surface occurs in the longitudinal direction.

A comparison of the economy and efficiency of the two methods indicates the overwhelming superiority of automatic computation. The time and cost figures are based on a twenty-model extension. If the time rate is .8 hour per model and the hourly cost is $2.62, then the total time for the graphical adjustment of a 20-model extension is 16 hours and the cost is $42.00. On the other hand, the total time for the automatic adjustment including preparation of the data on magnetic tape, the computing time, and the printing of the adjusted data would require approximately thirty minutes. The cost would be $8.75. Thus the automatic adjustment method would cost about one-fifth as much as the graphical method.

The application of automatic computation for vertical adjustment is in its infancy at Army Map Service; however, if the results from future use of this method prove equally encouraging, it will undoubtedly be adopted as part of the standard procedure.

*A New Look at Lens Distortion**

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ABSTRACT: *A new approach to the interpretation of distortion is presented. A curve of distortion based on variations of focal length rather than radial image displacement is proposed. A simple method is described relating height errors in the stereoscopic model to lens distortion.*

WHAT does lens distortion mean to a photogrammetrist? What is his conception of the term "distortion," particularly distortion in the photograph? The dictionary defines distortion as a twisting out of shape. If this definition be applied to the distorted photograph, we would say that the images are twisted out of shape relative to the object. This may seem to be a simple definition; however, there are two basic concepts in the definition, namely: "shape" and "twisted." What is meant by the former? The dictionary says: "Shape is the quality of a thing depending on the relative positions of all the points in its outline or external

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surface." If we were to go no further than the dictionary, we would conclude that distortion in the photograph is a twisting of the relative positions of all the points of the photograph. I emphasize that it is a twisting of the *relative* positions of *all* the points.

This is a good place to stop, for we have arrived at a good conception of distortion. From this we can conclude that the position or displacement of *one* point does not constitute distortion; neither does the displacement or position of *two* points, since no shape is involved. The twisting or changing of the relative positions of *three* points provide the *first measure* of distortion. The displacement or twisting of relative positions of *all* points gives a *true picture* of distortion.

Determination of Distortion

The determination of lens distortion is one of the measurements made by testing laboratories in the process of camera calibration. The distortion information is usually given to the photogrammetrist either as a tabulated list of values, a plotted curve, or both. Sometimes, such data may be merely filed by the photogrammetrist and forgotten. Or he may observe the distortion curve for a camera lens and note that the values do not exceed ± 0.10 mm. He may decide that this lens will work well with his plotting equipment. If he notes only the maximum positive and negative values, he may easily make a very bad guess, because each tabulated or plotted value of distortion is as important as the maximum and minimum. All values must be considered in evaluating the lens.

Taken as an example will be an aerial camera which is sent to a testing laboratory for calibration, with concern at this time only for the bare essentials of determining distortion. How does the distortion affect the photogrammetrist? Possibly some quantities will be called by names that are not in the good graces of the Society's Nomenclature Committee. Excuse on the grounds of striving for simplicity is hoped for.

The schematic diagram of a camera included in Figure 1, shows only the lens and the focal plane. The distance from the lens to the focal plane is so adjusted that a sharp photograph is obtained.

To determine the distortion of the camera lens, the procedure is as follows. The camera is placed above a series of collimators containing illuminated targets which have been carefully arranged to provide precisely known angles from the lens axis. The targets are photographed. The distance on the photograph between the central target and the first angular target is measured. This distance is then divided by the tangent of the known angle between the two collimators. (This angle is usually $7\frac{1}{2}$ degrees or 10 degrees.) The resulting value is commonly called the equivalent focal length. The distance on the photograph from the central target to the second target is then measured and this distance is divided by the tangent of the second angle, for example, 15 degrees. A second value or equivalent focal length is thus determined. The process is repeated until a value of equivalent focal length is obtained for each angular target position.

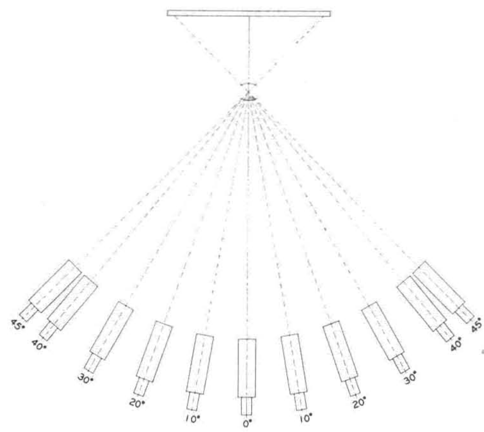

FIG. 1. Schematic diagram showing method of determining distortion.

In Curve *A* (Figure 2), the equivalent focal lengths are plotted as ordinates and the angles as abcissas. The relation between these values determines the distor-

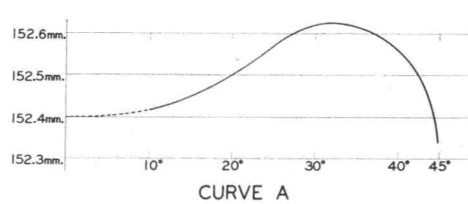

Fig. 2. Variation in equivalent focal length with angle of axis.

tion of the photograph. Assuming a symmetrical distribution, all the distortion data that are needed by the photogrammetrist are contained in this curve. These values are representative of the photograph taken with the camera being tested. The distortion data cannot be based on a particular focal length since it is a function of the relation between focal lengths. Those who like their distortion data based on the equivalent focal length, as well as those who favor the calibrated focal length, should not object since each focal length is included in the data.

Curve *B* (Figure 3), represents the usual method of plotting distortion based on the equivalent focal length, and Curve *C* the same lens with distortion data based on the calibrated focal length. To obtain

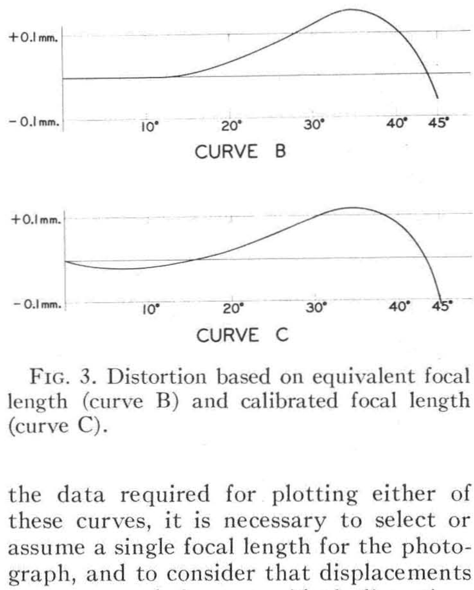

FIG. 3. Distortion based on equivalent focal length (curve B) and calibrated focal length (curve C).

the data required for plotting either of these curves, it is necessary to select or assume a single focal length for the photograph, and to consider that displacements are measured from an ideal distortion-free photograph of this focal length. The difficulty with this method is the fundamental one of deciding which focal length to use. The assumed focal length that is chosen determines the magnitude of the quantity that will be called displacement or distortion. Thus, there arises the unenviable situation of creating what appears to be a variable out of a physical characteristic of the photograph which ought to be considered a constant.

I do not agree that it is advantageous to rotate or mathematically adjust the distortion curve. Why should one want to adjust a quantity which is a characteristic of the photograph? We don't take two photographs with the camera—one good, the other distorted; we have only the distorted one with its changing scale. This changing scale could be plotted directly as shown in Curve *A*, Figure 2.

INTERPRETATION OF DISTORTION CURVES

Curves *B* and *C*, which are usually employed to represent distortion will be examined. There is nothing mathematically incorrect with these curves or with the usual method of treating distortion. Either Curve *B* or Curve *C* contains sufficient information for the photogrammetrist. I do say, however, that the usual methods may not be easy to interpret by the average photogrammetrist. I venture to say that even one skilled in engineering or mathematics is likely to misinterpret the data if care is not taken. As an illustration, let us consider Curve *B*, which is based on the equivalent focal length. The radial displacements are all positive until we get to the region near 43 degrees. In Curve *C*, which is based on the calibrated focal length, the radial displacements take on negative values from zero to 15 degrees. It might appear from this curve that images in this region are displaced toward the axis because of angular distortion of the lens. Such is not actually the case. Another common misinterpretation of this curve concerns the scale of the photograph. Since the curve touches or crosses the *X* axis at 0, 15, and 43 degrees, it might be assumed that at these three values the scale of the photograph is the same.

An examination of Curve *A* (Figure 2), which is a plot of the variation in focal length or changing scale of the photograph, shows that the scale at 15 degrees and 43 degrees is equal, but that the photograph has a different scale near the axis. This curve also shows that the focal length or scale increases continuously from near the axis to approximately 32 degrees and then decreases continuously to 45 degrees. There is no real displacement of images toward the axis in the region of 0 to 15 degrees.

It has been shown how Curve *A* accurately depicts the changing scale of the photograph. It will now be shown how this curve is, to a remarkable degree, directly related to the vertical deformation of the stereoscopic model. It is possible to determine the elevation error of many points of the stereo model by merely reading the ordinate of the curve at the proper point. It should be noted that this curve is a representative curve for the metrogon lens and is only used to illustrate the principle of determining elevation error in the stereo model. In most stereoscopic instruments a distortion compensation system is used; to find the height error under such conditions it would be necessary to plot the curve representing the residual distortion. After this is done, the distortion is treated in the same manner as in the example which follows. For purposes of illustration, it will be assumed that Curve *A* in Figure 4 represents uncompensated distortion where the distortion has been pictured as a change in focal length.

First, it is necessary to assign a base-height and width-height ratio for the

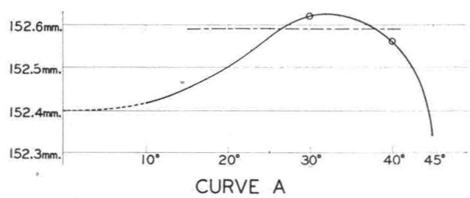

FIG. 4. Relation of change in equivalent focal length to deformation of stereoscopic model.

stereo model—that is, one must decide where the parallax is to be removed and the model leveled. Under normal conditions the location of each of the four points for parallax removal and leveling occurs approximately 30 degrees from one camera station and 40 degrees from the second station. The focal lengths corresponding to the 30 degree and 40 degree points have been marked on Curve *A* and a line parallel to the *X*-axis has been drawn midway between these focal lengths. The vertical error in the stereoscopic model for certain points can now be read directly from the curve. The center of the stereo model occurs at approximately 17 degrees and the difference in focal length between the base line and the curve in this region represents the vertical error at the scale of the photography. The curve from 20 degrees to 40 degrees is a profile of model error through the center of the model perpendicular to the air base. This curve can thus be used to determine directly errors in the central area, the most sensitive area of the stereoscopic model. From this curve, the photogrammetrist can tell quickly how well a particular camera or lens will work with his plotting equipment.

Form for Computing Model Deformation

A computation form (Figure 5) has been developed which allows the photogrammetrist to compute quickly the distortion error for 15 points well distributed throughout the stereoscopic model. Multiplication, addition and subtraction are the only required operations. The camera lens distortion is entered opposite the appropriate angles. The distortion may be based on the calibrated, equivalent or any other focal length; it makes no difference in the final answer. This is true because the computations are based entirely on differences in focal length as shown graphically by the focal length curve. The form supplies instructions for the proper use of multipliers for the various points in the model. It might be mentioned that two lines at the top of the form could be eliminated if distortion data were given as a change in focal length or scale.

In Figure 5, nominal values for the distortion of the metrogon lens are entered in the form. For this example, it is assumed that there is no instrument or printer compensation for this distortion. The instructions as shown at the right margin of the form are followed to compute the vertical error for the points indicated. Figure 7, shows the approximate location of each point in the stereoscopic model in relation to a 5 mm. grid. The angular coordinates of the points from each projector or camera station are shown in Figure 8. The value for vertical error for each point as computed in Figure 6 are at the scale of the aerial negative. For example, if the nominal camera focal length is 152.4 mm. and the stereoscopic model is to be measured in the Kelsh plotter at a 760 mm. projection distance, then the values in the form must be multiplied by 5 to be comparable with model readings.

In order to relate the computations shown in Figure 5 to a practical example, it is assumed that it is desired to determine the accuracy of distortion compensation for a Kelsh plotter using either a mechanical cam or diapositive printer type of distortion compensation. It is further

VERTICAL ERROR IN MODEL DUE TO RESIDUAL RADIAL DISTORTION IN PHOTOGRAMMETRIC SYSTEM

Column →	1	2	3	4	5	6	7	8	9	
Line	5°	10°	15°	20°	25°	30°	35°	40°	45°	Angle Degrees
↓	.001	.005	.016	.037	.067	.100	.119	.081	-.105	Camera Lens Distortion.
	.000	.000	.000	.000	.000	.000	.000	.000	.000	Instrument Compensation.
1	.001	.005	.016	.037	.067	.100	.119	.081	-.105	Camera minus Instrument.
2	11.43	5.67	3.73	2.75	2.15	1.73	1.43	1.19	1.00	Cotangent Angle
3	-.011	-.028	-.060	-.102	-.144	-.173	-.170	-.096	.105	Multiply Line 1 By Line 2. Change Sign.
4	-.173	-.173	-.173	-.173	-.173	X	-.173	X	X	Enter value of Col. 6 Line 3 in all spaces
5	.162	.145	.113	.071	.029	0.0	.003	X	X	Subtract Line 4 from Line 3
6	X	X	X	-.102	-.144	-.173	-.170	-.096	.105	Copy Line 3 on this line
7	X	X	X	-.096	-.096	-.096	-.096	-.096	-.096	Enter value of Col. 8 Line 3 in all spaces
8	X	X	X	-.006	-.048	-.077 Point M	-.074 Point N	0.0 Point O	.201	Subtract Line 7 from Line 6.
9	X	X	X	.035	.015	0.0	.001	X	X	Value in Line 5 divided by 2.
10	X	X	X	-.003	-.024	-.038	-.037	X	X	Value in Line 8 divided by 2.
11	X	X	X	.032 Point A	-.009 Point B	-.038 Point C	-.036 Point D	X	X	Sum of Line 9 and Line 10
12	X	.036	X	.018	X	0.0	.001 Point H	X	X	1/4 Value in Line 5
13	X	X	X	X	-.036	-.058	-.056 Point G	X	X	3/4 Value in Line 8
14	-.020	-.018	X	X	-.004 Point R	X	.000	X	X	Line 5 multiplied by − 1/8
15	X	X	X	X	X	X	-.083	X	.226	Line 8 multiplied by 9/8
16	X	X	.045	X	X	X	X	X	X	Line 5 multiplied by 0.4
17	X	X	X	-.004	X	X	X	X	X	Line 8 multiplied by 0.6

ENTER VALUES FROM TABLE ABOVE IN SPACE BELOW AS INDICATED

Line 12, Col. 2	.036	Line 12, Col. 4	.018	Line 14, Col. 1	-.020	Line 14, Col. 2	-.018	Line 14, Col. 7	.000	Line 16, Col. 3	.045
Line 13, Col. 5	-.036	Line 13, Col. 6	-.058	Line 15, Col. 7	-.083	Line 15, Col. 7	-.083	Line 15, Col. 9	.226	Line 17, Col. 4	-.004
ADD	.000 Point E	ADD	-.040 Point F	ADD	-.103 Point P	ADD	-.101 Point Q	ADD	.226 Point S	ADD	.041 Point T

$\frac{B}{H} = 0.62 \qquad \frac{W}{H} = 1.12$

FIG. 5. Effect of nominal metrogon distortion with no instrument compensation. (Values shown are at the scale of the aerial negative.)

assumed that the compensation device is designed to correct for nominal metrogon distortion as shown in Figure 5. If a mechanical cam compensation is used, a pair of precise 5 mm. grids would be placed in the instrument. The parallax would be removed at the four points marked *O* in Figure 7, and the model would be leveled to these four points. The elevation for each of the 15 points shown in Figure 7 would be read and recorded. The readings would be referenced to a zero reading for point *O*. The values computed in the form in Figure 5 would be multiplied by five. For example, point *T* has a value of +0.041 mm. from the form. When multiplied by five, the value becomes +0.2 mm. The value, read in the stereoscopic model for point *T*,

VERTICAL ERROR IN MODEL DUE TO RESIDUAL RADIAL DISTORTION IN PHOTOGRAMMETRIC SYSTEM

Column →	1	2	3	4	5	6	7	8	9	
Line ↓	5°	10°	15°	20°	25°	30°	35°	40°	45°	Angle Degrees
	0	0	.012	.035	.080	.124	.145	.120	-.055	Camera Lens Distortion.
	.001	.005	.016	.037	.067	.100	.119	.081	-.105	Instrument Compensation.
1	-.001	-.005	-.004	-.002	.013	.024	.026	.039	.050	Camera minus Instrument.
2	11.43	5.67	3.73	2.75	2.15	1.73	1.43	1.19	1.00	Cotangent Angle
3	.011	.028	.015	.006	-.028	-.042	-.037	-.046	-.050	Multiply Line 1 By Line 2. Change Sign.
4	-.042	-.042	-.042	-.042	-.042	X	-.042	X	X	Enter value of Col. 6 Line 3 in all spaces
5	.053	.070	.057	.048	.014	0.0	.005	X	X	Subtract Line 4 from Line 3
6	X	X	X	.006	-.028	-.042	-.037	-.046	-.050	Copy Line 3 on this line
7	X	X	X	-.046	-.046	-.046	-.046	-.046	-.046	Enter value of Col. 8 Line 3 in all spaces
8	X	X	X	.052	.018	.004 Point M	.009 Point N	0.0 Point O	-.004	Subtract Line 7 from Line 6.
9	X	X	X	.024	.007	0.0	.003	X	X	Value in Line 5 divided by 2.
10	X	X	X	.026	.009	.002	.005	X	X	Value in Line 8 divided by 2.
11	X	X	X	.050 Point A	.016 Point B	.002 Point C	.008 Point D	X	X	Sum of Line 9 and Line 10
12	X	.018	X	.012	X	0.0	.001 Point H	X	X	1/4 Value in Line 5
13	X	X	X	X	.014	.003	.007 Point G	X	X	3/4 Value in Line 8
14	-.007	-.009	X	X	-.002 Point R	X	-.001	X	X	Line 5 multiplied by −1/8
15	X	X	X	X	X	X	.010	X	-.005	Line 8 multiplied by 9/8
16	X	X	.023	X	X	X	X	X	X	Line 5 multiplied by 0.4
17	X	X	X	.031	X	X	X	X	X	Line 8 multiplied by 0.6

ENTER VALUES FROM TABLE ABOVE IN SPACE BELOW AS INDICATED

Line 12, Col. 2	.018	Line 12, Col. 4	.012	Line 14, Col. 1	-.007	Line 14, Col. 2	-.009	Line 14, Col. 7	-.001	Line 16, Col. 3	.023
Line 13, Col. 5	.014	Line 13, Col. 6	.003	Line 15, Col. 7	.010	Line 15, Col. 7	.010	Line 15, Col. 9	-.005	Line 17, Col. 4	.031
ADD	.032 Point E	ADD	.015 Point F	ADD	.003 Point P	ADD	.001 Point Q	ADD	-.006 Point S	ADD	.054 Point T

$\frac{B}{H} = 0.62$ $\frac{W}{H} = 1.12$

FIG. 6. Residual vertical error in stereoscopic model for a sample camera where instrument or printer compensation is used.

should be +0.2 mm., if the compensation is perfect. If the actual reading in the model varies from this figure, the difference represents the lack or error in compensation by the instrument. All points are then compared to give an over-all picture of the accuracy of compensation being obtained. The algebraic sign, shown in the form for each point, indicates the direction in the model above or below a datum through the four points marked *O*.

In Figure 6, the distortion for a metrogon lens has been entered in the form opposite camera-lens distortion. The values for nominal metrogon distortion are entered opposite instrument compensation. It is desired to determine the vertical error in the stereoscopic model due to de-

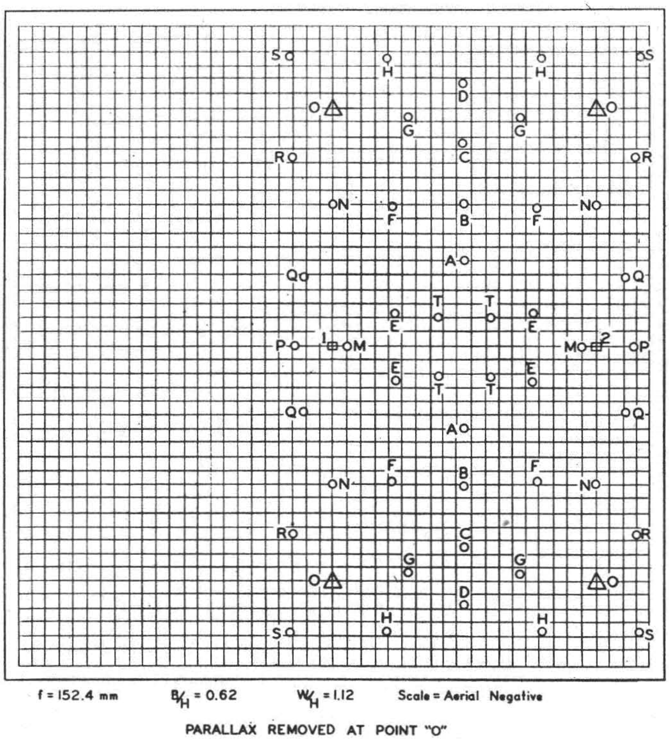

FIG. 7. Approximate model point locations in relation to 5 mm. grid.

POINT	COORDINATES		Vertical distortion from computation form.
	Angle from Projector 1	Angle from Projector 2	
A	20°	20°	
B	25°	25°	
C	30°	30°	
D	35°	35°	
E	10°	25°	
F	20°	30°	
G	30°	35°	
H	35°	40°	
M	0°	30°	
N	20°	35°	
O	30°	40°	
P	5°	35°	
Q	10°	35°	
R	25°	40°	
S	35°	45°	
T	15°	20°	

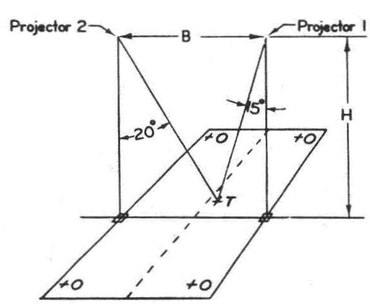

FIG. 8. Angular Coordinates of Points in the Stereoscopic Model.

parture of the distortion of this camera lens from the nominal used for compensation. The values computed for each point are multiplied by the ratio of the model scale to the aerial negative scale. The resulting values will then indicate what errors will be produced in the model by lack of compensation for this particular camera. For example, point T has a value from the form of +0.054 mm. If the photography is to be used in the Kelsh plotter, the model will be 5 times the negative scale and the value for point T becomes +0.22 mms. This value indicates that even after perfect compensation for a nominal lens there will remain an error of +.22 mms. at point T. The value here represents the magnitude of the lack of compensation for this particular camera. The algebraic sign indicates that the model reading must be increased by +.22 mms. to obtain a correct reading at this point. The model would thus read low by .22 mms. at point T. It is a striking fact that the error at point T is as great for this camera after compensation as the original error for a nominal lens using no instrument compensation. Of course, for a point such as S, the instrument compensation has reduced the error from an intolerable amount to an insignificant amount.

Conclusion

From this study, it can be concluded that the usual method of presenting distortion data carries with it a pitfall of misinterpretation into which the ordinary user is very likely to fall. By means of the approach presented in this paper, the effect of aerial camera lens distortion on the accuracy of map compilation can be reduced to uniform and readily-understood terms. The new procedure is simple and direct in application and gives results that leave no room for misinterpretation.