should be noted that these azimuths will always be positive angles from 0° to 90°, since the perspective grid only furnishes prime azimuths in this range. This precludes the possibility of simple algebraic addition. Instead, two simple rules must be followed. The first is that the angle determined by this method is always the angle lying between the two sides and the horizon. In Figure 10 it would be the angle a and not its supplement b.

The second rule is that the ground azimuths obtained from (16) are added when the vanishing points for the two sides fall on opposite sides of the principal vanishing point, and subtracted when they fall on the same side. The perspective grid may be used for determining which case is present, if necessary.

Problems of Relative Orientation

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THE problem of relative orientation, as generally understood, involves the solution of a number of equations relating five unknown orientation elements to Y-parallaxes.

Von Gruber devised empirical methods for the removal of parallaxes by successive approximation. These methods have enjoyed considerable popularity with photogrammetric operators because of their simplicity. The tendency in modern practice, however, is to adopt numerical methods which have the singular advantage of presenting solutions free from the personal bias of the observer. Moreover, the troublesome effects of parallaxes caused by irregular film stretch and residual lens distortions are recognized in the initial stages of relative orientation, and the best solution in the circumstances is obtained by simple calculation. Theoretically too, it is recognized that numerical methods are more conducive to the reliable extension of control by aero-triangulation.

The problem of outer orientation of a pair of photographs, resolves itself into two stages; the first of relative orientation, where the relationship of individual photographs to the airbase is established, and the second of absolute orientation, where the space model is rotated and is scaled to fit ground control. It is usual practice to associate the removal of Y-parallaxes with the first stage and the establishment of correct X-parallaxes with the latter.

It is not generally realized that certain problems of relative orientation can be solved by measuring height deformations in the overlap. The purpose of this paper is to introduce problems involving X and Y-parallaxes simultaneously. Since the placement of heights is necessary for absolute orientation, it will be possible, by placing them carefully, to carry out relative orientation, at the same time, in problems which otherwise would be insolvable. It should be mentioned here that complete relative orientation by measuring heights is not possible because height differences depend on relative and absolute orientation with influences that are not completely separable.

With the relative orientation of dependent pairs, we may assume the basic parallax equations to first order terms as follows:

$$P_x = dbx - \frac{X}{Z} dbz - \frac{XY}{Z} d\omega + \frac{Z^2 + X^2}{Z} d\varphi + Y d\kappa \tag{1}$$

$$P_y = dby + \frac{Y}{Z} dbz + \frac{Z^2 + Y^2}{Z} d\omega - \frac{XY}{Z} d\varphi + X d\kappa \tag{2}$$

Equation (1) expressed as height deformation and taking into account the effects of absolute orientation, becomes:

$$\Delta Z = \frac{h}{b}\left(dbx - \frac{X}{h} dbz - \frac{XY}{h} d\omega + \frac{h^2 + X^2}{h} d\varphi + Y d\kappa\right) + X d\phi + Y dw \tag{3}$$

For 6 points uniquely placed as in Figure 1, equation (1) becomes:

$$\Delta Z_1 = \frac{h}{b}\left(dbx - \frac{b}{h} dbz + \frac{h^2 + b^2}{h} d\varphi\right) + b d\phi \tag{4}$$

$$\Delta Z_2 = \frac{h}{b}(dbx + h d\varphi) \tag{5}$$

$$\Delta Z_3 = \frac{h}{b}\left(dbx - \frac{b}{h} dbz - \frac{bd}{h} d\omega + \frac{h^2 + b^2}{h} d\varphi + d d\kappa\right) + b d\phi + d dw \tag{6}$$

$$\Delta Z_4 = \frac{h}{b}(dbx + h d\varphi + d d\kappa) + d dw \tag{7}$$

$$\Delta Z_5 = \frac{h}{b}\left(dbx - \frac{b}{h} dbz + \frac{bd}{h} d\omega + \frac{h^2 + b^2}{h} d\phi - d d\kappa\right) + b d\phi - d dw \tag{8}$$

$$\Delta Z_6 = \frac{h}{b}(dbx + h d\phi - d d\kappa) - d dw \tag{9}$$

From equations (6), (7), (8), and (9)

$$d\omega = \frac{1}{2d}(\Delta Z_4 - \Delta Z_6 - \Delta Z_3 + \Delta Z_5) \tag{10}$$

For point 7 midway between 1 and 2 we have

$$\Delta Z_7 = \frac{h}{b}\left(dbx - \frac{b}{2h} dbz + \frac{4h^2 + b^2}{4h} d\varphi\right) + \frac{b d\phi}{2} \tag{11}$$

From equations (4), (5) and (11)

$$d\varphi = -\frac{2}{b}(2\Delta Z_7 - \Delta Z_1 - \Delta Z_2) \tag{12}$$

Equation (12) holds good for points similarly placed on lines parallel to the base.

The graphical significance of (10) and (12) is illustrated in Figures 2 and 3.

A relative tilt $d\omega_2$ about 1–2 as axis produces height deformations at 3 and 5, equal in quantity, but opposite in sign. From equation (3) it is clear that the amount of deformation varies as the product XY and the surface of deformation

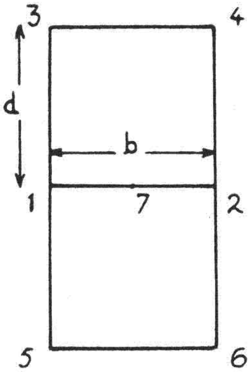

FIG. 1

is obviously a rectangular hyperboloid. Height differences due to $d\omega$ may be treated linearly along lines parallel and perpendicular to the base. Formula (10) is clearly evident from Figure 2, without further explanation.

The effect of $d\varphi_2$ about 4-2-6 as axis is to produce identical changes in height at 1, 3 and 5. From equation (3) it is clear that the deformation varies as X^2 and the surface of deformation is a paraboloid. The magnitude of the parabola symmetrically placed through 1 and 2 is given by the perpendicular distance of its vertex from the base.

The following unusual problems of relative orientation, illustrated in Figures 4 and 5, are solved by using Y-parallaxes and heights. Points of height control are indicated by circles and are selected with a view to relative and absolute orientation.

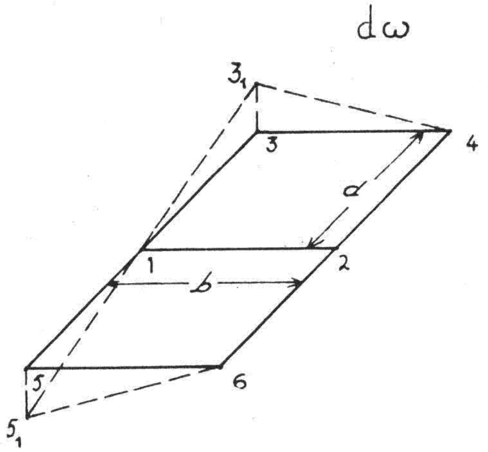

FIG. 2

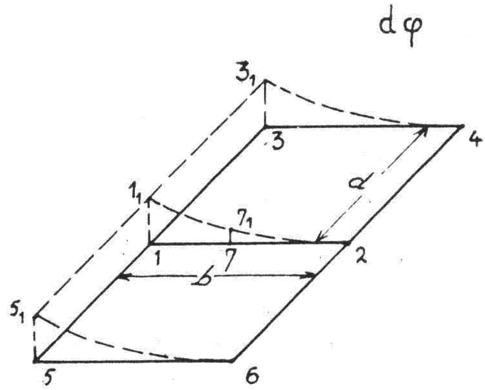

FIG. 3

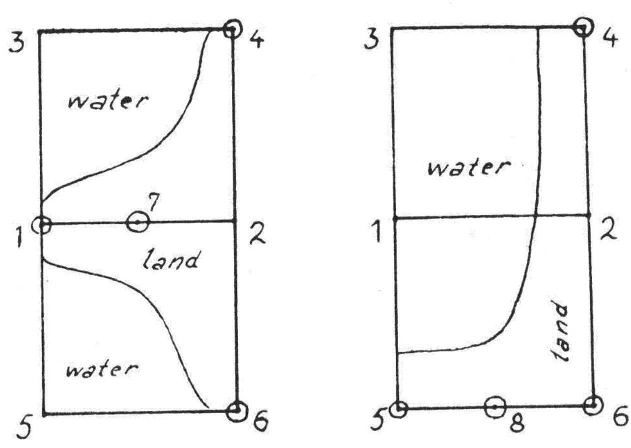

FIG. 4

STAGES OF RELATIVE ORIENTATION

Subject to repetition if necessary

FIG. 4a

1. Eliminate parallax at 2 with by
2. Eliminate parallax at 1 with $d\kappa$
3. Measure parallaxes at 2, 4, and 6. Determine elements $d\omega$ and dbz from (13) and (14) and set.
4. Derive ΔZ at 1, 4, 6 and 7. Interpolate between 4 and 6 for Δz_2. Determine $d\phi$ from (12) and set.
5. Measure by at 1, 2, 4 and 6 and set an average value.

FIG. 4b

1. As for stage 3 above.
2. Derive ΔZ at 5, 6 and 8. Determine $d\phi$ from (12) and set.
3. Set by at average reading for 2, 4 and 6.
4. Eliminate remaining parallax at 5 with $d\kappa$

Equations relevant to the use of Y-parallaxes are derived from (1) and given as follows:

$$d\omega = \frac{h}{2d^2}(2P_2 - P_4 - P_6) \tag{13}$$

$$dbz = \frac{h}{2d}(P_4 - P_6) \tag{14}$$

1. Measure parallaxes at 4 and 6 and heights at 3, 4, 5 and 6.
2. Determine $d\omega$ from (10) and set.
3. Determine dbz from (14) and set.
4. Set by at average of readings at 3 and 5 and remove parallax at 3 or 5 with $d\phi$
5. Set by at either 4 or 6 and remove parallax at 3 or 5 with $d\kappa$.

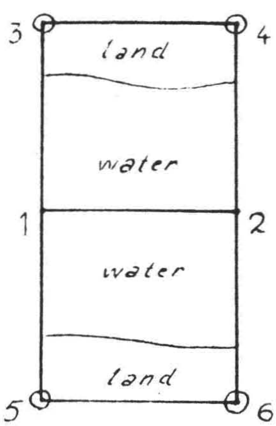

FIG. 5

In conclusion, it would be well to consider the comparative precision of derivations from Y-parallaxes and heights, in circumstances where either is possible.

Assuming measurement of parallaxes at points 1 to 6, the relevant weight numbers are given by

$$Q_{\omega\omega} = \frac{3h^2}{4d^4} Q_{byby} \quad \text{and} \quad Q_{\phi\phi} = \frac{h^2}{b^2d^2} Q_{byby}$$

From height measurements

$$Q_{\omega\omega} = \frac{1}{d^2} Q_{zz} \quad \text{and} \quad Q_{\phi\phi} = \frac{24}{b^2} Q_{zz}.$$

If heights are assumed measured twice as accurately as Y-parallaxes and a base/height ratio of 1.4, the mean square error in derived $d\omega$ will be six times smaller from height measurements. Where the ratio is less than 2.4 the use of Y-parallaxes for $d\varphi$ is to be preferred, otherwise a derivation from heights would be more precise.