Measuring Accuracy and Its Relation to Model Deformations and Other Measurements Made in a Stereo Model

WILLIAM C. MAHONEY, *Research Associate, The Ohio State University Research Foundation, Mapping and Charting Research Laboratory, Columbus 10, Ohio*

ABSTRACT: *In a paper entitled "Theorie des Erreurs de L'Orientation Relative," by Dr. W. K. Bachmann, equations were derived to describe model deformations in a stereo model. These equations however, differed from heretofore commonly accepted Von Gruber equations in the expression for Y model deformation. In this paper, an attempt is made to resolve these differences by constructing a more general theory of model deformations based upon the relative accuracy of measurement of a point in a photo and its projected image. Taking this factor into account, it is concluded that in a perfect projection system, the equations derived by the methods of Von Gruber and Bachmann were equivalent. In addition to demonstrating this equivalence, a method is also presented for weighting measurements of Y parallaxes and model coordinate measurements in a stereo model.*

1. INTRODUCTION

IN A paper entitled, "Theorie des Erreurs de L'Orientation Relative," by W. K. Bachmann[1], a set of equations were developed which were intended to describe the model deformation present in a stereo model after relative orientation had been completed in an autograph type plotting instrument. These equations however, were at variance with the heretofore commonly accepted Von Gruber[2] equations in the expression for δY model deformations. The reason for this difference is found in the way in which the principle of symmetry has been applied in the two derivations. In the relations developed by Von Gruber, the principle was applied in the model whereas in the relations developed by Bachman, it was applied in the plane of the photos themselves. Because the assumption of symmetry in the photo plane is more in agreement with the method of observation employed in an autograph, it was concluded by Bachmann that relations developed by Von Gruber were false in the general case for such an instrument.

Upon reviewing the theory behind these relations, it appeared to the author that a logical inconsistency was present because, assuming a perfect projection and measuring system, *it did not seem reasonable that the same source information (i.e., a photo image) should yield two different answers for the same quantity* simply because one set of measurements is made in the image plane and the other set made in the projection plane. Moreover, since the difficulty was not geometric—the results of the mathematical develop for both methods agreed perfectly with their geometric models—it was concluded that an important factor was

[1] Bachmann, W. K., "Theorie des Erreurs de L'Orientation Relative," Lausanne Imprimerie, La Concorde, 1943.

[2] Von Gruber, O., "Einfache- und Doppelpunkteinschaltung in Raum," 1924.

being neglected which was common to both methods, but which could only be uncovered by building a more general theory to describe the phenomena of model deformations. As will be shown in this paper, this factor concerns the relative accuracies of measurements made on a photo and its projection.

2. Definitions

Throughout this paper, the following defined quantities will be used:

$P(X, Y, Z)$ = Model coordinates of a point in a right hand coordinate system with Z positive upwards.

$P(x, y)$ = Photographic coordinates of a point on a photo positive in a right hand coordinate system with z positive upwards.

$R_k = \begin{bmatrix} X_k \\ Y_k \\ -(Z_k - h) \end{bmatrix}$ = Position vector from an exposure station L_k to any point in the stereo model.

$r_k = \begin{bmatrix} x_k \\ y_k \\ -f \end{bmatrix}$ = Position vector from an exposure station L_k to any point on a photographic positive.

A_k = An orthonormal matrix expressing the angular relationship between the axis of the photo coordinate system and the model coordinate system for a photo k.

A_{ki} = the ith row vector of A_k.

B_{kj} = the jth column vector of A_k.

A_k^* = Transpose of an orientation matrix A_k

$PS = (X_1 - X_2) = X$ parallax of a point in the stereo model

3. Fundamental Relationships between Photo and Ground Coordinate Systems:

In matrix notation, R_k and r_k are related as follows:

$$R_k = \frac{-(Z - h)}{z_{Tk}} A_k r_k \tag{1}$$

$$r_k = \frac{-f}{Z_{Tk}} A_k^* R_k \tag{2}$$

where

$$z_{Tk} = A_{k3} r_k \tag{3}$$

$$Z_{Tk} = B_{k3}^* R_k \tag{4}$$

Also, since $AA^* = I$ is the identity matrix, substituting (2) into (1) gives

$$\frac{Z - h}{z_{Tk}} = \frac{Z_{Tk}}{f} \tag{5}$$

Defining

$$dR_k = \begin{bmatrix} dX_k \\ dY_k \\ dZ_k \end{bmatrix}$$

and letting $dR_k|_{A_k}$ be the total differential of R_k regarding r_k constant and $dR_k|_{r_k}$ the total differential of R_k regarding A_k as constant, it can be shown that

$$dR_{k|A_k} = \frac{-(Z_k - h)}{z_{Tk}}\left[dA_k - \frac{dA_{k3}}{z_{Tk}} r_k A_k\right] r_k \tag{6}$$

$$dR_{k|r_k} = \frac{-(Z_k - h)}{z_{Tk}}\left[A_k + \frac{1}{(Z_k - h)} R_k A_{k3}\right] dr_k \tag{7}$$

$$dr_{k|A_k^*} = \frac{-f}{Z_{Tk}}\left[dA_k^* - \frac{dB_{k3}^*}{Z_{Tk}} R_k A_k^*\right] R_k \tag{8}$$

$$dr_{k|R_k} = \frac{-f}{Z_{Tk}} A_k^* \left[I - \frac{R_k B_{k3}^*}{Z_{Tk}}\right] dR_k \tag{9}$$

Substituting the appropriate orientation matrix for the particular system of rotation employed will result in expressions for dX, dY, and dx, dy found in any text book describing the theory of errors in photogrammetry.

4. Introduction to a General Theory of Model Deformations

Let it now be assumed that two photographs L_1 and L_2 have been relatively oriented. Defining the exposure station L_1 as the origin of a ground coordinate system, it is possible to determine the true position vector R with respect to L_1 by either of the following matrix relations:

$$R = R_1 \tag{10}$$

$$R = \overline{L}_2 + R_2 \tag{11}$$

where

R_1 is the position vector of P with respect to L_1
R_2 is the position vector of P with respect to L_2
$\overline{L}_2$ is the position vector of L_2 with respect to L_1

Assuming, for the time being, that relations (10) and (11) do not have the same weight when used to determine R, then to use the maximum information available, R should be the weighted mean of these relations, such that for,

$$W_1 R = W_1 R_1$$

$$W_2 R = W_2(\overline{L}_2 + R_2)$$

then

$$(W_1 + W_2)R = W_1 R_1 + W_2(\overline{L}_2 + R_2) \tag{12}$$

where W_k is in each case the diagonal matrix of the weights to be applied to X_k, Y_k, and Z_k respectively.

If R is the true vector as determined above, then the model deformations δR due to errors in R_1 and R_2 can be expressed by:

$$(W_1 + W_2)\delta R = W_1 \frac{\partial R}{\partial R_1} \Delta R_1 + W_2 \frac{\partial R}{\partial R_2} \Delta R_2 \tag{13}$$

Now the geometric condition for relative orientation using equations (10) and (11) is given by:

$$R_1 = \overline{L}_2 + R_2,$$

which after substitution of formula (1) becomes,

$$-\frac{(Z_1 - h)}{f}\frac{f}{z_{T1}}A_1r_1 = \overline{L}_2 - \frac{(Z_2 - h)}{f}\frac{f}{z_{T2}}A_2r_2$$

Factoring for (Z_1-h) gives:

$$\frac{-(Z_1 - h)}{f}\left[\frac{f}{z_{T1}}A_1r_1 - \frac{(Z_2 - h)}{(Z_1 - h)}\frac{f}{z_{T2}}A_2r_2\right] = \overline{L}_2 \tag{14}$$

Taking equation (14) out of matrix form and solving for (Z_1-h) and (Z_2-h) noting,

$$\overline{L}_2 = \begin{Bmatrix} B \\ 0 \\ 0 \end{Bmatrix}$$

the following results:

$$(Z_1 - h) = \frac{-Bf}{\dfrac{-f}{z_{T1}}A_{11}r_1 \quad \dfrac{(Z_2 - h)}{(Z_1 - h)}\dfrac{f}{zT_2}A_{21}r_2} = \frac{B(Z_1 - h)}{X_1 - X_2} \tag{15}$$

$$(Z_2 - h) = \frac{-Bf}{\dfrac{-(Z_1 - h)}{(Z_2 - h)}\dfrac{f}{z_{T1}}A_{11}r_1 + \dfrac{f}{z_{T2}}A_{21}r_2} = \frac{B(Z_2 - h)}{} \tag{16}$$

Substituting into (1), and recalling $P_X = X_1 - X_2$, then

$$R = R_1 = \frac{B}{P_X}R_1$$

$$R = \overline{L}_2 + R_2 = \overline{L}_2 + \frac{B}{P_X}R_2$$

therefore:

$$\Delta R_1 = \frac{B}{P_X}\left(dR_1 - \frac{R_1}{P_X}dP_X\right) \tag{17}$$

$$\Delta R_2 = \frac{B}{P_X}\left(dR_2 - \frac{R_2}{P_X}dP_X\right) \tag{18}$$

When (17) and (18) are substituted into (13), the following final expression is obtained relating weight and errors for X_k, Y_k, and Z_k to model errors δX, δY, δZ;

$$(W_1 + W_2)\delta R = W_1 dR + W_2 dR_2 - \frac{dP_X}{P_X}(W_1R_1 + W_2R_2) \tag{19}$$

5. Discussion of Factors Affecting W_k

In general, the determination of the elements of W_k depends on two factors:

a. The geometry of the scheme employed in making measurements of model deformations.
b. The accuracies of these measurements.

The first factor takes into account the geometry of the stereo model and the geometric place in space where the measurements will be made, whether it be in the plane of the photograph, in the model, or somewhere in between.

The second factor takes into account those influences which affect the accuracy of any measurement. These consist of such items as negative resolution, resolution of the projection system employed, accuracy of the measuring instruments, etc. It is this factor which was neglected by both Von Gruber and Bachmann and which led to their apparently different results.

Since the above two factors are independent of each other, they do not have to be treated simultaneously. However, before the influence of the second factor can be understood, it is necessary that the relations involved in the first factor be also understood. Therefore, the next two sections will be devoted to developing the matrix W_k based on the geometry of the Von Gruber and Bachmann methods.

6. Development of the Matrix W Based Upon the Geometry of the Von Gruber Method

The basic assumption of the Von Gruber method is that an unbiased estimate of the vector R is the arithmetic mean of the vectors R determined from the right and left photos. Referring to equation (12), this is in effect saying that W_1 and W_2 are identical. Letting $W_1 = W_2 = W$, and $Z_1 - h = Z_2 - h = Z - h$, and expanding equation (19) gives:

$$2W \begin{pmatrix} \delta X \\ \delta Y \\ \delta Z \end{pmatrix} = W \begin{pmatrix} dX_1 + dX_2 \\ dY_1 + dY_2 \\ 0 \end{pmatrix} - \frac{W dP_X}{B} \begin{pmatrix} X_1 + X_2 \\ Y_1 + Y_2 \\ -2(Z - h) \end{pmatrix} \tag{20}$$

Now if

$$W = \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{pmatrix},$$

solving for δR gives finally the well-known Von Gruber relations:

$$\delta Z = \frac{Z - h}{B} dP_X$$

$$\delta Y = -\frac{Y_1}{B} dP_X + \frac{dY_1 + dY_2}{2}$$

$$\delta X = -\frac{X_1}{B} dP_X + dX_1 \tag{21}$$

Since w_1, w_2 and w_3 cancel out, W can be any diagonal matrix, none of whose elements w_1, w_2, and w_3, equals zero. For purposes of convenience however, W is designated simply as the identity matrix. Therefore, for the Von Gruber method,

$$W_1 = W_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{22}$$

7. Development of the Matrix W_k Based upon the Geometry of the Bachmann Method

The basic assumption of the Bachmann method is that an unbiased estimate of the vector R is obtained by taking the arithmetic mean of the absolute values of the vectors dr_k which are the projections of dR_k in the negative plane of their respective photographs. The projected components of the vector R_k, designated as ξ_k and η_k respectively, are computed with the aid of equation (2). The errors in these components caused by errors in the vector R_k can be computed with the aid of equation (9) which is reproduced below:

$$dr_{k|R_k} = -\frac{f}{Z_{Tk}} A_k^* \left[I - \frac{R_k B_{k3}^*}{Z_{Tk}} \right] dR_k$$

Designating

$$dR_{Xk} = \begin{Bmatrix} dX_k \\ 0 \\ 0 \end{Bmatrix}$$

$$dR_{Yk} = \begin{Bmatrix} 0 \\ dY_k \\ 0 \end{Bmatrix}$$

and defining a new quantity Φ by,

$$\Phi_k = (\phi_{k1}, \phi_{k2}, \phi_{k3}) = A_k^* \left[I - \frac{R_k B_{k3}^*}{Z_{Tk}} \right] \tag{23}$$

then

$$d\xi_k = \frac{-f}{Z_{kTk}} \Phi_k dR_{Xk} \tag{24}$$

$$d\eta_k = \frac{-f}{Z_{Tk}} \Phi_k dR_{Yk} \tag{25}$$

Suppose now the mark, which represents the true position of the point in the plane of the negative, lies on the origin of the vectors $d\xi_k$ and $d\eta_k$. Then $d\eta_k$ and $d\xi_k$ represents the error in the position of a point with respect to the mark in the plane of the negative. Now in order to remove the discrepancies $d\eta_k$ and $d\xi_k$, it is required that the point be moved a distance $-d\eta_k$ and $-d\xi_k$ back to the mark. This is equivalent to saying that the image of the point in the model is moved through distances δX, δY and δZ which are related to dX_k and dY_k. To do this the following geometric conditions must be fulfilled,

$$-dX_k = -\delta X + \frac{X_k}{(H-h)} \delta Z$$

$$-dY_k = -\delta Y + \frac{Y_k}{(H-h)} \delta Z$$

$$\delta Z = \frac{(Z-h)}{B} dP_X \tag{26}$$

Substituting (26) into the negative of (24) and (25) gives then

$$
\begin{aligned}
-d\xi_k &= \frac{f}{Z_{Tk}}\,\Phi_k\left[\frac{-\delta Z}{(Z-h)}R_{Xk}+\delta R_X\right]\\
-d\eta_k &= \frac{f}{Z_{Tk}}\,\Phi_k\left[\frac{-\delta Z}{(Z-h)}R_{Yk}+\delta R_Y\right]
\end{aligned}
\tag{27}
$$

where

$$
\delta R_X = \begin{pmatrix}\delta X\\0\\0\end{pmatrix};\quad \delta R_Y = \begin{pmatrix}0\\\delta Y\\0\end{pmatrix};\quad R_{Xk} = \begin{pmatrix}X_k\\0\\0\end{pmatrix};\quad R_{Yk} = \begin{pmatrix}0\\Y_k\\0\end{pmatrix}
$$

Adding equations (27) to (24) and (25) gives then

$$
\begin{aligned}
\frac{-f}{Z_{Tk}}\,\Phi^4 dR_{Xk} &= \frac{f}{Z_{Tk}}\,\Phi_k\left[\frac{-\delta Z}{(Z-h)}R_{Xk}+\delta R_X\right]\\
\frac{-f}{Z_{Tk}}\,\Phi_k dR_{Yk} &= \frac{f}{Z_{Tk}}\,\Phi_k\left[\frac{-\delta Z}{(Z-h)}R_{Yk}+\delta R_Y\right]
\end{aligned}
\tag{28}
$$

Recalling $\Phi_k=(\phi_{k1},\phi_{k2},\phi_{k3})$ and taking (28) out of matrix form gives,

$$
\begin{aligned}
\frac{-f}{Z_{Tk}}\,\Phi_{k1}\delta X &= \frac{f}{Z_{Tk}}\,\phi_{k1}\left[dX_k - \frac{\delta Z}{(Z-h)}X_k\right]\\
\frac{-f}{Z_{Tk}}\,\Phi_{k2}\delta Y &= \frac{f}{Z_{Tk}}\,\phi_{k2}\left[dY_k - \frac{\delta Z}{(Z-h)}Y_k\right]
\end{aligned}
$$

Solving for δX and δY, noting that ϕ_{k1} and ϕ_{k2} are column vectors gives

$$
\begin{aligned}
\frac{f}{Z_{Tk}}\sqrt{\phi_{k1}{}^*\phi_{k1}}\;\delta X &= \frac{f}{Z_{Tk}}\sqrt{\phi_{k1}{}^*\phi_{k1}}\left[dX_k - \frac{\delta Z}{(Z-h)}X_k\right]\\
\frac{f}{Z_{Tk}}\sqrt{\phi_{k2}{}^*\phi_{k2}}\;\delta Y &= \frac{f}{Z_{Tk}}\sqrt{\phi_{k2}{}^*\phi_{k2}}\left[dY_k - \frac{\delta Z}{(Z-h)}Y_k\right]
\end{aligned}
\tag{29}
$$

Substituting further

$$
\frac{f}{Z_{Tk}}\sqrt{\phi_{k1}{}^*\phi_{k1}} = \psi_k \quad\text{and}\quad \frac{f}{Z_{Tk}}\sqrt{\phi_{k2}{}^*\phi_{k2}} = \Omega_k
$$

equations (29) can be written for both the left and right hand pictures by

$$
\begin{aligned}
\psi_1\delta X &= \psi_1\left[dX_1 - \frac{\alpha Z}{(Z-h)}X_1\right]\\
\psi_2\delta X &= \psi_2\left[dX_2 - \frac{\delta Z}{(Z-h)}X_2\right]
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\Omega_1\delta Y &= \Omega_1\left[dY_1 - \frac{\delta Z}{(Z-h)}Y_1\right]\\
\Omega_2\delta Y &= \Omega_2\left[dY_2 - \frac{\delta Z}{(Z-h)}Y_2\right]
\end{aligned}
\tag{31}
$$

Solving for δX and δY and recalling

$$\frac{\delta Z}{Z-h} = \frac{dP_X}{B}$$

$$\delta X = \frac{-X_1}{B} dP_X + dX_1$$

$$\delta Y = \frac{-Y_1}{B} dP_X + \frac{\Omega_1 dY_1 + \Omega_2 dY_2}{\Omega_1 + \Omega_2}$$

$$\delta Z = \frac{Z-h}{B} dP_X \tag{32}$$

It is clear from equations (30) and (31) that since weights have no effect on the determination of δZ, w_{k3} can be set equal to one, therefore, the weighting expression for the Bachmann relations in (32) which also satisfies (19), is given by

$$W_k = \begin{Bmatrix} \psi_k & 0 & 0 \\ 0 & \Omega_k & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

8. Development of the Matrix W_k for the Von Gruber Method Based upon Measuring Accuracy

The ultimate factor which limits measuring accuracy is the certainty with which a point can be identified in the plane of measurement. This certainty is affected by all the things which affect the image such as film resolution, lens resolution, film distortion, illumination, exposure conditions etc. The relationship between the weights of measurements made in the photo plane and in its projected image can be expressed mathematically as follows:

$$P_k \leqq K_k p_k \tag{33}$$

where

K_k is the weight of an image measurement in the projected plane.
p_k is the weight of an image measurement in the photo plane.
K_k is the ideal factor of proportionality between the two weights.

To show that this relation is reasonable, consider an errorless measuring system, and

$$p_k \geqq K_k^{-1} P_k$$

since p_k is given, then $P_k > K_k p_k$ cannot be true because this would imply that a projected image has a higher definition accuracy than the image being projected; this leads to the ridiculous conclusion that to improve his measurements all one would have to do would be to project his picture to a plane and reproject it back to the original photo plane and thereby obtain a better image than the one he started out with. Therefore, any relation between weights for measurements on a photo and its projection must be of the form given in (33).

But what of the condition $P_k < K_k p_k$. This condition exists if there are agents between the photo and its projection which cause the quality of the projected photo image to deteriorate to a degree greater than that given by the ideal factor of proportionality K_k. In this paper, however, it will be assumed these agents are not operative and that only the condition $P_k = K_k p_k$ exists. This is equivalent to saying there is a perfect geometric projection of the photo image to its projection plane, thereby eliminating any consideration of lens resolution, illumination, the method of measurement employed, etc.

8.1 DETERMINATION OF K_k

There are many different ways of defining K_k, depending upon how image quality is defined in the plane of the photo. In measurement theory, quality is generally defined in terms of the mean square error with which a point can be identified. This identification is the result of two measurements along different components which are generally considered independent. The variety of choices for these components permits K_k to be defined in many different ways. In the discussion to follow, it will be assumed that all photo measurements are made along those components whose projections in the projection plane are parallel to the X and Y axes. Under this assumption, the relations developed in paragraph (6) are applicable and can be rearranged into the following:

$$\left| d\xi_k \right| = \frac{-f}{Z_{Tk}} \sqrt{\phi_{k1}^* \phi_{k1}} \left| dR_{Xk} \right|$$

$$\left| d\eta_k \right| = \frac{-f}{Z_{Tk}} \sqrt{\phi_{k2}^* \phi_{k2}} \left| dR_{Yk} \right| \tag{34}$$

The above equations can be thought of as expressing the correlation between an error in a measurement in the image plane and an error in a measurement in its projection. Assuming now a series of independent measurements of $d\eta_k$, $d\xi_k$, dX_k and dY_k of the same point, the following statistical relationships can be written between these quantities.

$$E\{ \left| d\xi_k \right| \} = \frac{-f}{Z_{Tk}} \sqrt{\phi_{k1}^* \phi_{k1}}\, E\{ \left| dR_{Xk} \right| \}$$

$$E\{ \left| d\eta_k \right| \} = \frac{-f}{Z_{Tk}} \sqrt{\phi_{k2}^* \phi_{k2}}\, E\{ \left| dR_{Yk} \right| \} \tag{35}$$

where $\{\ \}$ is the expected value of the observed quantity within the braces $\{\ \}$.

If also assumed that each observation $d\xi_k$ and $d\eta_k$ is weighted by p_k, then since all observations are made on the same point, they all have the same weight, and

$$p_{\xi k} E\{ \left| d\xi_k \right| \} = \frac{-f}{Z_T} \sqrt{\phi_{k1}^* \phi_{k1} p_{\xi k} E\{ \left| dX_k \right| \}}$$

$$p_{\eta k} E\{ \left| d\eta_k \right| \} = \frac{-f}{Z_T} \sqrt{\phi_{k2}^* \phi_{k2} p_{\eta k} E\{ \left| dY_k \right| \}} \tag{36}$$

It is easily seen now, that if

$$K_{\xi k} = \frac{-f}{Z_{Tk}} \sqrt{\phi_{k1}^* \phi_{k1}} = \psi_k$$

and

$$K_{\eta k} = \frac{-f}{Z_{Tk}} \sqrt{\phi_{k2}^* \phi_{k2}} = \Omega_k$$

equation (33) is satisfied by

$$
\begin{aligned}
P_{Xk} &= \psi_k p_{\xi k} \\
P_{Yk} &= \Omega_k p_{\eta k}
\end{aligned}
\tag{37}
$$

8.2 APPLICATION OF MEASURING WEIGHTS TO VON GRUBER METHOD

From the above discussion, it has been shown that if measurements are made in the stereo model, they should be weighted according to the formulas of equation (37). Applying this principle to the general equation given in (12) and subsequently (19), leads to the conclusion that for the Von Gruber method, the final weighting matrix must be of the form

$$
W_k = \begin{Bmatrix} \psi_k p_{\xi k} & 0 & 0 \\ 0 & \Omega_k p_{\eta k} & 0 \\ 0 & 0 & 1 \end{Bmatrix} \tag{38}
$$

Therefore, the final form of the equations for the Von Gruber Method should be

$$
\begin{aligned}
\delta X &= \frac{-X_1}{B} dP_X + dX_1 \\
\delta Y &= \frac{-Y_1}{B} dP_X + \frac{p_{\eta 1}\Omega_1 dY_1 + p_{\eta 2}\Omega_2 dY_2}{p_{\eta 1}\Omega_1 + p_{\eta 2}\Omega_2} \\
\delta Z &= \frac{(Z-h)}{B} dP_X
\end{aligned}
\tag{39}
$$

9. DEVELOPMENT OF THE MATRIX W FOR THE BACHMANN METHOD BASED UPON MEASURING ACCURACY

As previously, let it be assumed that the weight of measurements ξ_k and η_k are $p_{\xi k}$ and $p_{\eta k}$ respectively. Then equations (30) and (31) become

$$
\begin{aligned}
\psi_1 p_{\xi 1}\delta X &= \psi_1 p_{\xi 1}\left(dX_1 - \frac{\delta Z}{Z-h} X_1\right) \\
\psi_2 p_{\xi 2}\delta X &= \psi_2 p_{\xi 2}\left(dX_2 - \frac{\delta Z}{Z-h} X_2\right) \\
\Omega_1 p_{\eta 1}\delta Y &= \Omega_1 p_{\eta 1}\left(dY_1 - \frac{\delta Z}{(Z-h)} Y_1\right) \\
\Omega_2 p_{\eta 2}\delta Y &= \Omega_2 p_{\eta 2}\left(dY_2 - \frac{\delta Z}{Z-h} Y_2\right)
\end{aligned}
\tag{40}
$$

Solving these equations in the same manner as (32) gives finally,

$$
W_k = \begin{Bmatrix} \psi_k p_{\xi k} & 0 & 0 \\ 0 & \Omega_k p_{\eta k} & 0 \\ 0 & 0 & 1 \end{Bmatrix} \tag{41}
$$

and

$$\delta X = \frac{-X_1}{B} dP_X + dX_1$$

$$\delta Y = \frac{-Y_1}{B} dP_X + \frac{p_{\eta 1}\Omega_1 dY_1 + p_{\eta 2}\Omega_2 dY_2}{p_{\eta 1}\Omega_1 + p_{\eta 2}\Omega_2}$$

$$\delta Z = \frac{(Z - h)}{B} dP_X \tag{42}$$

Note that the expressions for W_k and the model deformations δR are identical for the Von Gruber and Bachmann Methods after measuring accuracy has been taken into consideration. This means, of course, that based on the purely theoretical considerations given in this paper, it must be concluded that both the Bachmann and Von Gruber Methods are equivalent. Any differences which arise in the application of these methods can only be ascribed to those mechanical agencies which alter the quality of the projected image or to the methods of measurement employed.

(It is interesting to note also, that if $p_{\eta 1}$ and $p_{\eta 2}$ are equal, then equation (42) reduces to (32). It is this characteristic of the Bachmann equation which led to its differing from the Von Gruber Equation, for under this condition measuring accuracy is assumed in the geometric statement of symmetry in the photo plane. Such is not the case for the Von Gruber Method.)

10. Effect of Measuring Accuracy in the Weight of Y Parallax and Model Measurements

In the measurement of Y parallaxes and the X, Y, Z coordinates of a point in a stereo model, two major sources of error affect the actual value of those measurements. The first consists of the systematic errors introduced by errors in the elements of relative orientation, and the second, the random errors caused by an inability to make measurements of these quantities accurately. Since errors in relative orientation affect only the geometry of the stereo model, they cannot have any bearing on the accuracy with which measurements can be made of points in the stereo model. Hence it must be concluded that errors in relative orientation can have no influence on weighting measurements of Y parallax and coordinates X, Y, and Z in a stereo model. On the other hand, by its very definition, the random errors of measurement do affect such observations and therefore, must directly influence their weights. The following sections based upon the notions developed previously in this paper will develop the theory for determining such weights.

10.1 Effect of Measuring Accuracy on the Weight of Y Parallax Measurement

It will be recalled that Y parallax in a stereo model is defined by

$$P_Y = Y_2 - Y_1 \tag{43}$$

and

$$dP_Y = dY_2 - dY_1 \tag{44}$$

Applying the general law of error propagation, the relation between the weight coefficients of P_Y, dY_2, and dY_1 is given by

$$Q_{P_Y P_Y} = Q_{Y_2 Y_2} + Q_{Y_1 Y_1} - 2Q_{Y_1 Y}$$

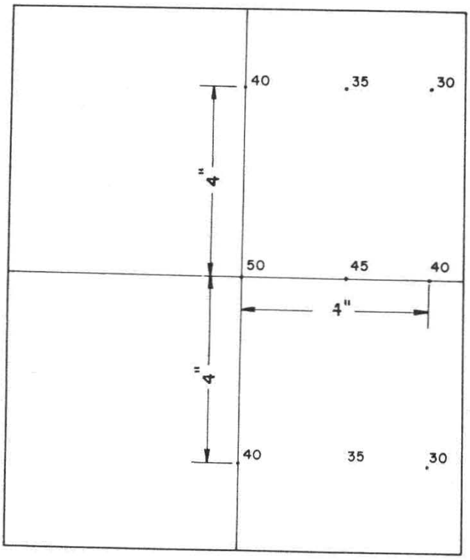

FIG. 1. Resolution in the photo plane in lines mm.

Since a measuring error for Y_1 is independent of Y_2, then

$$Q_{P_Y P_Y} = Q_{Y_2 Y_2} + Q_{Y_1 Y_1} \tag{45}$$

But from equation (37), the weights of Y_2 and Y_1 are given by $p_{\eta 1}\Omega_1$ and $p_{\eta 2}\Omega_2$, respectively. Therefore, the weight coefficient of a Y parallax measurement is given by

$$Q_{P_Y P_Y} = \frac{1}{p_{\eta 1}\Omega_1} + \frac{1}{p_{\eta 2}\Omega_2} \tag{46}$$

As a practical example in order to show the application of equation (46), the weight coefficients were computed for a model formed by 20 degree convergent photography. For this example $p_{\eta k}$ was assumed to be proportional to the resolution in terms of lines per millimeter as shown in Figure 1. Evaluating equation (34) for Ω_k for a ϕ, ω, κ system of orientation revealed for $\omega_k = \kappa_k = 0$,

$$\Omega_k = \frac{-f}{Z_{Tk}} = \frac{-f}{-X_k \sin\phi_k + (Z - h)\cos\phi_k} \tag{47}$$

In Figure 2 are shown the comparative dimensions of the neat model and its dimensions in the left hand photo.

Table I shows all the pertinent data for the computation of the weight coefficients for each point.

Noting $k/Q_{P_Y P_Y}$ are normalized weight coefficients based on the weight coefficient of point 9, Figure 3 shows a graph of lines of constant weight coefficients in a stereo model formed from convergent photography.

TABLE I
DATA FOR THE COMPUTATION OF WEIGHT COEFFICIENTS OF A Y-PARALLAX OBSERVATION IN A STEREO MODEL FORMED FROM 20° CONVERGENT PHOTOGRAPHS

Model Coordinates				Y_1		Y_2		$Q_{P_Y P_Y}$	Normalized weight k
Pt.	X	Y	Z_T	Ω_1	$p_{\eta 1}$	Ω_2	$p_{\eta 2}$		$Q_{P_Y P_Y}$
1	2.184	0	−6.385	.940	1.0	.712	.8	2.816	.97
2	8.165	0	−8.431	.712	.8	.940	1.0	2.816	.97
3	2.184	4.257	−6.385	.940	.8	.712	.6	3.669	.75
4	8.165	4.257	−8.431	.712	.6	.940	.8	3.669	.75
5	2.184	−4.257	−6.385	.940	.8	.712	.6	3.669	.75
6	8.165	−4.257	−8.431	.712	.6	.940	.8	3.669	.75
7	5.174	4.257	−7.408	.810	.7	.810	.7	3.524	.78
8	5.174	−4.257	−7.408	.810	.7	.810	.7	3.524	.78
9	5.174	0	−7.408	.810	.9	.810	.9	2.740	1.00

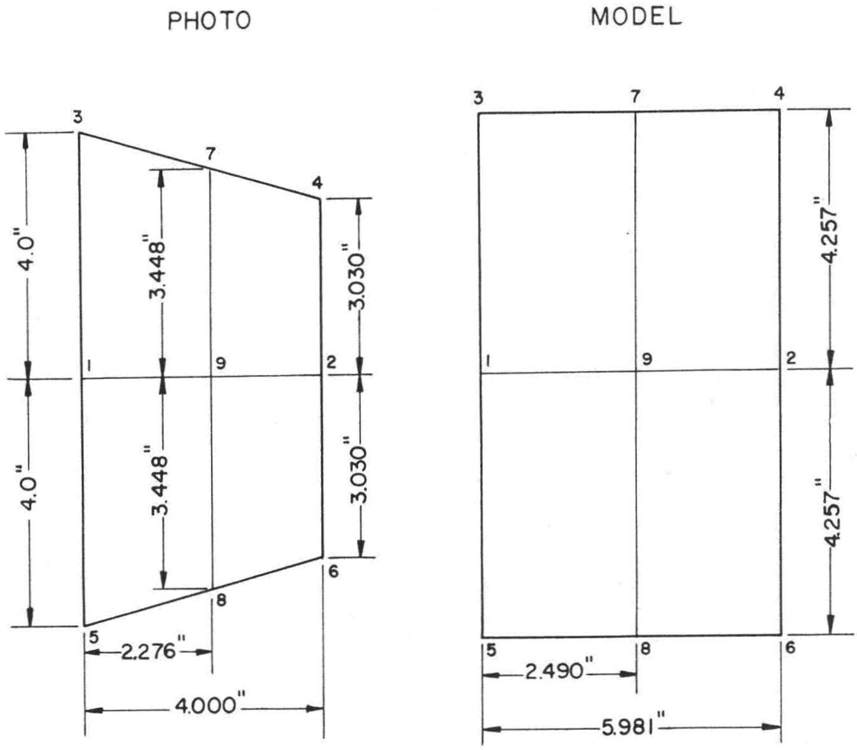

FIG. 2. Comparative dimensions of the neat model area and its corresponding dimensions in the left hand photo.

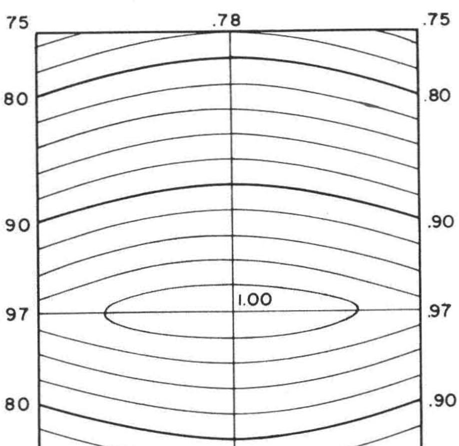

FIG. 3. Graph of lines of constant weight coefficients for Y parallax measurements in a 20° convergent photography model.

By studying Figure 3 it is readily seen that significant variations exist for the relative weight coefficients of measurement of Y parallaxes for the resolution model postulated. On the basis of this test example, it seems that further investigation should be considered in this area, particularly to study the effect of these differences on relative orientation as they apply to aerial triangulation.

10.2 EFFECT OF MEASURING ACCURACY ON THE WEIGHT OF MODEL DEFORMATION MEASUREMENTS

Recalling the δX, δY, and δZ are observed model deformations and that

$$\delta X = \frac{-X_1}{B} dP_X + dX_1$$

$$\delta Y = \frac{-Y_1}{B} dP_X + \frac{p_{\eta 1}\Omega_1 dY_1 + p_{\eta 2}\Omega_2 dY_2}{p_{\eta 1}\Omega_1 + p_{\eta 2}\Omega_2}$$

$$\delta Z = \frac{Z - h}{B} dP_X$$

and applying the laws of weight propagation and substituting from (37), observational weight coefficients of measurement of δX, δY, δZ are given by the following expressions

$$Q_{\delta X \delta X} = \left(1 - \frac{X_1}{B}\right)^2 \frac{1}{p_{\xi 1}\psi_1} + \left(\frac{X_1}{B}\right)^2 \frac{1}{p_{\xi 2}\psi_2}$$

$$Q_{\delta Y \delta Y} = \left(\frac{Y_1}{B}\right)^2 \left(\frac{1}{p_{\xi 1}\psi_1} + \frac{1}{p_{\xi 2}\psi_2}\right) + \frac{(p_{\eta 1}\Omega_1) + (p_{\eta 2}\Omega_2)}{(p_{\eta 1}\Omega_1 + p_{\eta 2}\Omega_2)^2}$$

$$Q_{\delta Z \delta Z} = \left(\frac{Z - h}{B}\right)^2 \left[\frac{1}{p_{\xi 1}\psi_1} + \frac{1}{p_{\xi 2}\psi_2}\right] \tag{48}$$

Though a complete evaluation of these expressions for a numerical example will not be given in this paper, preliminary work has shown that here also there are significant differences in the weight coefficients of each of the above quantities at various positions in the stereo model. These differences appear to be of the same order of magnitude shown for the weight coefficients of the Y parallax measurements given previously.

The importance of investigating the above relations further is self evident particularly with respect to work being performed in aerial triangulation where it would be extremely desirable to know the relative worth of model measurements made in various parts of a stereo model and their effect on presently employed procedures.

11. Discussion and Conclusions

Summarizing, this paper has attempted to show that the apparent discrepancies between the equations of model deformation developed by Von Gruber and Bachmann are in reality non-existent and are due only to the omission of considerations of measuring accuracy in their derivation. To support this contention, complete sets of equations involving these considerations were developed for both methods and demonstrated to be equivalent. Although the techniques used in this paper were specifically directed towards eliminating the differences between the two methods, it is quite apparent that other situations can also be studied in the same manner. For instance, the author has studied the weighting of dY and dX when measurements are made in the photo plane, along the photo x and y axes—a situation that arises in computational procedures. Another example that could be studied is what happens when the measuring mark falls in neither the model nor the image plane.

Also, the possibility that Y parallax and model coordinate observation weight coefficients can vary significantly throughout a model merits further investigation. Such an investigation might be directed towards evaluating these effects on the determination of the elements of relative and absolute orientation with respect to the needs of aerial triangulation.

Finally, it seems that a fruitful area of study might be opened by analyzing the conditions under which

$$P_k < K_k p_k$$

Results of such a study would have particular applicability to such instruments as the Kelsh and Multiplex plotters where the above condition is almost sure to be involved.

ACKNOWLEDGMENTS

The author is grateful for the time given by his colleagues in the Mapping and Charting Research Laboratory to criticize this paper and in particular to Dr. A. Brandenberger who, while the author was one of his students in the Institute of Photogrammetry, Geodesy and Cartography, stimulated and encouraged the development and publication of the ideas presented.