# *Nomographic Solution to Oblique Photo Mensuration*

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ABSTRACT

While serving as a photo interpreter in the Air Force, 1952–1955, the author became interested in the problem of performing physical measurements of ground objects from oblique aerial photographs. The known procedures encountered produced somewhat less than the expected results, so a study was made of the problem, and after over a year of research the following method was evolved. The principal solution is in the measurement of level-lying linear dimensions. However, arising from this solution is a simple method for measuring areas and angles lying in a level plane. And in order to make all measurements from oblique photographs more feasible and accurate, methods for obtaining the independent photo parameters, altitude and depression angle, have been incorporated into the system.

The principal advantages of the mensuration techniques described herein are simplicity, speed, and accuracy. With these techniques an unskilled technician can perform tasks previously restricted to a trained mathematician and beat him at his own job, particularly in speed and usually in accuracy.

## I. INTRODUCTION

OBTAINING level ground dimensions from oblique aerial photography has been a long standing problem to the photo interpreter. A number of different procedures have been devised to do the job. Some provide a strictly mathematical solution; others are a combination of mathematical and graphical means.

All methods have certain limitations. The mathematical solutions are rather long and tedious, and the chance for making errors in calculation are large. The graphical solutions are designed for use only with certain special combinations of focal length and tilt, and require changing charts, overlays, or scales for each different change in these parameters. Nearly all solutions require combining two components of a line by taking the square root of the sum of the squares, to obtain the resulting ground distance.

Lately, a number of people have shown an interest in overcoming the limitations just described, as witnessed by the various articles and comments on the subject printed in this journal. The following method is an attempt by the author to overcome these limitations, and although perhaps not the final answer, the method is simple, quick, and accurate.

The mathematics is solved graphically with nomographs. Lines running obliquely to the photo axes are measured directly, eliminating combination of components. The method will handle any value of focal length and depression angle. The average time per measurement should be from five to ten minutes, and the maximum additional error due to using this method about 2 per cent. Other errors will, of course, add to this.

## II. PROCEDURE FOR PERFORMING LINEAR MEASUREMENT

Three charts, Figures 1, 2, and 3, are used to which the photo interpreter applies parameters of the photograph and photo image. He arrives at a number

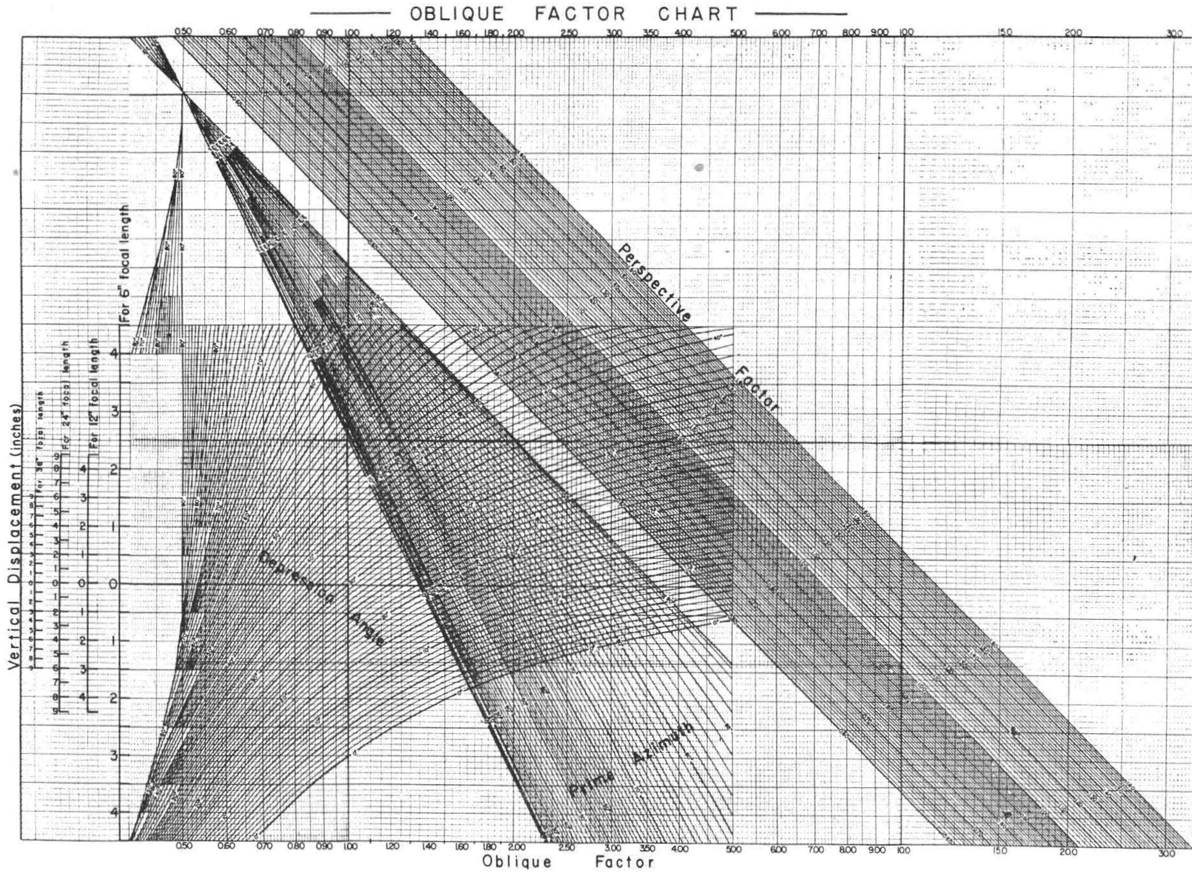


FIG. 1. The Oblique Factor Chart. Furnishes graphical solution for measurement of images having midpoints on the principal line. Also used for determining values for area and angular measurement.

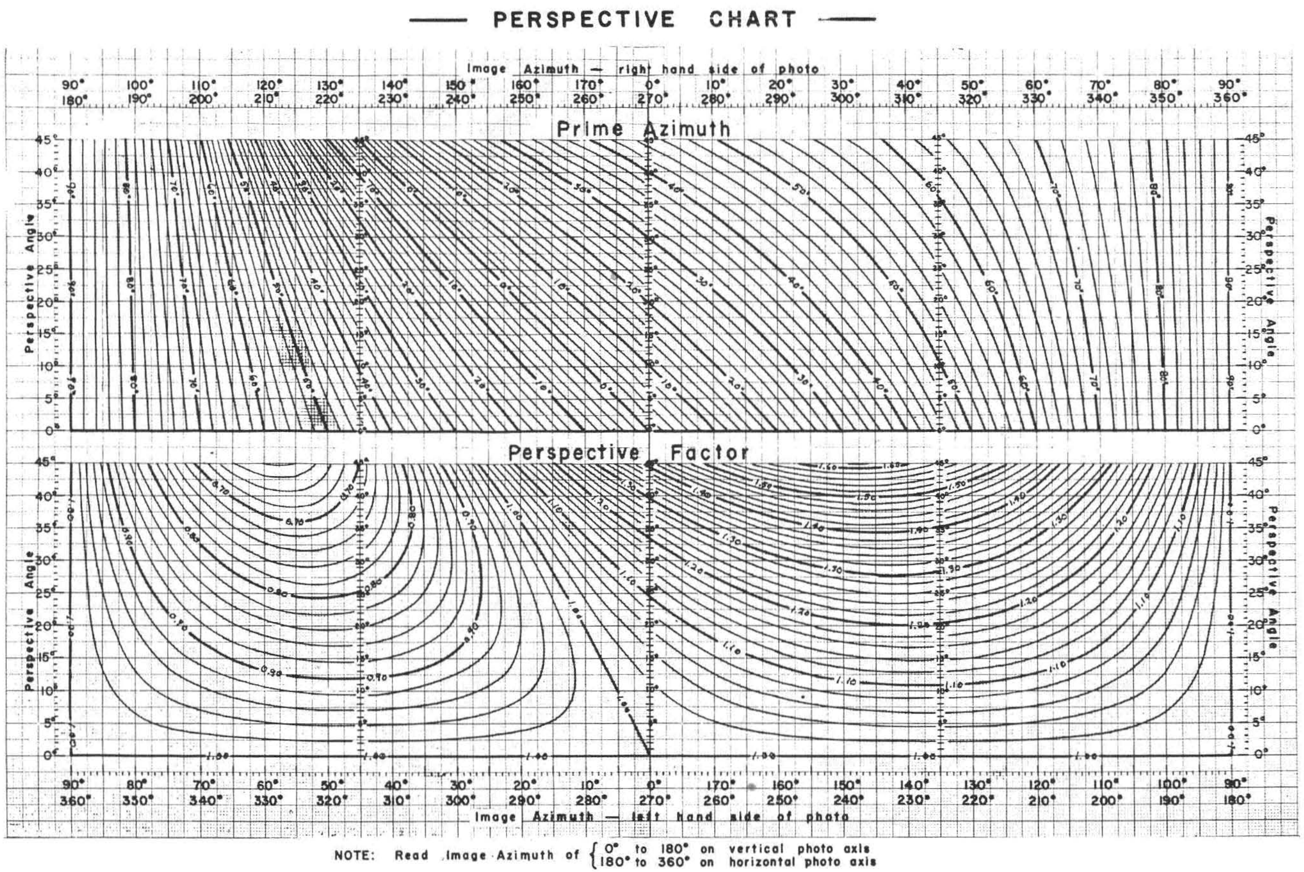


FIG. 2. Perspective Chart. Used to determine values for converting any image on an oblique photograph to measurable image having midpoint on the principal line.

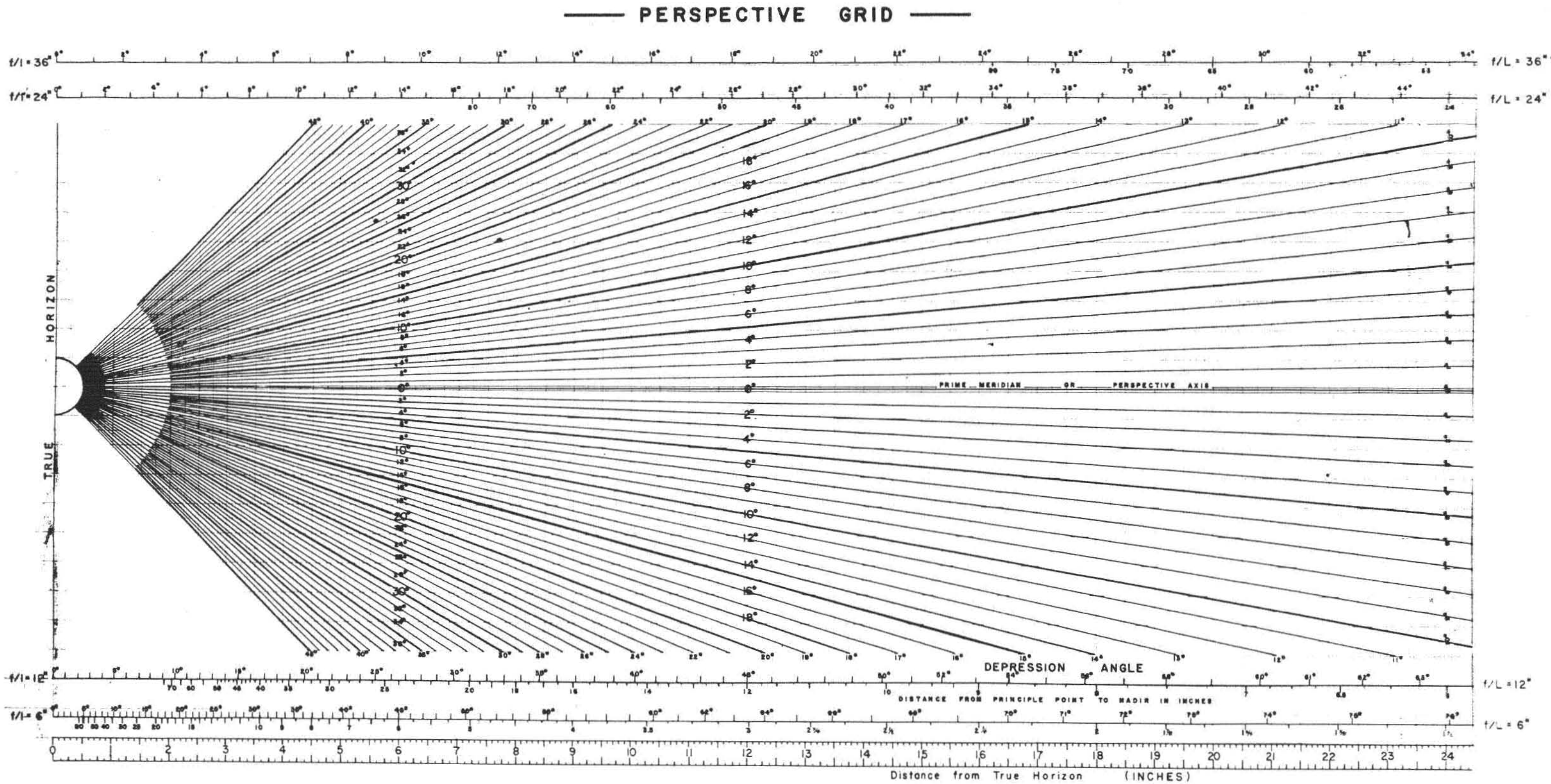


FIG. 3. Perspective Grid. Furnishes perspective angle necessary for use on perspective chart. In addition, scales at top and bottom are used for direct determination of depression angle from constructed horizon and nadir.

which represents the inverse ratio between the actual length of the photo image and the image length of the object on an equivalent vertical taken by the same camera at the same exposure station. This number is called the *oblique photo restitution factor*, or simply the *oblique factor*. By multiplying this ratio times the formula used to determine level ground dimensions on a vertical photograph, the photo interpreter arrives at the correct distance on the oblique. The resulting formula becomes simply:

$$\text{Ground length} = \frac{\text{altitude}}{\text{focal length}} \times \text{image length} \times \text{oblique factor} \qquad (1)$$

This comprises the extent of the mathematical computations necessary with the oblique factor method, at least for standard focal lengths with which the charts may be provided.

Determination of the proper oblique factor for a given line image is strictly a mechanical operation accomplished on the charts by following a simple procedure sheet. It should be realized that every different line image on an oblique photograph has a different oblique factor, depending both on direction, or *azimuth*, of the image and its location on the photograph. The parameters, therefore, which are required of an *image* are:

a. *Image length*—$l$
b. *Image azimuth*—$\alpha$
c. *Horizontal displacement*—$x_0$
d. *Vertical displacement*—$y_0$

These relationships are illustrated in Figure 4a. The origin is at the principal point of the photograph, and the $x$ and $y$ axes correspond to the principal parallel and principal line, respectively. The reason for the particular choice of image parameters will be explained in the derivations of the following section.

Image azimuth is determined in the same manner as a bearing on a chart—with protractor or Weems plotter. The positive $y$-axis serves as the north reference.

The *photographic* parameters (as distinct from image parameters) are illustrated in Figure 4b. They are:

a. *Altitude*—$H$
b. *Focal Length*—$f$
c. *Depression angle*—$\theta$
*or tilt angle*, $t = 90° - \theta$

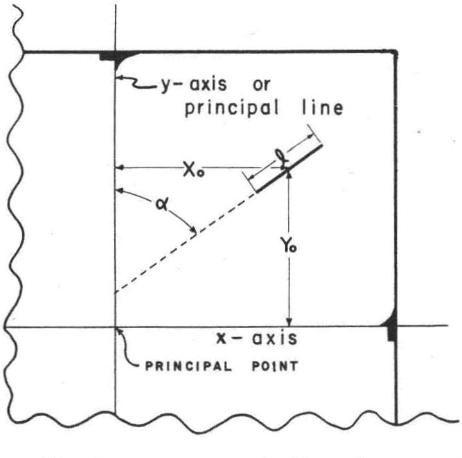


FIG. 4a. Parameters of a linear image.

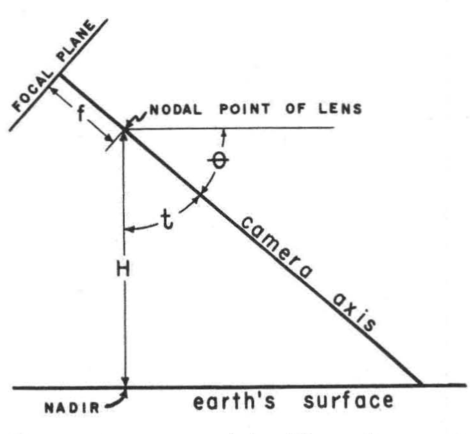


FIG. 4b. Parameters of the oblique photograph.

The determination of accurate values of altitude and depression angle is the primary difficulty in oblique photo mensuration. Once they are found, the battle is nearly won. A later section will be devoted to several means of determining these two photo parameters, some of which are greatly facilitated by using the charts provided for the oblique factor method. If accurate photo parameters are known, the oblique factor method will furnish accurate lengths of level ground dimensions.

The *oblique factor chart* (Figure 1) is a small-scale copy of the original. This multiple nomograph gives the value of the oblique factor for lines *whose mid-points fall on the principal line*, that is, lines for which $x_0 = 0$. In actual use it is several times larger, so that values of angles may be easily interpolated and precision in measurement is possible.

For the first step the vertical displacement $y_0$ of the image midpoint is measured, and this value located on the scale for the proper focal length at the left edge of the oblique factor chart. A working copy of the chart is generally reproduced at a 1:1 scale for a 12 inch focal length. Thus, for photographs of this focal length the vertical displacement may simply be taken from the photo with a pair of dividers and applied directly to the chart.

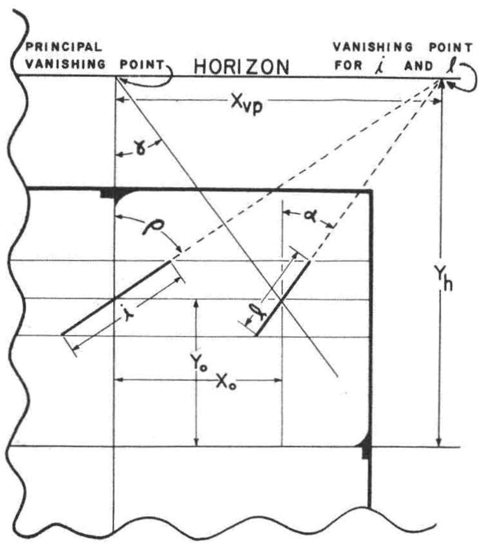


FIG. 5. Illustration of perspective correction. Line $l$ represents any image on oblique photograph and $i$ is its prime projection. The ground lines represented by $i$ and $l$ are parallel (they have a common vanishing point) and equal (included between the same photographic parallels).

Values of vertical displacement on the chart vary inversely with the focal length, so for the other standard focal lengths illustrated, multiples or submultiples of the vertical displacement are applied: 2 $y_0$ for 6 inch $f/L$, $\frac{1}{2}$ $y_0$ for 24 inch $f/L$, and so forth. Any focal length may be handled in this manner. Proportional dividers may be used to scale off these distances directly and may, in fact, be used to account for photo shrinkage.

From the point located on the proper vertical displacement scale a horizontal line is extended, but not necessarily drawn, with a straight edge until it intersects the proper value of depression angle. From this intersection a vertical line is extended to the proper azimuth value. At this point a horizontal line is extended to the "1.00" value of *perspective factor*. This value of perspective factor, unity, is used for image whose midpoints lie on the principal line. Then at this intersection the oblique factor is read between the vertical grid lines with scales at the top and bottom of the chart. Oblique factor, image length, and *scale factor reciprocal* (altitude divided by focal length) are multiplied together to obtain the ground dimension.

At first sight the oblique factor chart may appear a rather formidable device, but once understood, the simplicity of its use is readily apparent. It may be compared to a slide rule in its operation. Mathematical functions are represented by linear relationships and through certain manipulations these are combined to arrive at the proper answer. Since the chart is two dimensional, however, fairly complex functions can be presented. The use of straight edge and dividers on the chart serve much the same purpose as the hairline on the slide rule.

Most lines on an oblique photograph do not have midpoints lying on the principal line, as did the line just solved for. Indeed, this would be a special case. However, the oblique factor chart may be used if there is a means available to convert any image on a photograph to an equivalent one with midpoint on the principal line. That is the function of the charts illustrated in Figures 2 and 3. This conversion is called *perspective correction*, since the rules of perspective govern the change.

What these charts do, effectively, is to take an image $l$ (see Figure 5) at any location on an oblique photograph and convert it into an imaginary image $i$ of the same vertical displacement with midpoint on the principal line, which image is measurable on the oblique factor chart. The ground lengths of $i$ and $l$ are the same since they are parallel on the ground, and on the photograph are intercepted by lines parallel to the horizon. The differences for application to the oblique factor chart, then, are in image length and image azimuth. The *perspective factor* $p$ corrects for length, and the *prime azimuth* $\rho$ is the corrected azimuth. The image $i$ has been termed the *prime projection* of $l$.

The perspective factor is the ratio of the length of $i$ to the length of $l$. In using the oblique factor method the length of $l$ is measured and applied in formula (1), and the perspective factor applied to the oblique factor chart automatically corrects this to the measurable image $i$. Furthermore, it is $\rho$, the azimuth of $i$, and not the actual image azimuth $\alpha$ which must be applied to the oblique factor chart. To determine the above relationships, it is not necessary to perform a physical construction, as illustrated in Figure 5.

Determining the correct values of prime azimuth and perspective factor of a line is accomplished graphically on the *perspective chart* (Figure 2). Two parameters must be used. The first is the image azimuth $\alpha$, and the second is the *perspective angle* $\gamma$, a new parameter, dependent on horizontal and vertical displacement. Image azimuth values are located along the abscissa, and perspective angle values are located along the ordinate as shown in the illustration. The perspective chart is actually two charts in one. Intersection of the correct value of the two above parameters in each section of the chart locates the proper values of perspective factor and prime azimuth.

The last operation to be explained, before the actual image can be corrected to the measurable imaginary image, is the determination of the perspective angle, necessary in using the perspective chart. First, its definition. It is the angle subtended at the *principal vanishing point*, (see Figure 5) between the principal line and a line through the image midpoint.

This perspective angle is determined on the perspective grid, (Figure 3) which is essentially a huge protractor with lines radiating in 1 degree angular increments from the principal vanishing point at the intersection of the principal line with the horizon. If the perspective grid were reproduced as a positive transparency, similar to an ordinary protractor, it would merely have to be placed over the photograph in the correct relationship for the proper angle to be determined. In actual use the grid is reproduced as an opaque chart, and an annotated transparent templet scribed with a set of axes is used to transfer the image midpoint from the photograph to the grid.

The correct relationship of the transparent templet on the grid is determined by the scales at top and bottom of the chart for the four standard focal lengths provided. On the upper side of each scale are values of depression angle. These indicate the proper lateral positioning of the principal point of a photograph in relation to the true horizon and principal vanishing point on the left.

By aligning the templet properly on the photograph and marking the image midpoint, the templet may then be placed correctly over the perspective grid and the perspective angle of the image midpoint immediately read off. In marking the image midpoint on the templet the last of the listed image parameters, the horizontal displacement, is taken account of. An alternate procedure to the use of the templet would be to take the horizontal and vertical displacement from the photo with dividers, and apply to the indicated location of the principal point on the grid.

Distances from principal point to horizon are directly proportional to focal length. Therefore, the grid may be used for focal lengths other than those illustrated by simply scaling off the correct distance from one of the standard scales provided. For example, with a photograph of 7 inch focal length the distances of principal points from horizon are simply 7/6 times the distances for a 6 inch focal length.

In summarizing the foregoing procedure, the photo and image parameters are determined or measured and the perspective angle located on the perspective grid. This angle is applied to the perspective chart on which two values are determined for application to the oblique factor chart. On this chart the oblique factor is determined, and multiplied times image length and scale factor reciprocal ($H/F$) to obtain the correct ground length.

The foregoing explanation will doubtless seem confusing, presented as it was in tail-first fashion. However, once the initial steps are learned the operation may be quickly and simply applied. A mimeographed step-by-step procedure greatly facilitates following through the sequence of operations, and the various parameters, independent and dependent, are kept track of by entering them on a computation sheet. Results obtained from teaching this method to a variety of photo interpreters have shown that a man with no knowledge of trigonometry can learn the system in less than an hour of individual instruction.

Certain disadvantages of the oblique factor method may have become apparent to the reader, and others will be brought out in the following discussion, along with means of eliminating them. Perhaps the most obvious disadvantage is the large size of the present charts. However, the charts illustrated in the figures have not been engineered into final form. They are a preliminary design attempt and are strictly a solo operation by the author, both in computation of points and in drafting. Better design and more precise drafting would allow reproduction at a much smaller scale than the originals with little loss in accuracy.

Furthermore, these charts were designed to cover a fairly wide range of variations in focal length and depression angle. Charts for a specific use may be greatly simplified. Scales for any particular focal length may be added to the oblique factor chart or the perspective grid, or the oblique factor chart may be reproduced at a 1:1 scale for a particular focal length, allowing direct application of vertical displacement with dividers (for that focal length).

Limits of values on the different charts may be extended or restricted to suit a particular need. The perspective grid may be designed on a logarithmic scale to cover distances up to many feet from the horizon, or it may be drawn up in transparent templet form for one particular focal length and depression angle. The oblique factor chart may be designed for one particular depression angle with the proper modification. In this event only two families of curves would appear on it, and one operation would be eliminated. Little change could be made in design of the perspective chart other than changing the limit on perspective angle, since it is independent of both focal length and depression angle.

One important shortcoming should be discussed. This is the fact that the oblique factor of a line is exact only for a line of zero length. As the length of line becomes finite and increases to larger values, the oblique factor becomes less exact. However, the error involved for most practical cases is very small—a fraction of a per cent. And for cases where it might amount to several per cent, it may be minimized in two different ways; the line may be divided into two or more short segments which are treated as separate lines and then combined; or preferably, the position of its midpoint may be changed by rule of thumb to account for the difference. Wholly adequate error studies have not been performed to date, but work which has been done indicates that there is no great problem involved.

Another shortcoming of the oblique factor method for some purposes is that no corrections are made for earth curvature and refraction. All mathematics are derived under the assumption that the photograph is taken of a flat earth. Thought has been given to a method of accounting for this by modifying the photo parameters at a given point on a photograph similar to the manner in which image parameters are modified to handle perspective correction. It is hoped that this modification may also be determined graphically with nomographs.

In addition, it should be brought out that this method will give dimensions only of level objects. Heights of ground objects cannot be measured with these charts. However, the author has constructed similar charts which will solve the mathematics of height determination. Publication of the results of the work with heights should be forthcoming.

## III. Derivation of Mathematics

Derivation of mathematics for the oblique factor method may be divided into two parts. The first part is the correction of photo images with midpoint on the principal line to an equivalent vertical—solved on the oblique factor chart. The mathematics involved in this occurs primarily on the principal plane. The second part is the correction of perspective relationships—solved with the perspective grid and perspective chart. Mathematics here occurs on the focal plane.

### A. THE OBLIQUE FACTOR CHART

The derivation will begin with correction of images having midpoint on the principal line to an equivalent vertical. In Figure 5 consider a line on the ground represented by the image $i$ with the midpoint on the principal line. This image, as brought out previously, is the prime projection for all other images of equal, parallel ground lines (such as $l$) having the same vertical displacement on the photograph.

The image $i$ may be resolved into two components, $i_y$ parallel to the principal line, and $i_x$ perpendicular to the principal line, which represent similar components to the line $L$ on the earth (see Figure 6). The image $i$ is sufficiently short so that perspective effects due to its length may be neglected. $L_x$ and $L_y$ will be determined in terms of $i_x$ and $i_y$, respectively, and then combined to obtain the length $L$ in terms of $i$ and other parameters.

Consider first the determination of $L_x$ which is represented in Figure 7 by point $P$ and is perpendicular to the plane of the paper. A plane passed through $L_x$ perpendicular to the principal axis is considered as datum for an imaginary vertical photo of "altitude" $H'$. Since $L_x$ lies in the imaginary datum plane, the vertical photo distance formula may be applied:

$$L_x = \frac{H'}{f} i_x.$$

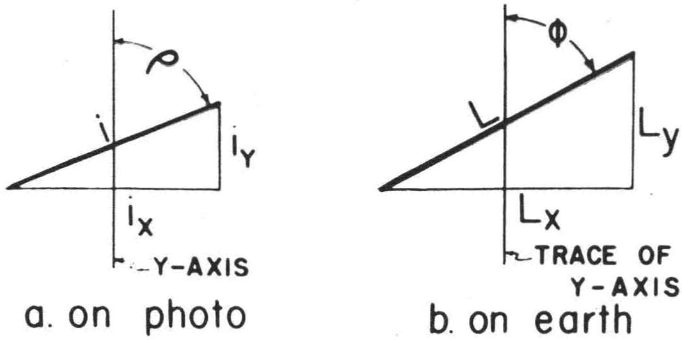


FIG. 6. Prime projection $i$ on photograph and line $L$ which it represents on earth, with $x$ and $y$ components.

It may be shown that

$$H' = H \frac{\cos \beta_1}{\cos (t + \beta_1)} \cdot$$

Therefore the resulting relationship becomes:

$$L_x = \frac{H}{f} i_x \frac{\cos \beta_1}{\cos (t + \beta_1)} \cdot \tag{2}$$

The derivation for $L_y$ also employs the imaginary datum plane. The image $i_y$ represents both $L_y$ on the earth and $L_y'$ lying in the imaginary datum. The formula just derived for $L_x$ may be applied to $L_y'$ or any line in the imaginary plane. Thus,

$$L_y' = \frac{H}{f} i_y \frac{\cos \beta_1}{\cos (t + \beta_1)} \cdot \tag{2a}$$

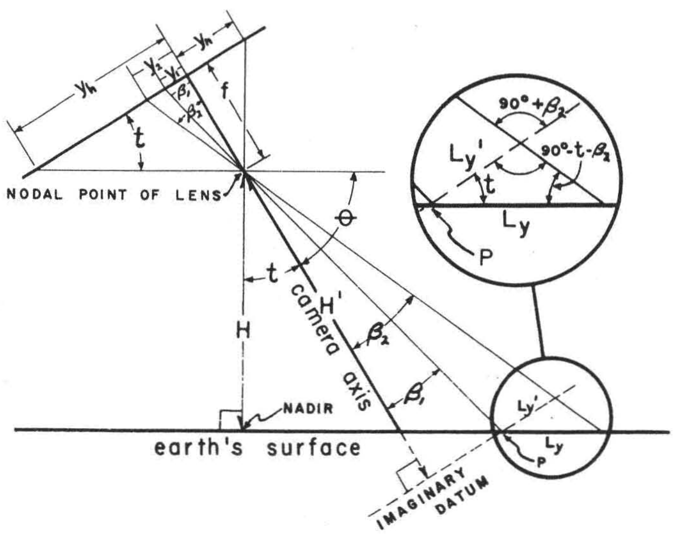


FIG. 7. Detailed side view of oblique photo geometry.

For determining the relationship between $L_y'$ and $L_y$ the law of sines is applied to the small triangle reproduced in the inset to obtain:

$$\frac{L_y}{\sin(90° + \beta_2)} = \frac{L_y'}{\sin(90° - t - \beta_2)}$$

which simplifies to

$$L_y = L_y' \frac{\cos \beta_2}{\cos(t + \beta_2)} .$$

Therefore, by substitution

$$L_y = \frac{H}{f} i_y \frac{\cos \beta_1}{\cos(t + \beta_1)} \frac{\cos \beta_2}{\cos(t + \beta_2)} . \tag{3}$$

As the length $i_y$ approaches zero, $\beta_1$ and $\beta_2$ approach a common value, $\beta_0$. Therefore for images of zero length, we may substitute in $\beta_1 = \beta_2 = \beta_0$ to obtain:

$$L_y = \frac{H}{f} i_y \left( \frac{\cos \beta_0}{\cos(t + \beta_0)} \right)^2 \tag{4}$$

For short lines of finite length, formula (4) may be used by considering the value of $\beta_0$ as corresponding to the image midpoint, approximately the average value of $\beta_1$ and $\beta_2$. This, of course, introduces an error. But, as mentioned before, preliminary studies show that for most practical cases the error may be neglected.

The same value of $\beta_0$ is used for determining the length of the $x$-component $L_x$. Therefore from formula (2)

$$L_x = \frac{H}{f} i_x \frac{\cos \beta_0}{\cos(t + \beta_0)} \tag{5}$$

The derivation just presented has been explained previously in different fashion by Katz for his scale number method.* In Katz' method a line image lying obliquely to the photo axes is divided into its two components, and the actual length of each component is measured. Then using formulas identical to (4) and (5) a ground length is computed for each, and the two are combined by taking the square root of the sum of the squares.

By going one step further in the mathematics and making use of the azimuth angle $\rho$ at which the prime projection $i$ intersects the principal line, the measuring of components and subsequent recombination is eliminated. The image parameters in this case are the actual length of the image and its azimuth, rather than the lengths of the $x$ and $y$ components.

Several advantages accrue from this choice of parameters. The first advantage is the relatively shorter length of time required to measure the parameters. A length measurement is made of an actual image and not of an imaginary component, and the angular measurement is quickly determined with a navigational type plotter or a protracter. Secondly, a higher degree of accuracy is possible when measuring short images, because errors in measuring a component approach the length of an image itself as shorter images are considered. And as already considered, recombination of components is eliminated. The value of this last feature to the time-conscious photo interpreter cannot be overstressed.

* Katz, Amrom H., "Contribution to the Theory and Mechanics of Photo-Interpretation from Vertical and Oblique Photographs," PHOTOGRAMMETRIC ENGINEERING, Vol. XVI, No. 2, pp. 339–386, June 1950.

From Figure 6 it may be seen that:

$$i_x = i \sin \rho$$

$$i_y = i \cos \rho$$

Letting

$$u = \frac{\cos \beta_0}{\cos (t + \beta_0)} \tag{6}$$

and substituting in (4) and (5):

$$L_x = \frac{H}{f} (i \sin \rho) u$$

$$L_y = \frac{H}{f} (i \cos \rho) u^2.$$

Since

$$L = \sqrt{L_x^2 + L_y^2}$$

then

$$L = \frac{H}{f} i \sqrt{u^2 \sin^2 \rho + u^4 \cos^2 \rho}$$

Letting

$$r = \sqrt{u^2 \sin^2 \rho + u^4 \cos^2 \rho} \tag{7}$$

then

$$L = \frac{H}{f} ir.$$

This last relationship may be recognized as the distance formula for a vertical multiplied by the factor $r$. The factor $r$ has been termed the *prime correction factor*, or simply *prime factor*, since it corrects the prime projection of an image to its actual length on an equivalent vertical.

Formulas (6) and (7) are the basis for the oblique factor chart. The depression angle curves are defined by formula (6), which must first be reduced to a form containing the vertical displacement $y_0$. Applying the sum of angles reduction for cos $(t+\beta_0)$ and simplifying:

$$u = \frac{1}{\cos t - \tan \beta_0 \sin t}$$

and since

$$\tan \beta_0 = \frac{y_0}{t}$$

then

$$u = \frac{1}{\cos t - \dfrac{y_0}{t} \sin t}. \tag{6a}$$

In the graphical representation of these curves, the ordinate is the $y_0/f$ axis, and the abscissa, the $u$-axis. The curves represent constant-$t$ values for different $t$ from 0° to 90°. In Figure 1, however, the curves are labeled in terms of depression angle, the complement of $t$.

Note how the focal length $f$ appears in equation (6a). It acts as a scale factor affecting inversely the appearance of $y_0$. Different $y$-scales may therefore be drawn along the ordinate to account for the different focal lengths as shown in Figure 1.

The second family of curves on the oblique factor chart represent values of prime azimuth and are defined by formula (7). The same $u$-axis is shared with the depression angle curves, and the values of prime factor $r$ appear along the negative ordinate. The curves represent $t$-values from 0° to 90°.

The graphical solution performed on the oblique factor chart may now be explained in mathematical terms. The vertical displacement $y_0$, represented by a distance along the ordinate, together with a particular tilt determines a $u$-value. A vertical line represents the same $u$-value to both families of curves. This $u$-value and an azimuth value determine the proper prime factor $r$. The family of straight lines (Figure 1) multiply the prime factor by the perspective factor to arrive finally at the correct oblique factor.

Since vertical lines represent $u$-values, the two families of curves may be overlapped in the vertical direction as was done with the oblique factor chart in Figure 1. Similarly, the perspective factor values have been overlapped and also folded in relation to the first two. The base grid represents only the first and last variables encountered in the operation of the chart—vertical displacement and oblique factor. A logarithmic presentation is employed along the axes for $u$, $r$, and oblique factor. Values of $y_0/f$ are presented linearly to allow direct application of $y_0$ with dividers.

### B. PERSPECTIVE CORRECTION

The mathematical relations between the lengths and azimuths of an image $l$ and its prime projection $i$, as previously stated, are functions of the image azimuth $\alpha$ and perspective angle $\gamma$ (refer to Figure 5). To reiterate, the image $l$ and its prime projection $i$ have the same verrical displacement $y_0$, represent equal ground lengths, and since they have a common vanishing point on the horizon, are parallel.

From Figure 5 it is seen that

$$\tan \rho = \frac{x_{vp}}{y_h - y_0}$$

$$\tan \alpha = \frac{x_{vp} - x_0}{y_h - y_0}$$

$$\tan \gamma = \frac{x_0}{y_h - y_0}$$

Therefore the relationship between these three angles becomes

$$\tan \alpha = \tan \rho - \tan \gamma \tag{8}$$

And the perspective factor, $p = i/l$, may be seen to be

$$p = \frac{\cos \alpha}{\cos \rho} \tag{9}$$

The upper section of the perspective chart (Figure 2) for determining prime azimuth is defined by formula (8). The lower section for determining perspective factor is a simultaneous solution of (8) and (9). In each section the abscissa is the $\alpha$-axis and the ordinate the $\gamma$-axis.

To account for positive and negative values of $\gamma$ corresponding to photo images lying on the right and left half of the photo, supplementary values of image azimuth are required. The image azimuth scale at the top of the chart must be used when $\gamma$ is positive—images in the right half of photo. The image azimuth scale at the bottom is for negative $\gamma$. The same scale is used for both top and bottom sections of the chart when measuring a given line.

Azimuth values of 0° to 180° are adequate to describe all possible directions of a linear dimension since linear dimensions are non-directional lines. Therefore, in measuring an azimuth of a dimension the inverse angles are never actually used. However, a problem arises in the measurement of photo images with a Weems plotter in which the inverse azimuths may be put to use. The problem is the difficulty of measuring images lying nearly parallel to the vertical photo axis, in the conventional use of the plotter with the vertical axis as reference. Such azimuths are usually more easily read by using the horizontal axis as reference. Therefore, the image scales on the perspective chart have been constructed so that azimuth values of 180° to 360° which are read against the horizontal axis correspond to the proper azimuth value of 0° to 180° read against the vertical photo axis. With the choice of the two procedures it is possible to read all azimuths quickly and simply, even from images lying in the extreme corners of the photograph.

### C. THE PERSPECTIVE GRID

Determination of perspective angle is accomplished, as previously stated, on the perspective grid (Figure 3) which is essentially a large protracter containing lines of 1 degree angular separation. The grid must be placed so that its origin corresponds to the principal vanishing point of the photograph, and the axes of grid and photograph are aligned.

The proper linear positioning of the photograph on the grid is determined by the scales along the top and bottom of the protracter, for four standard focal lengths. The distance, $y_h$, from the principal point to the principal vanishing point of a photograph of tilt $t$ (see Figure 7) is:

$$y_h = f \cos t. \qquad (10)$$

Based on this formula the values along the top of each scale in Figure 3 locate $y_h$ with respect to the horizon trace at left for each $t$. The actual values of $y_h$ in inches may be read from the scale at the bottom of the perspective grid.

Along the bottom of the same scales are numbers indicating the distance in inches from principal point to nadir for the depression angle values shown on top. This set of values is used in determination of depression angle when the nadir may be located on the photograph. The distance $y_n$ from principal point to nadir in terms of $y_h$ is

$$y_n = \frac{f^2}{y_h} \cdot \qquad (11)$$

## IV. Altitude and Depression Angle Determination

As previously stated, the greatest problem encountered in attempting to obtain accurate ground dimensions from aerial photography is determination of the photo parameters, altitude and depression angle. Altitude is usually obtained from information provided by the aircrew, and is generally accurate enough for mensuration purposes. This is fortunate, since only one other means is available for obtaining it. This method requires that the depression angle and the actual length of an object appearing on the photograph be known. From the known depression angle the correct oblique factor for that object length is obtained using the procedure for measuring the line. This oblique factor is then substituted in formula (1) solved for altitude

$$\text{Altitude} = \frac{\text{known ground length} \times \text{focal length}}{\text{image length} \times \text{oblique factor}} \tag{12}$$

A very accurate altitude may be determined in this manner provided the depression angle and object length are accurately known and the image length is sufficiently long.

Although altitude is generally supplied by the aircrew, an accurate depression angle is seldom known beforehand. Until aircraft are provided with dependable and precise stabilized camera mounts or tilt indicating devices, it will be necessary to determine depression angle from information on the photograph itself. It is fortunate that there are a variety of methods available for this, since each method is restricted in use to certain special conditions. The charts for the oblique factor method provide three means of determining depression angle fairly quickly and simply. Two of the methods have just been discussed in connection with the perspective grid.

In the first method parallel ground lines are extended to their intersection on the horizon, and the distance from principal point to horizon measured. This distance is then laid off along the correct scale on the perspective grid, at which point the correct depression angle is indicated. In the second method vertical lines are extended to their intersection at the nadir, and the nadir to principal point distance measured. This value is located on the bottom of the proper grid scale on the perspective grid, and the correct depression angle is indicated on the top of the scale.

The third method is similar to the determination of altitude from the photograph in that the length of a ground line must be known. The altitude is also required. The correct oblique factor is determined first by using formula (1). A trial and error process with the oblique factor method is then employed for different values of depression angle. The correct depression angle will supply the predetermined oblique factor. In this trial and error process, several steps may be combined so that one "run" may be made in much less time than that required to perform an initial measurement of a line. It is estimated that perhaps thirty minutes would be required to obtain an accurate depression angle in this manner.

A fourth method, employing the use of a known right angle, such as the corner of a building or other rectangular object on the photograph, was devised and tested. Although the method will furnish an accurate depression angle, it is doubtful if it will prove of more than academic value, since the sides of such an object would furnish a convenient construction horizon for determining depression angle more accurately and simply.

Another means of depression angle determination, although it does not employ the oblique factor method, is included here because it does make use of the same principle of solving a mathematical equation by graphical means. This method employs the visible horizon, and the formula is presented, among other places, in the MANUAL OF PHOTOGRAMMETRY:*

* Pp. 351–352, Second Edition. ($\tan^{-1} y_v/f$ is the apparent depression angle and $\Delta$, the dip angle, defined in the MANUAL. The constant $k$ depends upon the choice of units for $\Delta$.)

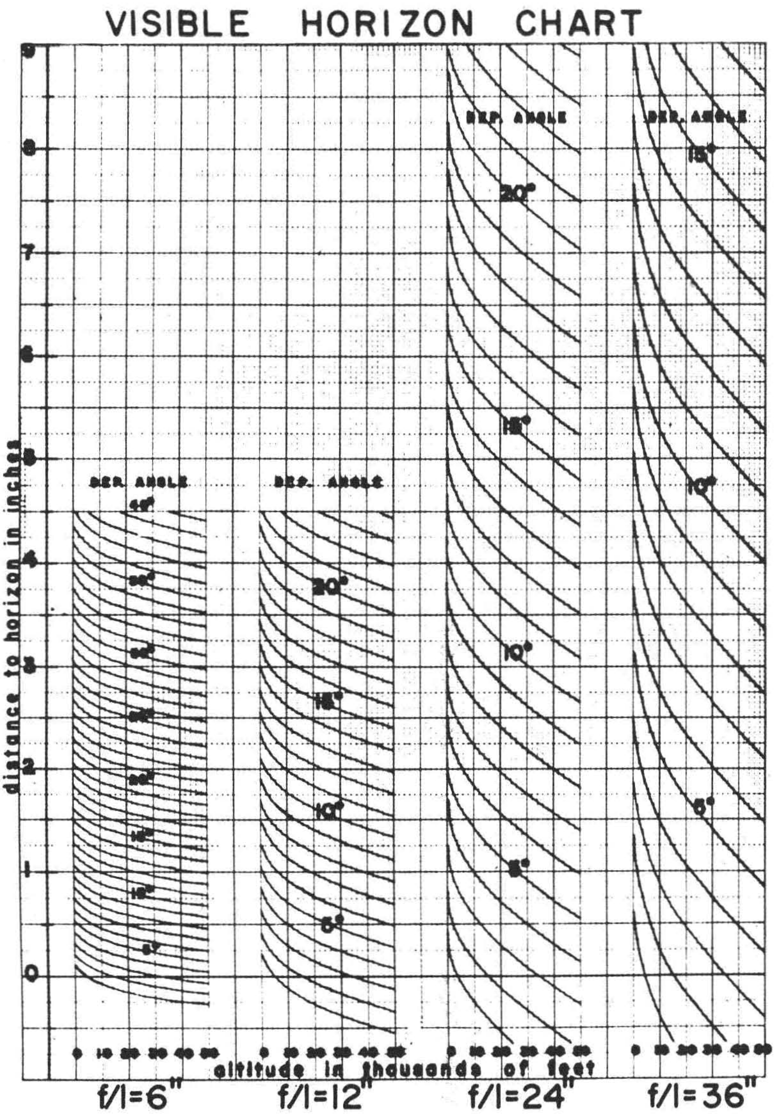


FIG. 8. Visible Horizon Chart provides direct graphical solution for depression angle when horizon appears on aerial photograph, eliminating use of mathematics.

$$\theta = \tan^{-1} \frac{y_v}{f} + \Delta \tag{13}$$

where

$$\Delta = k\sqrt{H}.$$

Figure 8 shows the graphical presentation of this formula. The distance from principal point to visible horizon, $y_v$, appears along the ordinate and the altitude above the terrain, $H$, along the abscissa. Curves are plotted for different depression angle $\theta$, and there are groups of curves for four different focal lengths. Here again dividers are employed to take the distance $y_v$ from the photograph and apply it to the graph along the vertical line for the proper altitude. The time saving value of this chart is self evident.

For the sake of completeness, other means of determining depression angle should be mentioned—known height, known nadir point distance, and the pass point method, which employs two overlapping oblique photos. There are undoubtedly many others which can and will be used.

## IV. Measurement of Area

Although the oblique factor method was primarily designed to measure level ground dimensions, the oblique factor chart may be used to solve for measurement of areas lying in the plane of the earth's surface. In deriving the relationship for measuring a ground area $A$ from an image area $a$, consider the infinitesimal area $da$ (Figure 9).

Since the quadrilateral $da$ is infinitesimal, opposite sides are parallel, and its shape is a parallelogram. Its area is therefore equal to the base $dx$ times the height $dy$. The ground lengths $dX$ and $dY$ are found from formulas (4) and (5):

$$dX = \frac{H}{f} u dx$$

$$dY = \frac{H}{f} u^2 dy.$$

Therefore the ground area $dA$ is

$$dA = dX \cdot dY = \left(\frac{H}{f}\right)^2 u^3 dx dy$$

For the finite ground area $A$ an integration of this function must be performed over the image area $a$.

$$A = \left(\frac{H}{f}\right)^2 \int_0^a \int u^3 dx dy. \tag{14}$$

Formula (14) is mathematically exact, but a good practical approximation may be obtained by considering $u$ to be constant over the image area and taking an approximate average $u$-value at the center of gravity of the area. In this case formula (14) becomes:

$$A = \left(\frac{H}{f}\right)^2 u^3 a \tag{15}$$

This value of $u$ is equivalent to the oblique factor for a line of 90 degrees azimuth. It may be determined graphically on the oblique factor chart by using the vertical displacement to the center of gravity, the depression angle of the photo, a 90 degree prime azimuth, and a perspective factor of unity.

The image area $a$ may be meas-

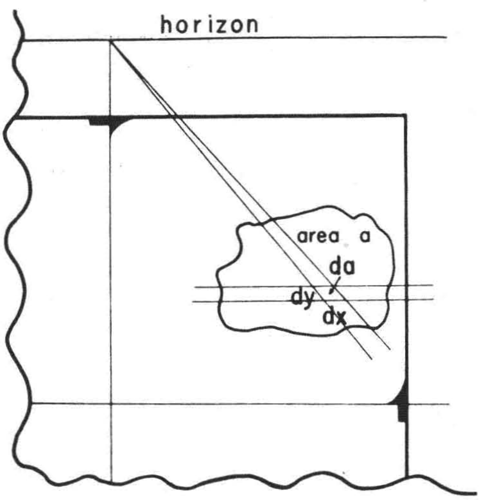


FIG. 9. Exact measurement of irregular shaped ground areas requires integration of area function over image area.

ured by placing over it a grid containing squares representing fractions of a square foot or other areal measurement and counting the number of squares it contains. The values of $u$ and $a$ together with the scale factor reciprocal are then inserted in formula (15) to obtain the ground area $A$.

## V. Measurement of Angles

The measurement of angles lying in a level plane requires the use of all three charts. For each side of an angle visible on an oblique photograph the image azimuth is measured and the prime azimuth determined in the manner discussed in Section II. Then using a simple trigonometric relationship the ground azimuth $\phi$ relative to the principal line (see Figure 6) is computed for each prime azimuth $\rho$. The two ground azimuths are then combined algebraically to obtain the actual value of the angle.

For the derivation refer to Figure 6. The prime projection $i$ has a prime azimuth $\rho$ and the line $L$ on the ground has an azimuth $\phi$ relative to the principal line. It may be seen that

$$\tan \rho = \frac{i_x}{i_y}$$

$$\tan \phi = \frac{L_x}{L_y}$$

Combining formulas (4) and (5) with the above, where

$$u = \frac{\cos \beta_0}{\cos (t + \beta_0)}$$

then

$$\tan \phi = \frac{1}{u} \tan \rho \tag{16}$$

This simple trigonometric formula converts the prime azimuth of a line into its relative ground azimuth.

In performing the necessary operations to convert image azimuths into relative ground azimuths a point is picked on each line. This point determines both a prime azimuth (using the perspective grid and chart) and a $u$-value (using the oblique factor chart) for use in formula (16). The amount of work is considerably reduced if the point common to both sides—the vertex—is utilized.

In combining the two relative ground azimuths found from formula (16) it

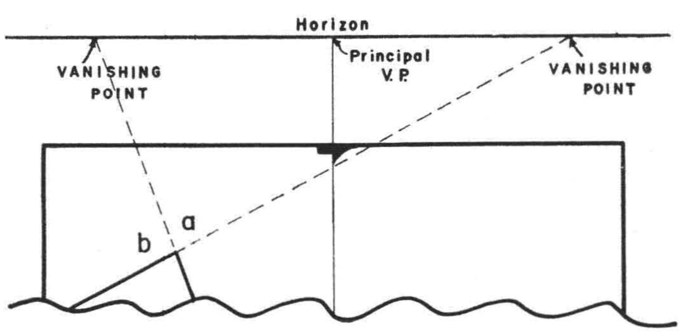


FIG. 10. Angular measurement with oblique factor method furnishes value of the angle $a$ included between the sides intersecting the horizon. Angle $b$ is the supplement of $a$.

should be noted that these azimuths will always be positive angles from 0° to 90°, since the perspective grid only furnishes prime azimuths in this range. This precludes the possibility of simple algebraic addition. Instead, two simple rules must be followed. The first is that the angle determined by this method is always the angle lying between the two sides and the horizon. In Figure 10 it would be the angle $a$ and not its supplement $b$.

The second rule is that the ground azimuths obtained from (16) are added when the vanishing points for the two sides fall on opposite sides of the principal vanishing point, and subtracted when they fall on the same side. The perspective grid may be used for determining which case is present, if necessary.

# *Problems of Relative Orientation*

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THE problem of relative orientation, as generally understood, involves the solution of a number of equations relating five unknown orientation elements to $Y$-parallaxes.

Von Gruber devised empirical methods for the removal of parallaxes by successive approximation. These methods have enjoyed considerable popularity with photogrammetric operators because of their simplicity. The tendency in modern practice, however, is to adopt numerical methods which have the singular advantage of presenting solutions free from the personal bias of the observer. Moreover, the troublesome effects of parallaxes caused by irregular film stretch and residual lens distortions are recognized in the initial stages of relative orientation, and the best solution in the circumstances is obtained by simple calculation. Theoretically too, it is recognized that numerical methods are more conducive to the reliable extension of control by aero-triangulation.

The problem of outer orientation of a pair of photographs, resolves itself into two stages; the first of relative orientation, where the relationship of individual photographs to the airbase is established, and the second of absolute orientation, where the space model is rotated and is scaled to fit ground control. It is usual practice to associate the removal of $Y$-parallaxes with the first stage and the establishment of correct $X$-parallaxes with the latter.

It is not generally realized that certain problems of relative orientation can be solved by measuring height deformations in the overlap. The purpose of this paper is to introduce problems involving $X$ and $Y$-parallaxes simultaneously. Since the placement of heights is necessary for absolute orientation, it will be possible, by placing them carefully, to carry out relative orientation, at the same time, in problems which otherwise would be insolvable. It should be mentioned here that complete relative orientation by measuring heights is not possible because height differences depend on relative and absolute orientation with influences that are not completely separable.

With the relative orientation of dependent pairs, we may assume the basic parallax equations to first order terms as follows: