A Theoretical Investigation of Aerial Triangulation as a Problem of Maxima and Minima

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ABSTRACT: *For planning aerial photography for aerial triangulation the main problem is to find such a flying altitude that a maximum of accuracy is obtained for a minimum of control points. This can be solved mathematically as demonstrated in the paper. The error propagation formulae in aerial triangulation are of fundamental importance for the procedure. It is proved that the most effective aerial triangulation is obtained from comparatively short strips. High flying altitudes will probably be used in the future for aerial triangulation over areas with sparse ground control.*

INTRODUCTION

IF AERIAL photography is to be planned for first-order aerial triangulation between control points of a certain density, how shall the scale of the photography and consequently the flying altitude be chosen in order to obtain maximum accuracy of the triangulation? Obviously too low an altitude will require too-many photographs and consequently the error accumulation in the triangulation procedure will become unfavorable. On the other hand, flying too high will give too-small a scale in the photographs and consequently too-strong an enlargement of the inevitable errors in the photographs. Such a problem can obviously be treated and solved only in a theoretical way.

THE PROCEDURE

First it is necessary to find the expressions for the error propagation in aerial triangulation strips. Such can be theoretically derived under certain assumptions. In particular, uniform conditions of photography and plotting must be assumed. Systematic errors in the photographs and of the plotting procedure (mainly disturbances of the fundamental central projections) must be assumed to be known to the extent that the residual errors can be treated as accidental errors. Systematic errors can be corrected during the plotting procedure by optical, mechanical or numerical means. It is also possible to correct known systematic errors by purely numerical procedures after a preliminary triangulation. Simultaneously, the inevitable discrepancies in certain elements, for instance, those of the relative orientation, can be adjusted and corrections applied to the results of the preliminary triangulation. The principles of such corrections were treated in [1].

The residual, inevitable errors of the fundamental operations of the procedure are then to be regarded as the cause of the errors found in the coordinates of the triangulation procedure. Assuming that the residual errors can be treated as accidental errors, the laws of their propagation can be derived from well-known procedures in accordance with the method of least squares.

It can be proved that the standard error of the y-parallax measurements in photogrammetric models can be regarded as a good approximation of the magnitude of the residual errors in question, after adjusted by the method of least squares. Consequently the errors of various triangulation procedures can be expressed as functions of this fundamental factor, which can be easily determined in each actual triangulation operation. If no strict adjustment according to the method of least squares were applied, the standard error of the y-parallax measurements could be approximated by the mean square values of the residual y-parallaxes in at least 15 points after the relative orientation.

When the formulae for the error distribution along the triangulation strip are derived, it is possible to determine such a relation between the triangulation distance and the flying altitude that the errors of the coordinates become as small as possible, for instance, in the middle of the strip.

Without presenting a complete derivation of the error distribution formulae, it will be demonstrated how such formulae systems can be used for a determination of the most effective relation between triangulation distance and flying altitude. Three different triangulation methods will be treated. In all three it will be assumed that necessary control points are present in the first and the last models of the triangulation strip only. No redundant control is assumed; see Figure 14.

1. Stereo-Radial Triangulation

In a strip consisting of n wide-angle pairs of photographs, the standard errors of the x- and y-coordinates of the nadir point of the photograph p can according to [2], with minor approximations, be expressed as:

$$m_x = \mu \frac{h}{c} \sqrt{\frac{p(n-p)}{6n} \{14p(n-p) + 25\}} \qquad (1)$$

$$m_y = \mu \frac{h}{c} \sqrt{\frac{p(n-p)}{6n} \{10p(n-p) + 107\}} \qquad (2)$$

μ is the standard error of image coordinate measurements. See Figures 1–4.

The maximum errors are to be expected in the middle of the strip or for

$$p = \frac{n}{2} \qquad (3)$$

After substituting this expression into

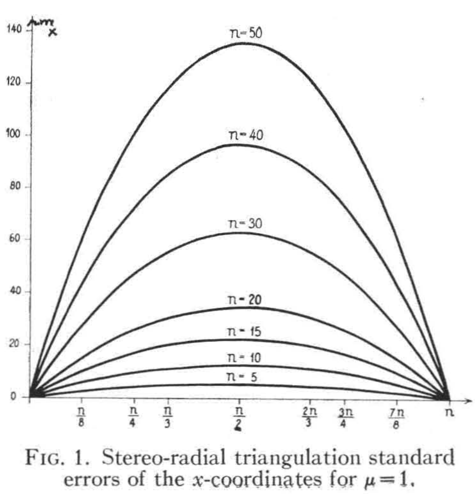

FIG. 1. Stereo-radial triangulation standard errors of the x-coordinates for $\mu=1$.

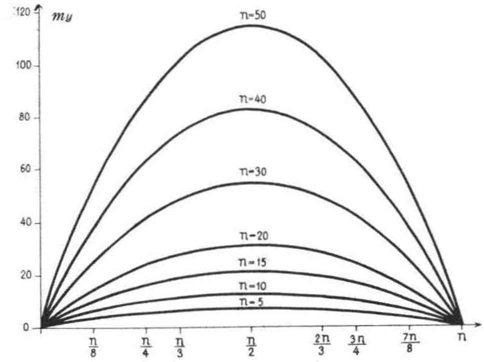

FIG. 2. Stereo-radial triangulation standard errors of the y-coordinates for $\mu=1$.

(1) and (2)

$$m_x = \mu \frac{h}{c} \sqrt{\frac{n}{24}\left(\frac{7n^2}{2} + 25\right)} \qquad (4)$$

$$m_y = \mu \frac{h}{c} \sqrt{\frac{n}{24}\left(\frac{5n^2}{2} + 107\right)} \qquad (5)$$

The radial standard error in the middle of the strip ($p=n/2$) is

$$m_r = \mu \frac{h}{c} \sqrt{\frac{n}{24} (6n^2 + 132)} \qquad (6)$$

If the triangulation distance (between the control points) is denoted by S and the average base length by b, the number n of

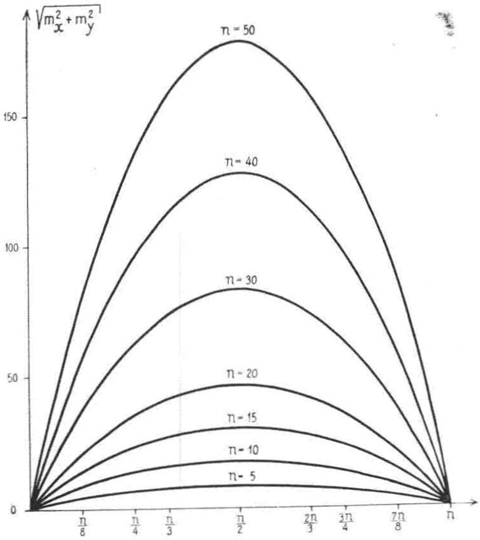

FIG. 3. Numerical stereo-radial triangulation. Radial standard errors $\mu h/c=1$.

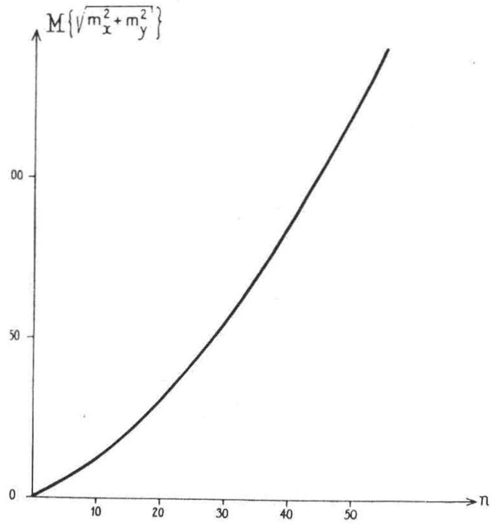

FIG. 4. Numerical stereo-radial triangulation. Mean values of radial standard errors $\mu h/c=1$.

bases will become

$$n = \frac{S}{b} \tag{7}$$

Further if the flying altitude is denoted by h and the base-height ratio by δ

$$b = h\delta \tag{8}$$

From (7) and (8)

$$n = \frac{S}{h\delta} \tag{9}$$

After substituting (9) into (6)

$$m_r = \frac{\mu}{c}\sqrt{\frac{S}{24\delta}\left(\frac{6S^2}{h\delta^2} + 132h\right)} \tag{10}$$

Now there is sought a determination of h so that m_r becomes as small as possible, finding the minimum of the expression

$$\min = \frac{6S^2}{h\delta^2} + 132h \tag{11}$$

Applying the usual procedure

$$\frac{d \min}{dh} = -\frac{6S^2}{h^2\delta^2} + 132 = 0 \tag{12}$$

Furthermore $d^2 \min/dh^2$ is always positive, indicating that there is a minimum.

From (12)

$$h = \frac{S\sqrt{22}}{22\delta} \tag{13}$$

For wide-angle photographs and normal overlap, $\delta = \frac{3}{5}$ and finally from (13)

$$h = 0.35S \tag{14}$$

or

$$S = 2.9h \tag{15}$$

Following these assumptions, the stereo-radial method will leave the smallest radial standard error in the middle of the strip, when the flying altitude is about one-third of the triangulation distance.

From (9)

$$n = 5 \tag{16}$$

or, in other words, the most favorable number of pairs of photographs is 5 for the stereo-radial triangulation.

The standard radial error can be found from (6). For $c=150$ mm.,

$$m_r = 0.0175\mu S \tag{17}$$

where μ is to be expressed in millimeters.

For $\mu=0.01$ mm.

$$m_r = 0.00017S$$

For example: for a triangulation distance of 20 km. the flying altitude would become 7,000 meters and the radial standard error in the middle of the strip about 3.4 meters.

2. MODEL TRIANGULATION. See [3]

In a strip consisting of n wide-angle pairs of photographs the standard errors of the x- and y-coordinates of the nadir point p of the photograph can be expressed approximately as (see [2])

$$m_x = \mu\frac{h}{c}\sqrt{\frac{p(n-p)}{3n}\{20p(n-p)+61\}} \tag{18}$$

$$m_y = \mu\frac{h}{c}\sqrt{\frac{p(n-p)}{9n}\{5p(n-p)+25\}} \tag{19}$$

μ is in this case the standard error of the y-parallax measurements if the relative orientation was adjusted in accordance with the method of least squares and if the procedure of the numerical corrections was applied. If these conditions were not satisfied the mean square value of the residual y-parallaxes be used as a close approximation for μ. See Figures 5–8.

From (18) and (19) the most favorable ratio between the triangulation distance and the flying altitude can be determined similarly to that used for the stereo-radial method.

For $p=n/2$ the radial standard error in the middle of the strip is

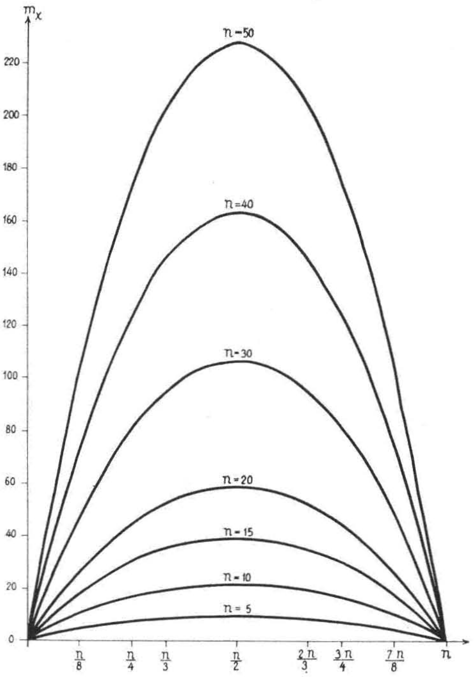

FIG. 5. Model triangulation standard errors of the x-coordinates for $\mu=1$.

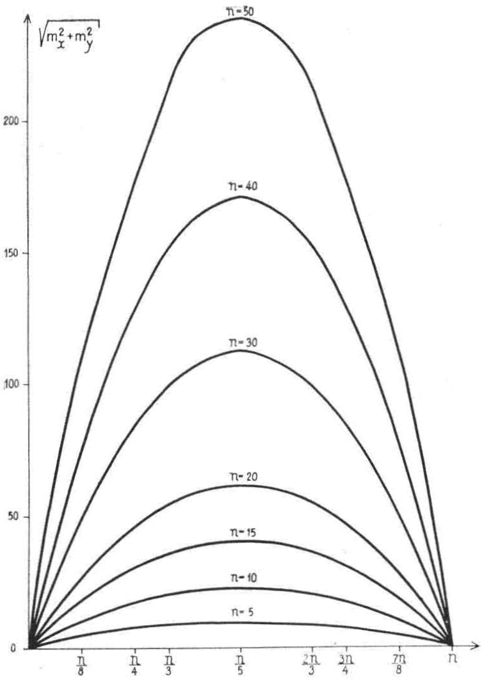

FIG. 7. Model triangulation. Radial standard errors $\mu h/c=1$.

$$m_r = \mu \frac{h}{12c} \sqrt{n(65n^2 + 832)} \tag{20}$$

Using the same procedure as demonstrated above, there is found a minimum of m_r for

$$h = 0.47S \quad \text{or} \quad S = 2.1h \tag{21}$$

Further

$$n = 4 \tag{22}$$

and for $c=150$ mm.

$$m_r = 0.0222\mu S \tag{23}$$

For $\mu=0.01$ mm.

$$m_r = 0.00022S$$

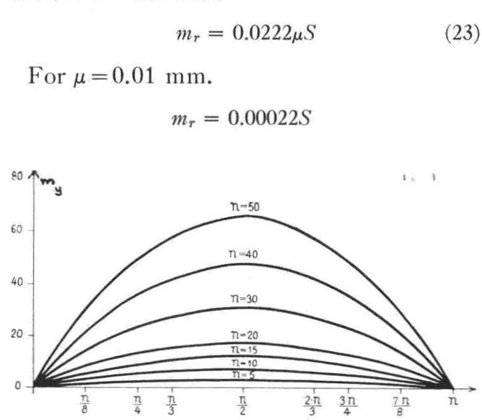

FIG. 6. Model triangulation standard errors of the y-coordinates for $\mu=1$.

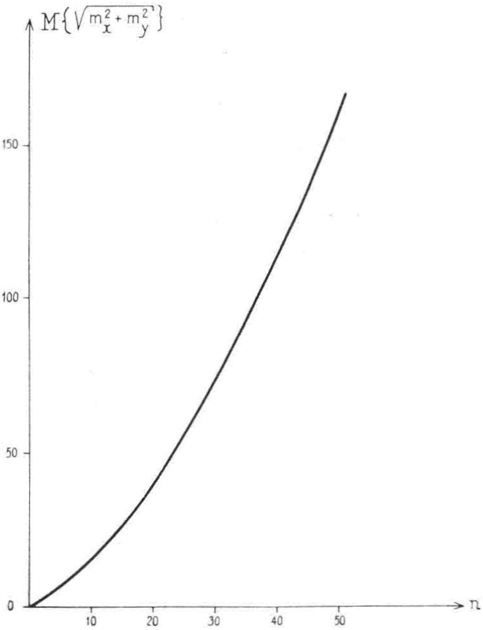

FIG. 8. Model triangulation. Mean values of radial standard errors $\mu h/c=1$.

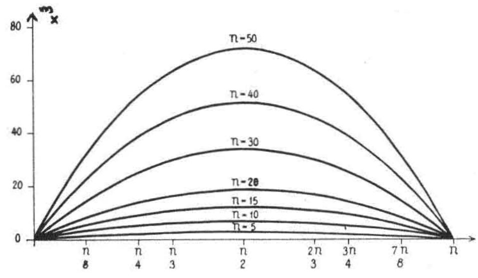

FIG. 9. Ordinary aerial triangulation standard errors of the x-coordinates for $\mu=1$.

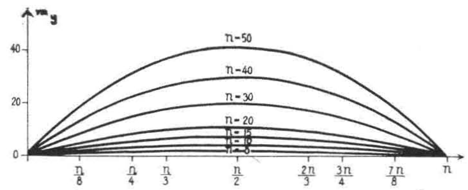

FIG. 10. Ordinary aerial triangulation standard errors of the y-coordinates for $\mu=1$.

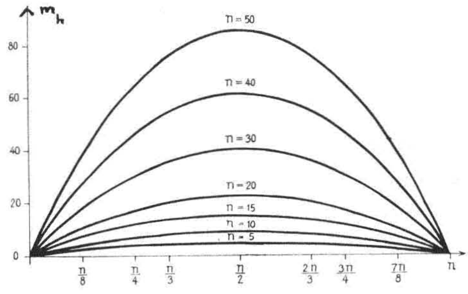

FIG. 11. Ordinary aerial triangulation standard errors of the elevations for $\mu=1$.

For $S=20$ km.

$$h = 9{,}400 \text{ meters}$$

and

$$m_r = 4.4 \text{ meters}$$

3. ORDINARY AERIAL TRIANGULATION

The standard errors of the three coordinates of the nadir point p of the photograph in a strip of n models of wide-angle photographs can be expressed with minor approximations as

$$m_x = \mu \frac{h}{c}\sqrt{\frac{p(n-p)}{3n}}\,\{2p(n-p)+1\} \qquad (24)$$

$$m_y = \mu \frac{h}{c}\sqrt{\frac{2p(n-p)}{9n}}\,\{p(n-p)+2\} \qquad (25)$$

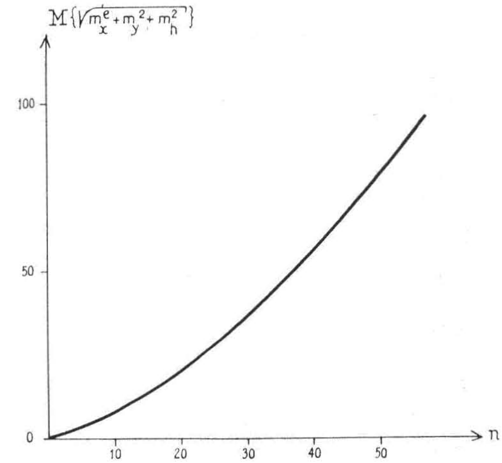

FIG. 12. Ordinary aerial triangulation. Mean values of radial standard errors $\mu h/c=1$.

$$m_z = \mu \frac{5h}{3c}\sqrt{\frac{p(n-p)}{6n}}\,\{2p(n-p)+13\} \qquad (26)$$

See Figures 9–13. μ is the standard error of the y-parallax measurements or the mean square value of the residual y-parallaxes. For $p=n/2$ (in the middle of the strip) the radial standard error m_r is found to be

$$m_r = \mu \frac{h}{12c}\sqrt{\frac{n}{3}\,(49n^2+734)} \qquad (27)$$

Using the same procedure as above there is found a minimum of m_r for

$$h = 0.43S \qquad (28)$$

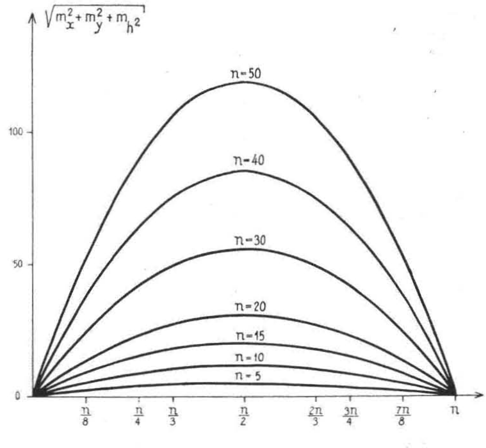

FIG. 13. Ordinary aerial triangulation. Radial standard errors $\mu h/c=1$.

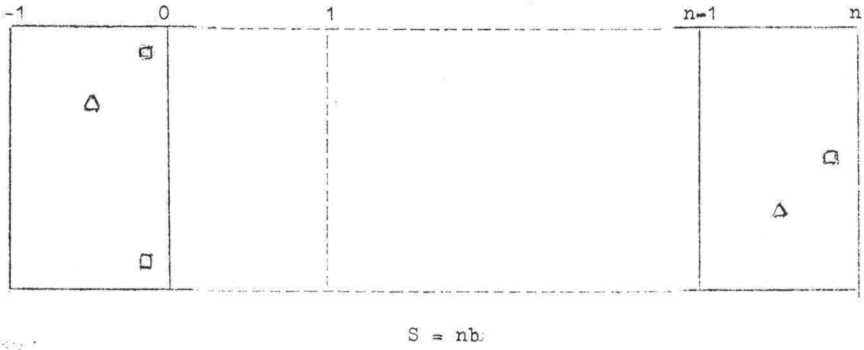

FIG. 14. The location of the control points in the strip consisting of the photographs $-1, 0, 1 \cdots n-1, n$. The triangles indicate planimetry control and the squares elevation control.

This means $n=4$.

For $c=150$ mm.

$$m_r = 0.0108\mu S \tag{29}$$

For $\mu=0.01$ mm.

$$m_r = 0.000108S \tag{30}$$

For $S=20$ km.

$$h = 8{,}600 \text{ meters}$$

and

$$m_r = 2.2 \text{ meters}$$

SUMMARY

The most effective aerial triangulation will be performed from comparatively few models. The flying altitude and the triangulation distance are mutually dependent and must be determined with respect to the tolerances of the final coordinates. These facts are of the greatest importance for the accuracy and economy of aerial triangulation.

REFERENCES

1. Hallert, B., "The Principles of Numerical Corrections in Aerial Photogrammetry," PHOTOGRAMMETRIC ENGINEERING, April 1956.
2. ———, "Felteori för Stereo-Radialtrianguler-ing och Modell-Triangulering." Manuscript. Stockholm 1957.
3. Ekelund, L., "Photogrammetric Triangulation with Separately Oriented Stereo-Models." *Photogrammetria*, 1951–1952: 4.

AUTHOR'S NOTE. Complete derivations of the fundamental expressions (1)–(2), (18)–(19) and (24)–(26) will be published in the near future.

*The Use of Astronomically Oriented Base Lines in Slotted Templet Triangulation**

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INTRODUCTION

IN MANY vast areas of the world no geodetic control whatever exists. For a first reconnaissance then, maps are often based on slotted templet triangulation, in which case a practical minimum of four control points, situated in the corners of a block, are required.

These control points are usually obtained by astronomical determination of latitude and longitude. This determination however,

* This paper was written during the time I was lecturing Photogrammetry in the International Training Centre for Aerial Surveys, at Delft, Netherlands.—The Author.